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Independent [1,2]-number versus independent domination number

Sahar A. Aleid, Mercè Mora and María Luz Puertas

Abstract

A $[1,2]$-set S in a graph G is a vertex subset such that every vertex not in S has at least one and at most two neighbors in it. If the additional requirement that the set be independent is added, the existence of such sets is not guaranteed in every graph. In this paper we provide local conditions, depending on the degree of vertices, for the existence of independent $[1,2]$-sets in caterpillars. We also study the relationship between independent $[1,2]$-sets and independent dominating sets in this graph class, that allows us to obtain an upper bound for the associated parameter, the independent $[1,2]$-number, in terms of the independent domination number.

1 Introduction

All the graphs considered here are finite, undirected, simple and connected. Undefined basic concepts can be found in introductory graph theory literature as in [2, 5]. Let $G=(V,E)$ be a graph, a vertex subset S is *independent* if no two vertices in S are adjacent and it is *dominating* if every vertex not in S has at least one neighbor in it. The minimum cardinality of a dominating set of a graph G is the *domination number* of G, denoted by $\gamma(G)$. The minimum cardinality of an independent dominating set is $i(G)$, the *independent dominating number*.

An *efficient dominating* set [1], also called *perfect code*, is an independent dominating set such that every vertex not in the set has a unique neighbor in

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it. It is well known that all efficient dominating sets in a graph G have the same cardinality, that always agrees with $\gamma(G)$ [1], so in this case $\gamma(G) = i(G)$. This means that perfect codes are minimum dominating sets and, in addition, they are independent, but unfortunately the existence of this type of sets is not guaranteed in every graph [6]. Less demanding properties would allow the existence of similar sets in a wider range of graphs.

In [3], Chellali et al. define a subset $S \subseteq V$ in a graph G to be a *[1,2]-set* if every vertex which is not in S is adjacent to at least one but not more than two vertices in S, in this case we will say that S *[1,2]-dominates* G and the cardinality of a minimum $[1,2]$-set of G is the *[1,2]-dominating number* $\gamma_{[1,2]}(G)$. In [4] a similar definition was introduced with the additional condition of independence, and the minimum cardinality of an independent $[1,2]$-set is denoted by $i_{[1,2]}(G)$. Note that the existence of an independent $[1,2]$-set is not guaranteed in every graph and $i(G) \leq i_{[1,2]}(G)$, if G has an independent $[1,2]$-set.

In [3] different graph families satisfying the domination number agrees with the $[1,2]$-dominating number, are shown. For instance, a *caterpillar* C is a tree such that the removal of its leaves gives a path and they obtain that $\gamma(C) = \gamma_{[1,2]}(C)$, for every caterpillar C. In this paper we focus on the relationship between the independent $[1,2]$-number and the independent domination number. It is not difficult to find examples of caterpillars satisfying $i(C) < i_{[1,2]}(C)$ and this means that the addition of independence provides a different behaviour of the related parameters.

We study how big the difference can be between the independent domination number and the independent $[1,2]$-number in the graph class of caterpillars. The problem of characterizing graphs that admit independent $[1,2]$-sets is open and a characterization of trees having such sets is obtained in [4]. On the other hand in [7] authors show an algorithm to determine whether a caterpillar has an independent $[1,2]$-set. None of them provide an explicit formula for the independent $[1,2]$-number. In the family of caterpillars, using the information about the neighborhood of each vertex, we will characterize the existence of independent $[1,2]$-sets and we will compute both i and $i_{[1,2]}$.

The *spine* E_C of a caterpillar C is the path resulting from the removing of its leaves. A vertex v in the spine is a *support vertex* if there is a vertex ℓ with degree one, such that v and ℓ are neighbors, so v is the unique neighbor of ℓ and we will say that v has ℓ as a leaf. In Section 2 we provide a characterization of certain caterpillars that admit an independent $[1,2]$-set, in terms of the number of leaves of the vertices in the spine. This characterization will allow us to obtain an upper bound for the independent $[1,2]$-number in terms of the independent number, in this graph family. To this end in Section 3 we present some technical results that will allow us to prove the upper bound in

Section 4. We also present a realization theorem that provides examples of all possible values that i and $i_{[1,2]}$ can take, in our caterpillar family. With that theorem we finally show that the difference $i_{[1,2]} - i$ can reach any non negative integer.

2 Caterpillars having independent $[1,2]$-sets

We begin this section showing a necessary condition for a caterpillar C to have an independent $[1,2]$-set, in terms of the degree of its vertices. This condition is also sufficient in a particular class of caterpillars. We first define a labeling of vertices in E_C and hereafter we will identify these vertices with their labels.

Definition 1. *Let C be caterpillar and let v be a vertex of the spine of C. Then:*

- $v = 0$ *if it is not a support vertex,*
- $v = 1$ *if it has exactly one leaf, denoted by ℓ_v,*
- $v = 2$ *if it has exactly two leaves, denoted by ℓ_v^1 and ℓ_v^2,*
- $v = 3$ *if it has at least three leaves.*

The following proposition shows that a caterpillar having an independent $[1,2]$-set has some restrictions in its spine.

Proposition 1. *Let C be a caterpillar having an independent $[1,2]$-set. Then E_C does not contain any of the sequences $33, 23, 32, 313$.*

Proof. Let S be any independent $[1,2]$-set of C. Clearly every vertex labeled as 3 belongs to S, so the sequence 33 is not possible in order to keep independence. On the other hand, if a vertex labeled as 1 or 2 does not belong to S, then its leaves must be in S. Hence sequences $32, 23, 313$ are not possible because in every case a vertex would have three neighbors in S. $\square$

Our next target is to show that these necessary conditions are also sufficient, not in every caterpillar, but in a wide range of them. To this end we first need the following lemma.

Lemma 1. *Let $P_m = a_1a_2 \dots a_m$ be a path with m vertices and $m \neq 1, 2, 4$. Then P_m has an independent $[1,2]$-set S such that $a_1, a_m \notin S$ and both vertices have just one neighbor in S.*

Proof. Assume that m is an odd number then define $S = \{a_i \colon i \text{ is even}\}$. On the contrary if m is an even number, then $m \geq 6$ and define $S = \{a_i \colon i \text{ is odd and } i \geq 5\} \cup \{a_2\}$. In both cases S is an independent $[1,2]$-set of P_m, $a_1, a_m \notin S$ and both vertices have just one neighbor in S. $\square$

Now we can characterize a family of caterpillars having an independent $[1,2]$-set, in a local way.

Proposition 2. *Let C be a caterpillar such that E_C does not contain any sequence with exactly one, two or four consecutive vertices labeled as zero. Then C has an independent $[1,2]$-set if and only if E_C does not contain any of the sequences $33, 32, 23, 313$.*

Proof. By Proposition 1, we just need to prove sufficiency. Assume that the spine $E_C = v_1 \dots v_n$ does not contain any of the sequences $33, 32, 23, 313$.

Firstly suppose that every vertex in E_C is a support vertex and define the set $S = \{v \in V(E_C) \colon v = 3\} \cup \{\ell_v \colon v \in V(E_C) \text{ and } v = 1\} \cup \{\ell_v^1, \ell_v^2 \colon v \in V(E_C) \text{ and } v = 2\}$. Note that leaves adjacent to a vertex labeled with 3 are dominated by its support vertex, each vertex with label 2 is just dominated by its leaves and each vertex with label 1 is dominated by its leaf and, at most, one neighbor in the spine with label 3. So S is an independent $[1,2]$-set of C.

Now suppose that E_C contains some vertex labeled as zero and consider the decomposition $E_C = E_1 Z_1 E_2 Z_2 \dots Z_{r-1} E_r$ in consecutive sequences, such that each E_i is a maximal sequence of consecutive vertices with non-zero labels and each Z_i is a maximal sequence of consecutive vertices labeled as zero. By hypothesis each Z_i has length different from one, two and four.

Using Lemma 1, each Z_j has an independent $[1,2]$-set R_j not containing the first vertex and the last vertex of Z_j and both vertices are dominated just once by vertices in R_j. On the other hand, using that E_i contains no vertex labeled as zero, we obtain that it has an independent $[1,2]$-set S_i. Finally the set $S = \left(\bigcup_{i=1}^{r} S_i\right) \bigcup \left(\bigcup_{j=1}^{r-1} R_j\right)$ is an independent $[1,2]$-set of C. □

We close this section showing that caterpillars with spines consisting of just support vertices, have an special behaviour related to both independent dominating sets and independent $[1,2]$-sets. So these caterpillars will provide a favorable environment to compute the associated parameters i and $i_{[1,2]}$.

Proposition 3.
1. *Let C be a caterpillar such that every vertex in E_C is a support vertex. Then C has an independent dominating set with minimum size, containing no vertex of E_C with label 1.*
2. *Let C be a caterpillar having an independent $[1,2]$-set and such that every vertex in E_C is a support vertex. Then every independent dominating set of C with minimum size contains all vertices in E_C labeled as 3. Moreover, C has an independent $[1,2]$-set with minimum size, containing no vertex of E_C with label 1.*

Proof. 1. Let R be an independent dominating set of C with minimum size, containing a vertex $v \in E_C$, labeled as 1. Then the neighbors of v in E_C

are not in R and using that they are support vertices, their leaves are in R. Thus the set $R' = (R \setminus \{v\}) \cup \{\ell_v\}$ is an independent dominating set with the same cardinality as R. Repeating this process with every vertex in R labeled as 1 we get the desired set.

2. Let R be an independent dominating set of C with minimum size and suppose on the contrary that there exists $v \in E_C$, labeled as 3 and such that $v \notin R$. Using the same reasoning as in the previous item, we may assume that R contains no vertex labeled as 1. By Proposition 1, the neighbors of v in E_C are labeled as 1. Therefore the set $R'' = (R \setminus \{\ell \colon \ell \text{ is a leaf and a neighbor of } v\}) \cup \{v\}$ is an independent dominating set with smaller cardinality, which is not possible.

   Finally if S is an independent $[1,2]$-set with minimum size, containing vertices of E_C with label 1, the set obtained by removing such vertices and adding their leaves, is an independent $[1,2]$-set with the same size and it contains no vertex with label 1.

□

3 Upper bound for $i_{[1,2]}$: partial results

The independent domination number is a natural lower bound of the independent $[1,2]$-number, that is $i(G) \leq i_{[1,2]}(G)$, if G has an independent $[1,2]$-set. In this section and the following one we focus on caterpillars whose spines have just support vertices, and for this graph class we provide the following general upper bound

$$i_{[1,2]}(C) \leq \frac{7}{5} i(C) + \frac{2}{5} \tag{1}$$

We devote this section to prove some previous results, showing that inequality is true in some particular caterpillars. They will allow us to approach the general case. We begin with three lemmas that consider caterpillars having vertices labeled as 2 in the spine, in different positions.

Notation 1. *Given a caterpillar C with $E_C = v_1 \dots v_n$, we will describe a vertex subset $S \subseteq V(C)$ in the following way. For each $v_i \in E_C$ we put a circle or a hat, where v_i means that $v_i \in S$ and its leaves are not in S, and $\widehat{v_i}$ means $v_i \notin S$ but all its leaves belong to S.*

Lemma 2. *Let C be a caterpillar let $E_C = v_1 \dots v_n$ be its spine.*

1. *Suppose that $n \geq 2$ and $v_i = 2$ for every $i \in \{1, \dots n\}$.*

   (a) *If n is even then $i(C) = \frac{3n}{2}$.*

(b) *If n is odd then $i(C) = \frac{3n-1}{2}$.*

(c) *In any case C has a unique independent $[1,2]$-set S, that satisfies $v_1, v_n \notin S$ and $|S| = 2n$. Moreover $|S| \leq \frac{7}{5}i(C) + \frac{2}{5}$.*

2. *Suppose that $n \geq 3$, $v_i = 2$ for every $i \in \{1, \ldots n-1\}$ and $v_n = 1$.*

(a) *If n is even then $i(C) = \frac{3n-2}{2}$.*

(b) *If n is odd then $i(C) = \frac{3n-1}{2}$.*

(c) *In any case C has a unique independent $[1,2]$-set S, that satisfies $v_1, v_n \notin S$, the unique neighbor of v_n in S is ℓ_{v_n} and $|S| = 2n-1$. Moreover $|S| \leq \frac{7}{5}i(C)$.*

Proof. 1. (a) Assume that $n = 2m$. Note that two consecutive vertices in the spine do not both belong to an independent set. It is also clear that, if a vertex in the spine is not in a dominating set, then its leaves must be in it. We construct the following set $R = 2\widehat{2}\ldots 2\widehat{2}$, (using notation 1). It is clear that R is an independent dominating set with minimum size. Each 2 means one vertex in R and each $\widehat{2}$ means two vertices in R, so $i(C) = |R| = m + 2m = 3m = \frac{3n}{2}$.

(b) If $n = 2m+1$, then the set $R = 2\widehat{2}\ldots 2\widehat{2}2$ is a minimum independent dominating set and $i(C) = |R| = (m+1) + 2m = 3m + 1 = \frac{3n-1}{2}$.

(c) By Proposition 2, C has an independent $[1,2]$-set S. If $v_i \in S$ for some $i \in \{1 \ldots n-1\}$, then $v_{i+1} \notin S$, $\ell^1_{v_{i+1}}, \ell^2_{v_{i+1}} \in S$ and v_{i+1} has three neighbors in S, that is not possible. If $v_n \in S$, repeat the same argument with v_{n-1}. Thus the unique independent $[1,2]$-set of C is $S = \widehat{2}\,\widehat{2}\ldots\widehat{2}$, that satisfies $v_1, v_n \notin S$ and $|S| = 2n$. Finally, in case n is even,

$$\frac{7}{5}i(C) + \frac{2}{5} = \frac{7}{5}\frac{3n}{2} + \frac{2}{5} = 2n + \frac{n+4}{10} \geq 2n = |S|.$$

If n is odd then $n \geq 3$ and

$$\frac{7}{5}i(C) + \frac{2}{5} = \frac{7}{5}\frac{3n-1}{2} + \frac{2}{5} = 2n + \frac{n-3}{10} \geq 2n = |S|.$$

2. (a) If $n = 2m$ then $R = 2\widehat{2}\ldots\widehat{2}2\widehat{1}$ is a minimum independent dominating set. Therefore $i(C) = |R| = m + 2(m-1) + 1 = 3m - 1 = \frac{3n-2}{2}$.

(b) If $n = 2m+1$, then $R = 2\widehat{2}\ldots 2\widehat{2}\,\widehat{1}$ is a minimum independent dominating set and $i(C) = |R| = m + 2m + 1 = 3m + 1 = \frac{3n-1}{2}$.

(c) By Proposition 2, C has an independent $[1,2]$-set. Same considerations as above provide the unique independent $[1,2]$-set of C is the set $S = \widehat{2}\,\widehat{2}\ldots\widehat{2}\,\widehat{1}$. It has $2n-1$ elements, $v_1, v_n \notin S$ and the unique neighbor of v_n in S is ℓ_{v_n}. If n is even then $n \geq 4$ and

$$\frac{7}{5}i(C) = \frac{7}{5}\frac{3n-2}{2} = 2n - 1 + \frac{n-4}{10} \geq 2n - 1 = |S|.$$

If n is odd then

$$\frac{7}{5}i(C) = \frac{7}{5}\frac{3n-1}{2} = 2n - 1 + \frac{n+3}{10} \geq 2n - 1 = |S|.$$

□

Lemma 3. *Let C be a caterpillar with $E_C = v_1 \ldots v_n$, such that $n = 2m \geq 4$, $v_1 = 3, v_2 = 1$, $v_{2k-1} = 2, v_{2k} = 1, 2 \leq k \leq m$. Then $i(C) = n$ and C has an independent $[1,2]$-set S such that $v_n \notin S$ and its unique neighbor in S is ℓ_{v_n}. Moreover*

1. *if $n \equiv 0 \pmod 4$ then $|S| = \frac{5n}{4}$,*
2. *if $n \equiv 2 \pmod 4$ then $|S| = \frac{5n+2}{4}$.*

In both cases $|S| \leq \frac{7}{5}i(C)$.

Proof. At the least n vertices are needed to dominate C and the set $R = 3\widehat{1}2\widehat{1}\ldots 2\widehat{1}$ is an independent dominating set with n vertices, so $i(C) = |R| = n$.

1. If $n \equiv 0 \pmod 4$ then $n = 4s$ $(s \geq 1)$, and E_C consists of an initial pair 31 and $2s-1$ consecutive copies of the pair 21. The set $S = 3\widehat{1}\ \widehat{2}\,\widehat{1}\ 2\widehat{1}\ldots 2\widehat{1}\ \widehat{2}\,\widehat{1}$ is an independent $[1,2]$-set such that $v_n \notin S$ and its unique neighbor in S is ℓ_{v_n}. This set contains one copy of $3\widehat{1}$, s copies of $\widehat{2}\,\widehat{1}$ and $s-1$ copies of $2\widehat{1}$, therefore $|S| = 2 + 3s + 2(s-1) = 5s = \frac{5n}{4}$. Clearly $|S| = \frac{5n}{4} \leq \frac{7n}{5} = \frac{7}{5}i(C)$.

2. If $n \equiv 2 \pmod 4$ then $n = 4s + 2$ $(s \geq 1)$, and E_C consists of an initial pair 31 and $2s$ consecutive copies of the pair 21. The desired independent $[1,2]$-set in this case is $S = 3\widehat{1}\,\widehat{2}\,\widehat{1}\,2\widehat{1}\ldots 2\widehat{1}\,\widehat{2}\,\widehat{1}\,\widehat{2}\,\widehat{1}$, that satisfies $v_n \notin S$ and its unique neighbor in S is ℓ_{v_n}. The set contains

one copy of $\textcircled{3}\widehat{1}$, $s+1$ copies of $\widehat{2}\,\widehat{1}$ and $s-1$ copies of $\textcircled{2}\widehat{1}$, so $|S| = 2+3(s+1)+2(s-1) = 5s+3 = \frac{5n+2}{4}$. Finally

$$|S| \leq \frac{7}{5}i(C) \iff \frac{5n+2}{4} \leq \frac{7n}{5} \iff 25n+10 \leq 28n \iff 10 \leq 3n$$

and the last inequality is true because, in this case, $n \geq 6$.

□

Lemma 4. *Let C be a caterpillar let $E_C = v_1 \dots v_n$ be its spine.*

1. *Suppose that $n = 2m \geq 2$, $v_{2k-1} = 2, v_{2k} = 1$, $1 \leq k \leq m$. Then $i(C) = n$ and C has an independent $[1,2]$-set S such that $v_n \notin S$ and*
   (a) *if $n \equiv 0 \ (mod\ 4)$ then $|S| = \frac{5}{4}n$,*
   (b) *if $n \equiv 2 \ (mod\ 4)$ then $|S| = \frac{5n-2}{4}$.*

   In both cases $|S| \leq \frac{7}{5}i(C)$.
2. *Suppose that $n = 2m+1 \geq 3$, $v_{2k-1} = 2, v_{2k} = 1$, $1 \leq k \leq m$ and $v_{2m+1} \in \{2,3\}$. Then $i(C) = n$ and C has an independent $[1,2]$-set S such that $v_1 \notin S$ and*
   (a) *if $n \equiv 1 \ (mod\ 4)$ then $|S| = \frac{5n+3}{4}$,*
   (b) *if $n \equiv 3 \ (mod\ 4)$ then $|S| = \frac{5n+1}{4}$.*

   In both cases $|S| \leq \frac{7}{5}i(C)$.

Proof. 1. At the least n vertices are needed to dominate C and the set $R = \textcircled{2}\widehat{1} \dots \textcircled{2}\widehat{1}$ is an independent dominating set with n elements, so $i(C) = |R| = n$.

(a) If $n \equiv 0 \ (mod\ 4)$ then $n = 4s$ $(s \geq 1)$, and E_C consists of $2s$ consecutive copies of the pair 21. The set $S = \textcircled{2}\widehat{1}\ \widehat{2}\,\widehat{1} \dots \textcircled{2}\widehat{1}\ \widehat{2}\,\widehat{1}$ is an independent $[1,2]$-set of C satisfying $v_n \notin S$. Each pair $\textcircled{2}\widehat{1}$ has 2 vertices in S and each pair $\widehat{2}\,\widehat{1}$ has 3 vertices in S, so $|S| = 2s+3s = 5s = \frac{5}{4}n$. Moreover $|S| = \frac{5}{4}n \leq \frac{7}{5}n = \frac{7}{5}i(C)$.

(b) If $n \equiv 2 \ (mod\ 4)$ then $n = 4s+2$ $(s \geq 0)$, and in this case E_C consists of $2s+1$ consecutive copies of the pair 21. We construct the independent $[1,2]$-set $S = \textcircled{2}\widehat{1}\ \widehat{2}\,\widehat{1} \dots \widehat{2}\,\widehat{1}\ \textcircled{2}\widehat{1}$, that satisfies

$v_n \notin S$. Note that S has $s+1$ copies of $2\widehat{1}$ and s copies of $\widehat{2}\,\widehat{1}$, so $|S| = 2(s+1) + 3s = 5s + 2 = \frac{5n-2}{4}$. In this case

$$|S| \leq \frac{7}{5} i(C) \iff \frac{5n-2}{4} \leq \frac{7n}{5} \iff 25n - 10 \leq 28n$$

and the last inequality is true for any $n \geq 1$.

2. Again, a dominating set of C must have at least n vertices and $R = 2\widehat{1} \dots 2\widehat{1}\, v_n$ is an independent dominating set with size n, so $i(C) = |R| = n$.

   (a) If $n \equiv 1$ $(mod\ 4)$ then $n = 4s + 1$ $(s \geq 1)$, and E_C consists of $2s$ consecutive copies of pair 21 followed by vertex v_n. The set $S = \widehat{2}\,\widehat{1}\ 2\widehat{1}\ \dots 2\widehat{1}\ \widehat{2}\,\widehat{1}\ \widehat{2}\,\widehat{1} v_n$ is an independent $[1,2]$-set such that $v_1 \notin S$. Note that S contains $s+1$ copies of $\widehat{2}\,\widehat{1}$, $s-1$ copies of $2\widehat{1}$, and v_n, so $|S| = 3(s+1) + 2(s-1) + 1 = 5s + 2 = \frac{5n+3}{4}$. Moreover

   $$|S| \leq \frac{7}{5} i(C) \iff \frac{5n+3}{4} \leq \frac{7n}{5} \iff 25n + 15 \leq 28n \iff 15 \leq 3n$$

   and the last inequality is true because, in this case, $n \geq 5$.

   (b) If $n \equiv 3$ $(mod\ 4)$ then $n = 4s + 3$ $(s \geq 0)$, and E_C consists of $2s + 1$ consecutive copies of pair 21 followed by vertex v_n. Then $S = \widehat{2}\,\widehat{1}\ 2\widehat{1}\ \dots 2\widehat{1}\ \widehat{2}\,\widehat{1} v_n$ is an independent $[1,2]$-set that satisfies $v_1 \notin S$. Note that S contains $s+1$ copies of $\widehat{2}\,\widehat{1}$, s copies of $2\widehat{1}$, and v_n, so $|S| = 3(s+1) + 2s + 1 = 5s + 4 = \frac{5n+1}{4}$. Finally

   $$|S| \leq \frac{7}{5} i(C) \iff \frac{5n+1}{4} \leq \frac{7n}{5} \iff 25n + 5 \leq 28n \iff 5 \leq 3n$$

   and the last inequality is true because in this case $n \geq 3$.

$\square$

In the following theorem we characterize caterpillars attaining the natural lower bound, $i(C) = i_{[1,2]}(C)$. They will also be useful to prove the upper bound given in Equation 1.

Theorem 1. *Let C be a caterpillar having an independent $[1,2]$-set and such that every vertex in E_C is a support vertex. Then $i(C) = i_{[1,2]}(C)$ if and only if E_C does not contain any of sequences $22, 212, 213, 312$.*

Proof. Let C be a caterpillar and let $E_C = v_1 \dots v_n$ be its spine. Suppose that $i(C) < i_{[1,2]}(C)$, and let R be an independent dominating set of C with minimum size. By Proposition 3, we may assume that every vertex in E_C with label 3 belongs to R and that R does not contain vertices labeled as 1. By hypothesis R is not an independent $[1,2]$-set, so there exists one vertex $u \in V(C) \setminus R$ having at least three neighbors in R. Note that leaves belonging to $V(C) \setminus R$ have exactly one neighbor in R, so $u = v_i \in E_C$. Using that $v_i \notin R$ we know that $v_i \neq 3$. If $v_i = 1$ then it has exactly three neighbors in C, $v_{i-1}, v_{i+1}, \ell_{v_i}$, so all of them belong to R, therefore $v_{i-1}, v_{i+1} \in \{2, 3\}$. Thus $v_{i-1}v_iv_{i+1} = 212$ or $v_{i-1}v_iv_{i+1} = 213$ or $v_{i-1}v_iv_{i+1} = 312$ (note that 313 is not allowed).

If $v_i = 2$ then $v_i \in V(C) \setminus R$ implies $\ell^1_{v_i}, \ell^2_{v_i} \in R$. Using that v_i has at least three neighbors in R, we may assume without loss of generality that $v_{i-1} \in R$. This means that $v_{i-1} = 2$, because 32 is not allowed, so $v_{i-1}v_i = 22$.

Conversely, assume that $E_C = v_1 \dots v_n$ $(n \geq 2)$ contains at least one of the sequences 22, 212, 213, 312. We consider the following cases.

1. $v_i = 2$, for every $i \in \{1, \dots, n\}$. Then, by Lemma 2, $i(C) \leq \frac{3n}{2}$ and C has a unique independent $[1, 2]$-set, that has $2n$ elements, so $i_{[1,2]}(C) = 2n$. Therefore $i(C) \leq \frac{3n}{2} < 2n = i_{[1,2]}(C)$.

2. E_C contains the pair 22, but not every vertex in E_C is labeled as 2. Then E_C must contain the sequence 221 or the sequence 122. Assume without loss of generality that $v_iv_{i+1}v_{i+2} = 221$. Let S be an independent $[1, 2]$-set of C, with minimum size such that $v_{i+2} \notin S$. Clearly leaves of both v_i, v_{i+1} must be in S and we define the set $R = \left(S \setminus \{\ell^1_{v_{i+1}}, \ell^2_{v_{i+1}}\}\right) \cup \{v_{i+1}\}$, that is an independent dominating set of C such that $|R| = |S| - 1$. Therefore $i(C) \leq |R| < |S| = i_{[1,2]}(C)$.

3. E_C contains no sequence 22. Then, by hypothesis $v_iv_{i+1}v_{i+2} = 212$ or $v_iv_{i+1}v_{i+2} = 213$ or $v_iv_{i+1}v_{i+2} = 312$ are sequences of E_C. Let S be an independent $[1, 2]$-set C, with minimum size containing no vertices with label 1 (and containing every vertex with label 3). Clearly at most one vertex among v_i, v_{i+2} belongs to S, so assume, without loss of generality that $v_i = 2$, $v_i \notin S$ and $\ell^1_{v_i}, \ell^2_{v_i} \in S$. We define $R = \left(S \setminus \{\ell^1_{v_i}, \ell^2_{v_i}\}\right) \cup \{v_i\}$. If v_i is not the first vertex of E_C, then $v_{i-1} = 1$, because E_C contains no sequence 22, so $v_{i-1} \notin S$. Therefore in that case, and also if $i = 1$, R is an independent dominating set of C such that $|R| = |S| - 1$. Therefore $i(C) \leq |R| < |S| = i_{[1,2]}(C)$. □

This corollary is an immediate consequence of the above theorem.

Corollary 1. *Let C be a caterpillar having an independent $[1,2]$-set and let $E_G = v_1 \dots v_n$ be its spine. If $v_i \in \{1,3\}$, for every $i \in \{1,\dots,n\}$, then $i(C) = i_{[1,2]}(C)$.*

Hereinafter we will use the following notation for caterpillars generated by sequences of consecutive vertices of the spine of C.

Notation 2. *Let C be a caterpillar and let F be a sequence of consecutive vertices in E_C. The caterpillar C_F associated to F is the subgraph generated by all vertices in F and all their leaves. Note that F is the spine of its associated caterpillar.*

This lemma shows that an appropriate partition of the spine of a caterpillar is a key tool to compute its independent domination number.

Lemma 5. *Let C be a caterpillar such that every vertex in E_C is a support vertex. Let $F_1 = v_1 \dots v_{t_1}, F_2 = v_{t_1+1} \dots v_{t_2}, \dots, F_k = v_{t_{k-1}+1} \dots v_n$ a partition of E_C into sequences of consecutive vertices such that $v_{t_i} = 1$ or $v_{t_i+1} = 1$, for each $i \in \{1,\dots,k-1\}$. Then $i(C) = i(C_{F_1}) + i(C_{F_2}) + \dots + i(C_{F_k})$.*

Proof. Let $S_1, S_2, \dots, S_k$ be minimum independent dominating sets of caterpillars $C_{F_1}, C_{F_2}, \dots, C_{F_k}$ respectively. Clearly $\bigcup_{i=1}^k S_i$ is a dominating set of C. Consider an index $i \in \{1,\dots,k-1\}$, if $v_{t_i} = 1$ then, by Proposition 3, we may assume that $v_{t_i} \notin S_i$. If on the contrary $v_{t_i} \neq 1$ then, by hypothesis, $v_{t_i+1} = 1$ and $v_{t_i+1} \notin S_{i+1}$. Therefore $\bigcup_{i=1}^k S_i$ is also independent and $i(C) \le |\bigcup_{i=1}^k S_i| = |S_1| + |S_2| + \dots + |S_k| = i(C_{F_1}) + i(C_{F_2}) + \dots + i(C_{F_k})$.

Conversely, let S be a minimum independent dominating set of C. By hypothesis, every vertex in E_C is a support vertex, so it is in S or its leaves are in S. This means that $S_i = S \cap V(C_{F_i})$ is a dominating set of C_{F_i} and clearly it is also independent. Therefore $i(C_{F_1}) + i(C_{F_2}) + \dots + i(C_{F_k}) \le |S_1| + |S_2| + \dots + |S_k| = |S| = i(C)$. □

Let C be a caterpillar having an independent $[1,2]$-set, such that every vertex in the spine $E_C = v_1 \dots v_n$ is a support vertex and $v_n \neq 2$. We define the *canonical partition* of E_C in the following way. First of all select all the sequences, with length at least three, of consecutive vertices $v_i \dots v_{i+r}$ such that $v_j = 2$ for every $j \in \{i,\dots,i+r-1\}$, $v_{i+r} = 1$ and the length is maximal. We call them Type I sequences and, using that $v_n \neq 2$, every pair of consecutive vertices in E_C labeled as 2, belongs to some Type I sequence. Therefore, among the remaining vertices of the spine, every vertex with label 2 must be preceded and followed by vertices with label 1.

Among remaining vertices of the spine, now select the sequences of consecutive vertices, that we will call Type II, of even length at least four, consisting of an initial copy of 31 followed by consecutive copies of 21 and having maximal length. Type III sequences consist of consecutive copies of 21 followed by a final 3, selected among vertices that do not belong to any Type I or Type II sequence, and having maximal length.

Among remaining vertices, select the sequences of even length at least two, consisting of consecutive copies of 21 and having maximal length. They are Type IV and by construction a sequence of this type is not preceded by the pair 31 nor followed by 3. Note that every vertex with label 2 belongs to some sequence of types I, II, III or IV. Finally, select all the remaining maximal sequences of consecutive vertices that will be Type V. Each Type V sequence consists of vertices with labels 1 or 3.

We now provide an upper bound, slightly smaller than the one shown in Equation 1, for the independent $[1, 2]$-number of caterpillars with last vertex in the spine non labeled as 2. The general case will be deduced from this one.

Proposition 4. *Let C be a caterpillar having an independent $[1, 2]$-set and such that every vertex in $E_C = v_1 \dots v_n$ is a support vertex. If $v_n \in \{1, 3\}$, then C has an independent $[1, 2]$-set S, not necessarily minimum, such that $|S| \leq \frac{7}{5} i(C)$. Moreover, if $v_n = 1$ then $v_n \notin S$.*

Proof. The result is trivially true if $n = 1$, that is, the spine consists of a single vertex, so assume that $n \geq 2$. Consider the canonical partition of E_C into k sequences of consecutive vertices $F_1 = v_1 \dots v_{t_1}, F_2 = v_{t_1+1} \dots v_{t_2}, \dots, F_k = v_{t_{k-1}+1} \dots v_n$. Let v_{t_i} the final vertex of F_i for $i \in \{1, \dots, k-1\}$. If F_i is Type I, II or IV, then $v_{t_i} = 1$ by construction. If F_i is Type III then $v_{t_i} = 3$ so $v_{t_i+1} = 1$. Finally if F_i is Type V then $v_{t_i} \in \{1, 3\}$ and F_{i+1} must be type I, II, III or IV, by maximality of F_i. In any case $v_{t_i+1} \in \{2, 3\}$ so $v_{t_i} \neq 3$ and therefore $v_{t_i} = 1$. Using Lemma 5, we obtain that $i(C) = i(C_{F_1}) + i(C_{F_2}) + \cdots + i(C_{F_k})$.

By hypothesis, C has an independent $[1, 2]$-set, so Proposition 2 provides E_C does not contain the sequences $33, 32, 23, 313$. Clearly subsequences of consecutive vertices of E_C inherit this property and each C_{F_i} has an independent $[1, 2]$-set S_i. For Type I sequences select S_i according to Lemma 2. For Type II sequences, take S_i given by Lemma 3. For Type III and IV sequences, S_i is given by Lemma 4. Finally for Type V sequences, Corollary 1 gives the appropriate S_i. In all cases $|S_i| \leq \frac{7}{5} i(C_{F_i})$.

We now define $S = \bigcup_{i=1}^{n} S_i$. It is clear that S dominates C, so let us see that it is also an independent set that dominates at most twice every vertex not in it. To this end, for any pair of consecutive sequences $F_i = v_t \dots v_{t+r}, F_{i+1} = v_{t+r+1} \dots v_{t+s}$, we need to ensure that edge $v_{t+r}v_{t+r+1}$ keeps independence and $[1, 2]$-domination. We consider the following cases:

1. F_i is Type I or Type II, then $v_{t+r} \notin S_i$ and it has just one neighbor in S_i.

2. F_i is Type III, then $v_{t+r} = 3$ and $v_{t+r+1} = 1$. Therefore F_{i+1} must be Type V (the first vertex in the rest of types is not labeled as 1) and, if S_{i+1} has at least two vertices, then $v_{t+r+2} = 1$, because sequence 313 is not allowed. Therefore $v_{t+r+1} \notin S_{i+1}$ and it has a unique neighbor in S_{i+1}.

   If S_{i+1} consists of the single vertex v_{t+r+1} then $v_{t+r+1} \notin S_{i+1}$ and it has a unique neighbor in S_{i+1}. In case $i+1 < n$, consider the sequence F_{i+2}. It can not be of Type II because sequence 313 is not allowed. It can not be of Type IV because, by construction, a Type IV sequence is not preceded by the pair 31 and it is not of Type V by the maximality of F_i. So F_{i+1} is of Type I or III and its first vertex v_{t+r+2} does not belong to S_{i+2}. This means that v_{t+r+1} has no neighbors in S_{i+2} and edge $v_{t+r}v_{t+r+1}$ keeps independence and $[1,2]$-domination.

3. F_i is Type IV, then $v_{t+r} = 1$ and $v_{t+r} \notin S_i$. Note that $v_{t+r+1} \neq 3$, because a Type IV sequence can not be followed by 3. If $v_{t+r+1} = 1$ then F_{t+1} must be Type V, $v_{t+r+1} \notin S_{i+1}$. If on the contrary $v_{t+r+1} = 2$ then, F_{i+1} must be Type I by the maximality of F_i. Therefore $v_{t+r+1} \notin S_{i+1}$.

In all cases the edge $v_{t+r}v_{t+r+1}$ keeps independence and $[1,2]$-domination. We now analyze what happens if F_i is Type V. In this case $v_{t+r} \in \{1,3\}$, but if $v_{t+r} = 3$ then $v_{t+r+1} = 1$ and F_{t+1} must be Type V, that contradicts the maximality of F_i. Thus $v_{t+r} = 1$ and $v_{t+r} \notin S_i$.

1. Suppose that $r \geq 1$, that is, F_i has at least two vertices. If $v_{t+r-1} = 1$ then $v_{t+r-1} \notin S_i$ and v_{t+r} has a unique neighbor in S_i.

   If on the contrary $v_{t+r-1} = 3$, then F_{i+1} can not be of Type II because sequence 313 is not allowed. Note also that F_{i+1} can not be of Type IV because it is preceded by the sequence 31. Therefore F_{i+1} must be of Type I or III and in both cases $v_{t+r+1} \notin S_{i+1}$.

2. Assume now that $r = 0$, that is, F_i consists of the single vertex $v_t = 1$, that does not belong to S_i and that has a unique neighbor in S_i.

   If $i \geq 2$ consider the previous sequence F_{i-1} and its last vertex v_{t-1}, that has label different from 2, because no sequence of our types ends with 2. If $v_{t-1} = 1$ then $v_{t-1} \notin S_{i-1}$ and edge $v_{t-1}v_t$ does not increase the number of neighbors of v_t in S.

   If $v_{t-1} = 3$ then F_{i-1} is Type III. This means that F_{i+1} is not of Type II, because sequence 313 is not allowed. Note also that F_{i+1} is not of

Type IV, because it is preceded by 31. Therefore, if $v_{t-1} = 3$ then F_{i+1} is of Type I or III and in both cases $v_{t+1} \notin S_{i+1}$.

In all cases the edge $v_{i+r}v_{i+r+1}$ keeps independence and $[1,2]$-domination. This means that $S = \bigcup_{j=1}^{n} S_j$ is an independent $[1,2]$-set of C and

$$|S| = |S_1| + \cdots + |S_k| \leq \frac{7}{5}i(C_1) + \cdots + \frac{7}{5}i(C_k) = \frac{7}{5}i(C).$$

Finally, if the last vertex of the spine v_n has label 1 then, v_n is a vertex in F_k that is of Type I, II, IV or V. In all cases $v_n \notin S_k$, so $v_n \notin S$. □

4 Main results

Our main result, that proves inequality given in Equation 1, is shown now. The proof uses all previous results and it consists of dividing the spine into two sequences, such that the first one satisfies $i_{[1,2]} \leq \frac{7}{5}i$ and the second one satisfies $i_{[1,2]} \leq \frac{7}{5}i + \frac{2}{5}$. The proof also shows the cases where the addend $\frac{2}{5}$ can not be avoided.

Theorem 2. *Let C be a caterpillar having an independent $[1,2]$-set and such that every vertex in the spine $E_C = v_1 \dots v_n$ is a support vertex. Then $i_{[1,2]}(C) \leq \frac{7}{5}i(C) + \frac{2}{5}$.*

Proof. By Proposition 4, we just need to consider the case $v_n = 2$. First suppose that $v_i = 2$ for each $i \in \{1, \dots, n\}$, then by Lemma 2, $i_{[1,2]}(C) \leq \frac{7}{5}i(C) + \frac{2}{5}$.

Now assume that there exists a vertex in the spine labeled as 1 and denote by t the greatest index such that $v_t = 1$. If $n = 2$ then $E_C = 12$, $i_{[1,2]}(C) = i(C) = 2$, so we may assume that $n \geq 3$. We consider the following cases:

1. $t \leq n-2$: define $F_1 = v_1 \dots v_t$ and $F_2 = v_{t+1} \dots v_n$. By Proposition 4, C_{F_1} has an independent $[1,2]$-set S_1 satisfying $|S_1| \leq \frac{7}{5}i(C_{F_1})$ and such that $v_t \notin S_1$. On the other hand, by Lemma 2, C_{F_2} has a unique independent $[1,2]$-set S_2, that satisfies $|S_2| \leq \frac{7}{5}i(C_{F_2}) + \frac{2}{5}$ and $v_{t+1} \notin S_2$. Thus $S = S_1 \cup S_2$ is an independent $[1,2]$-set of C, not necessarily minimum and, using Lemma 5

$$i_{[1,2]}(C) \leq |S| = |S_1| + |S_2| \leq \frac{7}{5}i(C_{F_1}) + \frac{7}{5}i(C_{F_2}) + \frac{2}{5} = \frac{7}{5}i(C) + \frac{2}{5}.$$

2. $t = n-1$ and $v_{n-2} = 1$: define $F_1 = v_1 \dots v_{n-1}$ and $F_2 = v_n$. Again by Proposition 4, C_{F_1} has an independent $[1,2]$-set S_1 satisfying $|S_1| \leq$

$\frac{7}{5}i(C_{F_1})$, $v_{n-2}, v_{n-1} \notin S_1$ and v_{n-1} has a unique neighbor in S_1. Then $S = S_1 \cup \{v_n\}$ is an independent $[1,2]$-set of C, not necessarily minimum. Note that $i(C_{F_2}) = 1$ and

$$i_{[1,2]}(C) \leq |S| = |S_1| + 1 \leq \frac{7}{5}i(C_{F_1}) + 1 \leq \frac{7}{5}i(C_{F_1}) + \frac{7}{5}i(C_{F_2}) = \frac{7}{5}i(C).$$

3. $t = n-1$ and $v_{n-2} = 2$: if $n = 3$ then $E_C = 212$, $i_{[1,2]}(C) = 4$, with $S = \widehat{2}\,\widehat{1}\,2$ a minimum independent $[1,2]$-set, and $i(C) = 3$, satisfying $i_{[1,2]}(C) \leq \frac{7}{5}i(C)$.
If $n \geq 4$ and $v_i = 2$ for $1 \leq i \leq n-2$ then define $F_1 = v_1 \dots v_{n-1}$ and $F_2 = v_n$. Lemma 2 ensures that C_{F_1} has an independent $[1,2]$-set S_1 such that $|S| \leq \frac{7}{5}i(C_1)$, $v_1 \notin S_1$, $v_{n-1} \notin S_1$ and v_{n-1} has a unique neighbor in S_1. Then $S = S_1 \cup \{v_n\}$ is an independent $[1,2]$-set of C, not necessarily minimum and

$$i_{[1,2]}(C) \leq |S| = |S_1| + 1 \leq \frac{7}{5}i(C_{F_1}) + 1 \leq \frac{7}{5}i(C_{F_1}) + \frac{7}{5}i(C_{F_2}) = \frac{7}{5}i(C).$$

If on the contrary there exists a vertex $v_r \neq v_{n-1}$ with label 1, define $F_1 = v_1 \dots v_s$ and $F_2 = v_{s+1} \dots v_{n-2}v_{n-1}v_n$ such that $v_s = 1$ and $v_{s+1}, \dots, v_{n-2} = 2$. Then by Proposition 4, C_{F_1} has an independent $[1,2]$-set S_1 satisfying $|S_1| \leq \frac{7}{5}i(C_{F_1})$ and by the case described in the preceding paragraph, C_{F_2} has an independent $[1,2]$-set S_2 satisfying $|S_2| \leq \frac{7}{5}i(C_{F_2})$ and $v_{s+1} \notin S_2$. Then $S = S_1 \cup S_2$ is an independent $[1,2]$-set of C, not necessarily minimum that satisfies

$$i_{[1,2]}(C) \leq |S| = |S_1| + |S_2| \leq \frac{7}{5}i(C_{F_1}) + \frac{7}{5}i(C_{F_2}) = \frac{7}{5}i(C).$$

4. $t = n-1$ and $v_{n-2} = 3$: if $n = 3$ then $E_C = 312$, $i_{[1,2]}(C) = 4$ and $i(C) = 3$, satisfying $i_{[1,2]}(C) \leq \frac{7}{5}i(C)$.
If $n \geq 4$, using that C has an independent $[1,2]$-set, we obtain that $v_{n-3} = 1$.
If $n = 4$ then $E_C = 1312$, $i_{[1,2]}(C) = 5$ and $i(C) = 4$, satisfying $i_{[1,2]}(C) \leq \frac{7}{5}i(C)$.
If $n \geq 5$, again using Proposition 2, $v_{n-3}v_{n-2} = 13$ implies $v_{n-4} \neq 3$.
In case that $v_{n-4} = 1$, let $F_1 = v_1 \dots v_{n-3}$ and $F_2 = v_{n-2}v_{n-1}v_n = 312$ and, as in the preceding cases, Proposition 4 ensures that C_{F_1} has an independent $[1,2]$-set S_1 satisfying $|S_1| \leq \frac{7}{5}i(C_{F_1})$, $v_{n-3}, v_{n-4} \notin S_1$ and v_{n-3} has a unique neighbor in S_1. Then $S = S_1 \cup \{v_{n-2}, \ell_{v_{n-1}}, \ell^1_{v_n}, \ell^2_{v_n}\}$ is an independent $[1,2]$-set of C, not necessarily minimum. Clearly

$i(C_{F_2}) = 3$ and using Lemma 5

$$i_{[1,2]}(C) \le |S| = |S_1| + 4 \le \frac{7}{5} i(C_{F_1}) + 4 = \frac{7}{5} i(C_{F_1}) + \frac{4}{3} i(C_{F_2}) \le \frac{7}{5} i(C).$$

If on the contrary $v_{n-4} = 2$, assume firstly that $n = 5$, then $E_C = 21312$, $i_{[1,2]}(C) = 7$ and $i(C) = 5$, satisfying $i_{[1,2]}(C) \le \frac{7}{5} i(C)$.
If $n \ge 6$ and $v_{n-5} = 2$, let $F_1 = v_1 \dots v_{n-3}$ and $F_2 = v_{n-2}v_{n-1}v_n = 312$. Then C_{F_1} has an independent $[1,2]$-set S_1 satisfying $|S_1| \le \frac{7}{5} i(C_{F_1})$, $v_{n-3} \notin S_1$ and it has a unique neighbor in S_1, (note that $v_{n-5}v_{n-4} = 22$, so $v_{n-4} \notin S_1$). Then $S = S_1 \cup \{v_{n-2}, \ell_{v_{n-1}}, \ell^1_{v_n}, \ell^2_{v_n}\}$ is an independent $[1,2]$-set of C, not necessarily minimum. Using that $i(C_{F_2}) = 3$, we obtain

$$i_{[1,2]}(C) \le |S| = |S_1| + 4 \le \frac{7}{5} i(C_{F_1}) + 4 = \frac{7}{5} i(C_{F_1}) + \frac{4}{3} i(C_{F_2}) \le \frac{7}{5} i(C).$$

If $v_{n-5} = 1$, let $F_1 = v_1 \dots v_{n-5}$ and $F_2 = v_{n-4}v_{n-3}v_{n-2}v_{n-1}v_n = 21312$. Then C_{F_1} has an independent $[1,2]$-set S_1 with $|S_1| \le \frac{7}{5} i(C_{F_1})$, $v_{n-5} \notin S_1$. Then $S = S_1 \cup \{\ell^1_{v_{n-4}}, \ell^2_{v_{n-4}}, \ell_{v_{n-3}}, v_{n-2}, \ell_{v_{n-1}}, \ell^1_{v_n}, \ell^2_{v_n}\}$ is an independent $[1,2]$-set of C, not necessarily minimum. Using that $i(C_{F_2}) = 5$, we obtain

$$i_{[1,2]}(C) \le |S| = |S_1| + 7 \le \frac{7}{5} i(C_{F_1}) + 7 = \frac{7}{5} i(C_{F_1}) + \frac{7}{5} i(C_{F_2}) \le \frac{7}{5} i(C).$$

□

Our final result is a realization-type theorem showing that every pair of suitable values for both parameters, according to lower and upper bounds, is realizable, except one particular case. In addition we will obtain that the difference $i_{[1,2]} - i$ can attain any nonnegative integer value.

Theorem 3. *Given two integers a, b such that $1 \le a \le b \le \frac{7}{5}a + \frac{2}{5}$, there exists a caterpillar C such that $i(C) = a$ and $i_{[1,2]}(C) = b$, except for the case $a = 2, b = 3$.*

Proof. If $a = 1$ then $b = 1$ and $C = P_2$, the path with two vertices, satisfies $i(C) = i_{[1,2]}(C) = 1$. If $a = 2$, then the caterpillar D with spine $E_D = 11$ satisfies $i(D) = i_{[1,2]}(D) = 2$. Moreover, let C be any caterpillar with $i(C) = 2$, then an independent dominating set with size two is trivially a $[1,2]$-set, so $i(C) = i_{[1,2]}(C) = 2$ and the case $a = 2, b = 3$ is not realizable. For the rest of the proof we may assume that $a \ge 3$.

Denote by C_i, $1 \le i \le 7$, the caterpillars with spines $E_{C_1} = 2221$, $E_{C_2} = 2211$, $E_{C_3} = 11111$, $E_{C_4} = 22$, $E_{C_5} = 222$, $E_{C_6} = 22222$, $E_{C_7} = 221$

respectively. It is clear that $i(C_i) = 5$ for $1 \leq i \leq 3$ and examples of minimum independent dominating sets for each of then are $R_1 = 2\,\widehat{2}\,2\,\widehat{1}$, $R_2 = 2\,\widehat{2}\,\widehat{1}\,\widehat{1}$, $R_3 = \widehat{1}\,\widehat{1}\,\widehat{1}\,\widehat{1}\,\widehat{1}$. Note also that $i(C_4) = 3$ with $R_4 = \widehat{2}\,2$ a minimum independent dominating set, $i(C_5) = 4$ with $R_5 = 2\,\widehat{2}\,2$ a minimum independent dominating set, $i(C_6) = 7$ with $R_6 = 2\,\widehat{2}\,2\,\widehat{2}\,2$ a independent dominating set and $i(C_7) = 4$ with $R_7 = 2\,\widehat{2}\,\widehat{1}$ a minimum independent dominating set.

Regarding the independent $[1,2]$-number, if $i \in \{1,2,4,5,6,7\}$ then leaves of vertices with label 2 belong to every independent $[1,2]$-set and $i_{[1,2]}(C_1) = 7$, $i_{[1,2]}(C_2) = i_{[1,2]}(C_5) = 6$, $i_{[1,2]}(C_4) = 4$, $i_{[1,2]}(C_6) = 10$ and $i_{[1,2]}(C_7) = 5$. Minimum independent $[1,2]$-sets for each of them are $S_1 = \widehat{2}\,\widehat{2}\,\widehat{2}\,\widehat{1}$, $S_2 = \widehat{2}\,\widehat{2}\,\widehat{1}\,\widehat{1}$, $S_4 = \widehat{2}\,\widehat{2}$, $S_5 = \widehat{2}\,\widehat{2}\,\widehat{2}$, $S_6 = \widehat{2}\,\widehat{2}\,\widehat{2}\,\widehat{2}\,\widehat{2}$ and $S_7 = \widehat{2}\,\widehat{2}\,\widehat{1}$. Clearly $i_{[1,2]}(C_3) = 5$ and $S_3 = \widehat{1}\,\widehat{1}\,\widehat{1}\,\widehat{1}\,\widehat{1}$ is a minimum independent $[1,2]$-set.

Let C be a caterpillar with $E_C = H_1 H_2 \dots H_k$ and H_i equal to some of the sequences $2221, 2211$ or 11111 for $i \in \{1, \dots, k-1\}$ and H_k equal to 2221, 2211, 11111, 22, 222, 22222, 221 or r vertices with label 1 ($r \geq 1$). Then Lemma 5 gives

$$i(C) = i(C_{H_1}) + i(C_{H_2}) + \dots + i(C_{H_k}).$$

On the other hand, note that vertices with label 2 in E_C always have a neighbor in E_C also labeled as 2, so every independent $[1,2]$-set contains all the leaves of vertices with label 2. This means that

$$\begin{aligned} i_{[1,2]}(C) =& 2 \times (\text{number of vertices with label 2}) + \\ &+ (\text{number of vertices with label 1}) = \\ =& i_{[1,2]}(C_{H_1}) + i_{[1,2]}(C_{H_2}) + \dots + i_{[1,2]}(C_{H_k}) \end{aligned}$$

All the caterpillars that we show as examples follow this construction and we will use the formulas above to compute both i and $i_{[1,2]}$.

Let a, b be integers such that $3 \leq a \leq b \leq \frac{7}{5}a + \frac{2}{5}$. Let k be an integer such that $a = 5k + r$ with $r = 0, 1, 2, 3, 4$. Then the relationship between a and b is $a = 5k + r \leq b \leq \lfloor \frac{7(5k+r)}{5} + \frac{2}{5} \rfloor = 7k + r + \lfloor \frac{2r+2}{5} \rfloor$.

For cases $a = 5k + r \leq b \leq 6k + r$, let $d = b - (5k + r)$, that satisfies $0 \leq d \leq k$. The caterpillar C, such that E_C consists of $k - d$ consecutive copies of $E_{C_3} = 11111$ followed by d consecutive copies of $E_{C_2} = 2211$ and r vertices with label 1, satisfies

$$i(C) = 5(k-d) + 5d + r = 5k + r = a$$

$$i_{[1,2]}(C) = 5(k-d) + 6d + r = 5k + d + r = b.$$

We now show models for cases $6k + r + 1 \leq b \leq 7k + r + \lfloor \frac{2r+2}{5} \rfloor$.

1. If $r = 0$ or $r = 1$ then $k \geq 1$ and $\lfloor \frac{2r+2}{5} \rfloor = 0$, so suppose $6k + r + 1 \leq b \leq 7k + r$. Let $d = 7k + r - b$, that satisfies $0 \leq d \leq k - 1$. The caterpillar C, such that E_C consists of $k - d$ consecutive copies of $E_{C_1} = 2221$ followed by d consecutive copies of $E_{C_2} = 2211$ and r vertices with label 1, satisfies

$$i(C) = 5(k - d) + 5d + r = 5k + r = a$$
$$i_{[1,2]}(C) = 7(k - d) + 6d + r = 7k - d + r = b.$$

2. If $r = 2$ then $k \geq 1$ and $\lfloor \frac{2r+2}{5} \rfloor = 1$. If $b = 6k + 3$ then the caterpillar C, such that E_C consists of $k - 1$ copies of $E_{C_2} = 2211$ followed by a copy of $E_{C_1} = 2221$ and two vertices with label 1, satisfies

$$i(C) = 5(k - 1) + 5 + 2 = 5k + 2 = a$$

$$i_{[1,2]}(C) = 6(k - 1) + 7 + 2 = 6k + 3 = b.$$

   Assume now that $6k + 4 \leq b \leq 7k + 3$ and let $d = 7k + 3 - b$, that satisfies $0 \leq d \leq k - 1$. The caterpillar C, such that E_C consists of $k - 1 - d$ consecutive copies of $E_{C_1} = 2221$ followed by d consecutive copies of $E_{C_2} = 2211$ and one copy of $E_{C_6} = 22222$, satisfies

$$i(C) = 5(k - 1 - d) + 5d + 7 = 5k + 2 = a$$
$$i_{[1,2]}(C) = 7(k - 1 - d) + 6d + 10 = 7k - d + 3 = b.$$

3. If $r = 3$, then $\lfloor \frac{2r+2}{5} \rfloor = 1$, so assume that $6k + 4 \leq b \leq 7k + 4$ let $d = 7k + 4 - b$, that satisfies $0 \leq d \leq k$. The caterpillar C, such that E_C consists of $k - d$ consecutive copies of $E_{C_1} = 2221$ followed by d consecutive copies of $E_{C_2} = 2211$ and one copy of $E_{C_4} = 22$, satisfies

$$i(C) = 5(k - d) + 5d + 3 = 5k + 3 = a$$
$$i_{[1,2]}(C) = 7(k - d) + 6d + 4 = 7k - d + 4 = b.$$

4. If $r = 4$, then $\lfloor \frac{2r+2}{5} \rfloor = 2$. If $b = 6k + 5$ then the caterpillar C, such that E_C consists of k copies of $E_{C_2} = 2211$ followed by one copy of $E_{C_7} = 221$, satisfies

$$i(C) = 5k + 4 = a$$

$$i_{[1,2]}(C) = 6k + 5 = b.$$

Assume now that $6k+6 \leq b \leq 7k+6$ and let $d = 7k+6-b$, that satisfies $0 \leq d \leq k$. The caterpillar C such that E_C consists on $k-d$ consecutive copies of $E_{C_1} = 2221$ followed by d consecutive copies of $E_{C_2} = 2211$ and one copy of $E_{C_5} = 222$ satisfies

$$i(C) = 5(k-d) + 5d + 4 = 5k + 4 = a$$
$$i_{[1,2]}(C) = 7(k-d) + 6d + 6 = 7k - d + 6 = b.$$

□

Remark 1. *Note that $\frac{7}{5}i(C) + \frac{2}{5}$ is an integer if and only if $i(C) \equiv 4(mod\ 5)$, so the upper bound provided by Equation 1 is just reached in this case. The caterpillar C with spine consisting of k consecutive copies of sequence 2221 ($k \geq 0$) followed by one copy of sequence 222 satisfies $i(C) = 5k + 4$ and $i_{[1,2]}(C) = 7k + 6$, so $i_{[1,2]}(C) = \frac{7}{5}i(C) + \frac{2}{5}$.*

Corollary 2. *For any integer $m \geq 0$ there exists a caterpillar C such that $i_{[1,2]}(C) - i(C) = m$.*

Proof. Take an integer $a \geq 3$ such that $m \leq \frac{2}{5}(a+1)$, then $a \leq a + m \leq a + \frac{2}{5}(a+1) = \frac{7}{5}a + \frac{2}{5}$ and by Theorem 3, there exists a caterpillar C such that $i(C) = a$ and $i_{[1,2]}(C) = a + m$, therefore $i_{[1,2]}(C) - i(C) = m$. □

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Sahar A. ALEID,
Department of Mathematics,
Universidad de Almería,
Carretera Sacramento s/n, 04120 Almería, Spain.
Email: sahar.aleid@ual.es

Mercè MORA,
Department of Mathematics,
Universitat Politècnica de Catalunya
Jordi Girona, 1-3, 08034 Barcelona, Spain.
Email: merce.mora@upc.edu

María Luz PUERTAS,
Department of Mathematics,
Universidad de Almería,
Carretera Sacramento s/n, 04120 Almería, Spain.
Email: mpuertas@ual.es