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Some Combinatorial Properties of the k-Fibonacci and the k-Lucas Quaternions

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Abstract

In this paper, we define the k-Fibonacci and the k-Lucas quaternions. We investigate the generating functions and Binet formulas for these quaternions. In addition, we derive some sums formulas and identities such as Cassini's identity.

1 Introduction

The Fibonacci numbers and their generalizations have many interesting properties and applications in many fields of science and art (see, e.g., [20]). The Fibonacci numbers F_n are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \ n \geqslant 1.$$

The first few terms are 0, 1, 1, 2, 3, 5, 8, 13,... (sequence A000045)*. Another important sequence is the Lucas sequence. This sequence is defined by the recurrence relation

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+1} = L_n + L_{n-1}, \ n \geqslant 1.$$

The first few terms are 2, 1, 3, 7, 11, 18, 29, 37... (sequence A000032).
Many kinds of generalizations of the Fibonacci sequence have been presented

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*Many integer sequences and their properties are to be found electronically on the On-Line Encyclopedia of Sequences [27].

in the literature (see, e.g., [20, 21]). In particular, there exist a generalization called the k-Fibonacci and the k-Lucas numbers.
For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined by

$$F_{k,0} = 0,\ F_{k,1} = 1, \quad \text{and} \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1},\ n \geqslant 1, \tag{1}$$

and the k-Lucas sequence, say $\{L_{k,n}\}_{n\in\mathbb{N}}$, is defined by

$$L_{k,0} = 2,\ L_{k,1} = k, \quad \text{and} \quad L_{k,n+1} = kL_{k,n} + L_{k,n-1},\ n \geqslant 1.$$

These sequences were studied by Horadam in [12]. Recently, Falcón and Plaza [6] found the k-Fibonacci numbers by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. The interested reader is also referred to [2, 3, 4, 5, 6, 7, 22, 23, 24, 25], for further information about these sequences.

On the other hand, Horadam [13] introduced the n-th Fibonacci and the n-th Lucas quaternion as follow:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + \kappa F_{n+3}, \tag{2}$$
$$K_n = L_n + iL_{n+1} + jL_{n+2} + \kappa L_{n+3}, \tag{3}$$

respectively. Here the basis i, j, κ satisface the following rules:

$$i^2 = j^2 = \kappa^2 = ij\kappa = -1. \tag{4}$$

Note that the rules (4) imply

$$ij = \kappa = -ji, \quad j\kappa = i = -\kappa j, \quad \kappa i = j = -i\kappa.$$

In general, a quaternion is a hyper-complex number and is defined by the following equation:

$$q = q_0 + iq_1 + jq_2 + \kappa q_3,$$

where i, j, κ are as in (4). Note that we can write $q = q_0 + u$ where $u = iq_1 + jq_2 + \kappa q_3$. The conjugate of the quaternion q is denoted by q^* and $q^* = q_0 - u$.

The Fibonacci and Lucas quaternions have been studied in several papers. For example, Swamy [26] gave some relations for the n-th Fibonacci quaternion. Horadam [14] studied some recurrence relations associated with the Fibonacci quaternions. Iyer [18, 19] derived relations connecting the Fibonacci and Lucas quaternions. Iakin [15, 16, 17] introduced the higher order quaternions and Binet formulas. Halici [11] investigated some combinatorial properties of Fibonacci quaternions and in [10] she studied the complex Fibonacci quaternions. Flaut and Shpakivskyi [8] studied some properties of generalized and Fibonacci quaternions and Fibonacci-Narayana quaternions, and in [9] they studied the left and right real matrix representations for the complex quaternions and Fibonacci quaternions. Akyiğit et.al. [1] introduced the split Fibonacci quaternions.

In analogy with (2) and (3), we introduce the k-Fibonacci and k-Lucas quaternions. We give some properties, the generating functions and Binet formulas for k-Fibonacci and k-Lucas quaternions. Moreover, we obtain some sums formulas for these quaternions and some identities such as Cassini's identity to k-Fibonacci quaternions.

2 Some properties of the k-Fibonacci and k-Lucas Numbers

The characteristic equation associated with the recurrence relation (1) is $z^2 - kz - 1 = 0$. The roots of this equation are

$$\alpha = \frac{k + \sqrt{k^2+4}}{2}, \qquad \beta = \frac{k - \sqrt{k^2+4}}{2}.$$

Then we have the following basic identities:

$$\alpha + \beta = k, \qquad \alpha - \beta = \sqrt{k^2+4}, \qquad \alpha\beta = -1.$$

Some of the properties that the k-Fibonacci numbers verify are summarized bellow (see [6, 7] for the proofs).

Binet formula: $$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ n \geq 0. \tag{5}$$

$$F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1},\ n \geq 0. \tag{6}$$

Generating function: $$f_k(z) = \frac{z}{1 - kz - z^2}. \tag{7}$$

$$\alpha^n = \alpha F_{k,n} + F_{k,n-1}. \tag{8}$$

Some properties that the k-Lucas numbers verify are summarized bellow (see [3] for the proofs).

Binet formula: $L_{k,n} = \alpha^n + \beta^n,\ n \geq 0.$

$$L_{k,n} = F_{k,n-1} + F_{k,n+1},\ n \geqslant 1.$$

$$L_{k,n}^2 + L_{k,n+1}^2 = (k^2+4)F_{k,2n+1}.$$

Generating function: $l_k(z) = \dfrac{2-kz}{1-kz-z^2}.$

3 Some properties of the k-Fibonacci and k-Lucas Quaternions

Definition 1. The k-Fibonacci quaternion $D_{k,n}$ is defined by

$$D_{k,n} = F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3},\ n \geq 0,$$

where $F_{k,n}$ is the n-th k-Fibonacci number.
The k-Lucas quaternion $P_{k,n}$ is defined by

$$P_{k,n} = L_{k,n} + iL_{k,n+1} + jL_{k,n+2} + \kappa L_{k,n+3},\ n \geq 0,$$

where $L_{k,n}$ is the n-th k-Lucas number.

Proposition 2. *The following identities hold:*

(i) $D_{k,n}D_{k,n}^* = (k^2+2)F_{k,2n+3}$.

(ii) $P_{k,n}P_{k,n}^* = (k^2+2)(k^2+4)F_{k,2n+3}$.

(iii) $D_{k,n}^2 = 2F_{k,n}D_{k,n} - D_{k,n}D_{k,n}^*$.

(iv) $P_{k,n}^2 = 2L_{k,n}P_{k,n} - P_{k,n}P_{k,n}^*$.

(v) $D_{k,n} + D_{k,n}^* = 2F_{k,n}$.

(vi) $P_{k,n} + P_{k,n}^* = 2L_{k,n}$.

(vii) $D_{k,n+2} = kD_{k,n+1} + D_{k,n}$.

(viii) $P_{k,n+2} = kP_{k,n+1} + P_{k,n}$.

Proof. (i) From Equations (6) and (1)

$$\begin{aligned} D_{k,n}D^*_{k,n} &= F^2_{k,n} + F^2_{k,n+1} + F^2_{k,n+2} + F^2_{k,n+3} \\ &= F_{k,2n+1} + F_{k,2n+5} \\ &= F_{k,2n+1} + k(kF_{k,2n+3} + F_{k,2n+2}) + F_{k,2n+3} \\ &= (k^2+1)F_{k,2n+3} + kF_{k,2n+2} + F_{k,2n+1} \\ &= (k^2+1)F_{k,2n+3} + F_{k,2n+3} \\ &= (k^2+2)F_{k,2n+3}. \end{aligned}$$

(ii) The proof is similar to (i).

(iii) From Proposition 2(i)

$$\begin{aligned} D^2_{k,n} &= (F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3})^2 \\ &= F^2_{k,n} - F^2_{k,n+1} - F^2_{k,n+2} - F^2_{k,n+3} + i(2F_{k,n}F_{k,n+1}) \\ &+ j(2F_{k,n}F_{k,n+2}) + \kappa(2F_{k,n}F_{k,n+3}) \\ &= 2F_{k,n}(F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + \kappa F_{k,n+3}) \\ &- F^2_{k,n} - F^2_{k,n+1} - F^2_{k,n+2} - F^2_{k,n+3} \\ &= 2F_{k,n}D_{k,n} - D_{k,n}D^*_{k,n}. \end{aligned}$$

(iv) The proof is similar to (iii).
The other identities are clear from definition.

□

4 Main Results

Theorem 3 (Binet's Formula)**.** *For $n \geq 0$, the Binet formulas for the k-Fibonacci and k-Lucas quaternions are as follow:*

$$D_{k,n} = \frac{1}{\sqrt{k^2+4}}(\hat{\alpha}\alpha^n - \hat{\beta}\beta^n) = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta}, \tag{9}$$

and

$$P_{k,n} = \hat{\alpha}\alpha^n - \hat{\beta}\beta^n, \tag{10}$$

respectively, where $\hat{\alpha} = 1 + i\alpha + j\alpha^2 + \kappa\alpha^3$ and $\hat{\beta} = 1 + i\beta + j\beta^2 + \kappa\beta^3$.

Proof. The characteristic equation of recurrence relation in Proposition 2(vii) is $z^2 - kz - 1 = 0$. Moreover, the initial values are $D_{k,0} = (0, 1, k, k^2 + 1)$ and $D_{k,1} = (1, k, k^2 + 1, k^3 + 2k)$. Hence,

$$D_{k,n} = A\alpha^n + B\beta^n.$$

Then, $D_{k,0} = A + B$ and $D_{k,1} = A\alpha + B\beta$, and from Equation (8) we obtain that

$$A = \frac{1}{\alpha - \beta}(D_{k,1} - \beta D_{k,0}) = \frac{1}{\sqrt{k^2 - 4}}(1 + i\alpha + j\alpha^2 + \kappa\alpha^3).$$

Analogously, $B = \frac{1}{\sqrt{k^2-4}}(1 + i\beta + j\beta^2 + \kappa\beta^3)$. Therefore,

$$D_{k,n} = \frac{1}{\sqrt{k^2 + 4}}(\hat{\alpha}\alpha^n - \hat{\beta}\beta^n) = \frac{\hat{\alpha}\alpha^n - \hat{\beta}\beta^n}{\alpha - \beta}.$$

Similarly, we can get Equation (10). □

Note that if $k = 1$ (see Equations (3.1) and (3.2) in [11]), then

$$D_{1,n} = \frac{1}{\sqrt{5}}(\hat{\alpha}\alpha^n - \hat{\beta}\beta^n),$$

and

$$P_{1,n} = \hat{\alpha}\alpha^n - \hat{\beta}\beta^n.$$

Theorem 4 (Cassini's identity)**.** *For $n \geq 1$, we have the following formula:*

$$D_{k,n-1}D_{k,n+1} - D_{k,n}^2 = (-1)^n(2D_{k,1} - (k^2 + 2k)\kappa).$$

Proof. We proceed by induction on n. If $n = 1$,

$$\begin{aligned} D_{k,0}D_{k,2} - D_{k,1}^2 =& (F_{k,0} + F_{k,1}i + F_{k,2}j + F_{k,3}\kappa)(F_{k,2} + F_{k,3}i + F_{k,4}j + F_{k,5}\kappa) \\ &- (F_{k,1} + F_{k,2}i + F_{k,3}j + F_{k,4}\kappa)(F_{k,1} + F_{k,2}i + F_{k,3}j + F_{k,4}\kappa) \\ &= -(2 + 2(k + 1)i + (k^2 + 1)j + (k^3 + 2k)\kappa) \\ &= (-1)^1(2D_{k,1} - (k^2 + 2k)\kappa). \end{aligned}$$

It is not difficult to show that the proposition is true for $n + 1$. □

Note that if $k = 1$ (see Equation (3.9) in [11]), then

$$D_{1,n-1}D_{1,n+1} - D_{1,n}^2 = (-1)^n(2D_{1,1} - 3\kappa).$$

From a numerical test in *Mathematica* we obtained the following conjecture:

Conjecture 5. *For $n \geq r \geq 1$, we conjecture the following formula:*

$$D_{k,n-r}D_{k,n+r} - D_{k,n}^2 = (-1)^{n-r}(2F_{k,r}D_{k,r} - G_{k,r}\kappa), \tag{11}$$

where $G_{k,r}$ is a sequence defined by

$$G_{k,0} = 0, G_{k,1} = k^2 + 2k, \text{ and } G_{k,n} = (k^2+2)G_{k,n-1} - G_{k,n-2},\ n \geq 2.$$

Example 6. If $n = 10$ and $r = 3$ in (11), then

$$\begin{aligned}&D_{k,n-r}D_{k,n+r} - D_{k,n}^2\\ &= (2+4k^2+2k^4) + (4k+6k^3+2k^5)i + (2+8k^2+8k^4+2k^6)j + (3k^3+4k^5+k^7)\kappa\end{aligned}$$

$$\begin{aligned}&2F_{k,r}D_{k,r}\\ &= (2+4k^2+2k^4)+(4k+6k^3+2k^5)i+(2+8k^2+8k^4+2k^6)j+(6k+14k^3+10k^5+2k^7)\kappa,\end{aligned}$$

and $G_{k,r} = 6k + 11k^3 + 6k^5 + k^7$. Then,

$$D_{k,7}D_{k,13} - D_{k,10}^2 = 2F_{k,3}D_{k,3} - G_{k,3}\kappa.$$

Note that, if this conjecture is true, then Cassini's identity is a particular case, $r = 1$.

Theorem 7. *For the k-Fibonacci quaternions $D_{k,n}$, we have*

$$\sum_{i=0}^{n} D_{k,mi+j} = \begin{cases} \frac{(-1)^m D_{k,nm+j} - D_{k,nm+m+j} + (-1)^j D_{k,m-j} + D_{k,j}}{(-1)^m - L_{k,m} + 1}, & \text{if} j < m; \\ \frac{(-1)^m D_{k,nm+j} - D_{k,nm+m+j} - (-1)^m D_{k,j-m} + D_{k,j}}{(-1)^m - L_{k,m} + 1}, & \text{otherwise.} \end{cases} \tag{12}$$

Proof.

$$\begin{aligned}\sum_{i=0}^{n} D_{k,mi+j} &= \sum_{i=0}^{n} \frac{\hat{\alpha}\alpha^{mi+j} - \hat{\beta}\beta^{mi+j}}{\sqrt{k^2+4}} = \frac{1}{\sqrt{k^2+4}}\left(\hat{\alpha}\alpha^j \sum_{i=0}^{n} \alpha^{mi} - \hat{\beta}\beta^j \sum_{i=0}^{n} \beta^{mi}\right)\\
&= \frac{1}{\sqrt{k^2+4}}\left(\hat{\alpha}\frac{\alpha^{nm+m+j} - \alpha^j}{\alpha^m - 1} - \hat{\beta}\frac{\beta^{nm+m+j} - \beta^j}{\beta^m - 1}\right)\\
&= \frac{1}{\sqrt{k^2+4}}\frac{1}{(\alpha\beta)^m - (\alpha^m + \beta^m) + 1}\big(\hat{\alpha}\alpha^{nm+m+j}\beta^m - \hat{\alpha}\alpha^{nm+m+j}\\
&\quad -\hat{\alpha}\alpha^j\beta^m + \hat{\alpha}\alpha^j - \hat{\beta}\beta^{nm+m+j}\alpha^m + \hat{\beta}\beta^{nm+m+j} + \hat{\beta}\beta^j\alpha^m - \hat{\beta}\beta^j\big)\\
&= \frac{1}{\sqrt{k^2+4}}\frac{1}{(-1)^m - L_{k,m} + 1}\Big((\hat{\alpha}\alpha^{nm+m+j}\beta^m - \hat{\beta}\beta^{nm+m+j}\alpha^m)\\
&\quad - (\hat{\alpha}\alpha^{nm+m+j} - \hat{\beta}\beta^{nm+m+j}) - (\hat{\alpha}\alpha^j\beta^m - \hat{\beta}\beta^j\alpha^m) + (\hat{\alpha}\alpha^j - \hat{\beta}\beta^j)\Big)\\
&= \frac{(-1)^m D_{k,nm+j} - D_{k,nm+m+j} - \frac{\hat{\alpha}\alpha^j\beta^m - \hat{\beta}\beta^j\alpha^m}{\sqrt{k^2+4}} + D_{k,j}}{(-1)^m - L_{k,m} + 1}.\end{aligned}$$

But

$$\hat{\alpha}\alpha^j\beta^m - \hat{\beta}\beta^j\alpha^m = \begin{cases} (-1)^{j+1}\sqrt{k^2+4}D_{k,m-j}, & \text{if } j < m; \\ (-1)^m\sqrt{k^2+4}D_{k,j-m}, & \text{otherwise.} \end{cases}$$

Therefore, Equation (12) is clear. □

From Theorem 7 we obtain the following corollary.

Corollary 8. *For the k-Fibonacci quaternions $D_{k,n}$, we have*

$$\sum_{i=0}^{n} D_{k,mi} = \frac{(-1)^m D_{k,nm} - D_{k,nm+m} + D_{k,m} + D_{k,0}}{(-1)^m - L_{k,m} + 1},$$
$$\sum_{i=0}^{n} D_{k,i} = \frac{1}{k}(D_{k,n} + D_{k,n+1} - D_{k,1} - D_{k,0}).$$

Theorem 9. *For $n \geq 0$, we have the following summation formulas:*

$$\sum_{i=0}^{n} \binom{n}{i} D_{k,i}k^i = D_{k,2n},$$
$$\sum_{i=0}^{n} \binom{n}{i} P_{k,i}k^i = P_{k,2n}.$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} D_{k,i}k^i &= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{\hat{\alpha}\alpha^i - \hat{\beta}\beta^i}{\alpha-\beta}\right) k^i \\
&= \frac{\hat{\alpha}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(k\alpha)^i - \frac{\hat{\beta}}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}(k\beta)^i \\
&= \frac{\hat{\alpha}}{\alpha-\beta}(1+k\alpha)^n - \frac{\hat{\beta}}{\alpha-\beta}(1+k\beta)^n \\
&= \frac{\hat{\alpha}}{\alpha-\beta}(\alpha^2)^n - \frac{\hat{\beta}}{\alpha-\beta}(\beta^2)^n \\
&= \frac{\hat{\alpha}\alpha^2 - \hat{\beta}\beta^2}{\alpha-\beta} = D_{k,2n}.
\end{aligned}$$

The proof of the second sum is analogously. □

Theorem 10. *The generating function for the k-Fibonacci and k-Lucas quaternions are*

$$G_k(z) = \frac{z + i + j(k+z) + \kappa(k^2+1+kz)}{1-kz-z^2}, \tag{13}$$

and

$$J_k(z) = \frac{2 - kz + i(k+2z) + j(k^2+2+kz) + \kappa(k^3+3k+(k^2+2)z)}{1-kz-z^2}, \tag{14}$$

respectively.

Proof. We begin with the formal power series representation of the generating function for $\{D_{k,n}\}_{n=0}^{\infty}$,

$$G_k(z) = D_{k,0} + D_{k,1}z + D_{k,2}z^2 + \cdots + D_{k,l}z^k + \cdots .$$

Then

$$\begin{aligned} kzG_k(z) &= kD_{k,0}z + kD_{k,1}z^2 + kD_{k,2}z^3 + \cdots + kD_{k,l}z^{k+1} + \cdots \\ z^2G_k(z) &= D_{k,0}z^2 + D_{k,1}z^3 + D_{k,2}z^4 + \cdots + D_{k,l}z^{k+2} + \cdots . \end{aligned}$$

Therefore

$$(1-kz-z^2)G_k(z) = D_{k,0} + (D_{k,1} - kD_{k,0})z.$$

So

$$G_k(z) = \frac{D_{k,0} + (D_{k,1} - kD_{k,0})z}{1-kz-z^2}.$$

The proof of Equation (14) runs like this. $\square$

Theorem 11. *For $m, n \in \mathbb{Z}$ the generating function of the k-Fibonacci quaternion $D_{k,m+n}$ and k-Lucas quaternion $P_{k,m+n}$ are*

$$\sum_{n=0}^{\infty} D_{k,n+m}z^n = \frac{D_{k,m} + D_{k,m-1}z}{1-kz-z^2},$$

and

$$\sum_{n=0}^{\infty} P_{k,n+m}z^n = \frac{P_{k,m} + P_{k,m-1}z}{1-kz-z^2}.$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{k,n+m} z^n &= \sum_{n=0}^{\infty} \left(\frac{\hat{\alpha}\alpha^{n+m} - \hat{\beta}\beta^{n+m}}{\alpha - \beta} \right) z^n \\
&= \frac{1}{\alpha - \beta} \left(\hat{\alpha}\alpha^m \sum_{n=0}^{\infty} \alpha^n z^n - \hat{\beta}\beta^m \sum_{n=0}^{\infty} \beta^n z^n \right) \\
&= \frac{1}{\sqrt{k^2 - 4}} \left(\hat{\alpha}\alpha^m \frac{1}{1 - \alpha z} - \hat{\beta}\beta^m \frac{1}{1 - \beta z} \right) \\
&= \frac{1}{\sqrt{k^2 - 4}} \left(\frac{(\hat{\alpha}\alpha^m - \hat{\beta^m}) + (\hat{\alpha}\alpha^{m-1} - \hat{\beta}\beta^{m-1})}{1 - kz - z^2} \right) \\
&= \frac{D_{k,m} + D_{k,m-1} z}{1 - kz - z^2}.
\end{aligned}$$

□

5 Conclusion

In this pear, we study a generalization of the Fibonacci and Lucas quaternions. Particularly, we define the k-Fibonacci and k-Lucas quaternions, and we find some combinatorial identities.

The k-Fibonacci sequence is a special case of a sequence called s-bonacci sequence which is defined recursively as a linear combination of the preceding s terms:

$$a_{n+s} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{s-1} a_{n+s-1},$$

where $c_0, c_1, \ldots, c_{s-1}$ are real constants. It would be interesting to introduce a s-bonacci quaternions.

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