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Modules which are self-p-injective relative to projection invariant submodules

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Abstract

In this article, we focus on modules M such that every homomorphism from a projection invariant submodule of M to M can be lifted to M. Although such modules share some of the properties of PI-extending (i.e., every projection invariant submodule is essential in a direct summand) modules, it is shown that they form a substantially bigger class of modules.

1 Introduction

Throughout this paper, let R be a ring with identity and let all modules be unitary right R-modules. Let M be a module. The injective hull of M is denoted by $E(M)$. A submodule K of M is *projection invariant* (denoted by $K \trianglelefteq_p M$) provided K is invariant under every idempotent endomorphism of M (see [3], [5]). Note that the set of projection invariant submodules of a module M forms a sublattice of the lattice of all submodules of M.

A module M is called an *extending module*, or a *CS-module*, if every submodule of M is essential in a direct summand, or, equivalently, if every closed submodule of M is a direct summand. This condition has proven to be an important common generalization of the injective and semisimple module notions (see, [4], [11]). In [3], the extending condition relative to various sets of submodules have been investigated. Recall that a module M is called *PI-extending*

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if every projection invariant submodule is essential in a direct summand of M. In the papers [14] and [16], the authors studied the following property, for a module M:

(P_n) : For every submodule K of M such that K can be written as a finite direct sum $K_1 \oplus K_2 \oplus ... \oplus K_n$ of complements $K_1, K_2, ..., K_n$ of M, every homomorphism $\alpha : K \to M$ can be lifted to a homomorphism $\beta : M \to M$. Following an idea from [16], we are concerned with the study of *self-p-injective* modules, i.e., modules M that satisfy the condition that every homomorphism from a projection invariant submodule of M to M can be lifted to M. Observe that the aforementioned property is equivalent to that of every homomorphism from a finite direct sum of projection invariant submodules of M to M lifts to M. Extending and PI-extending modules are examples of self-p-injective modules. Our investigation focuses on the behavior of self-p-injective modules with respect to direct sums and direct summands. To this end, we provide algebraic geometrical examples which show that being self-p-injective is not inherited by direct summands. In contrast, we prove that any direct sum of self-p-injective modules enjoys with the property. Moreover we obtain useful characterizations and direct sum property on relatively p-injective modules. Finally, we give examples which show that there is no implication between self-p-injective and tight concepts. Recall that a module M is said to be *right tight* (resp.,*right M-tight*) if every finitely generated (resp., cyclic) submodule of $E(M)$ can be embedded in M (see [1], [6]).

Recall the following conditions for a module M.

(C_2): every submodule of M can be embedded in a direct summand of M.

(C_3): for all direct summands K and L of M with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M.

Observe that C_2 implies C_3 by [11, Proposition 2.2]. Recall further that, a ring is called *Abelian* if every idempotent is central. Other terminology and notation can be found in [2], [4], [10], and [11].

2 Direct Summands and Direct Sums

In this section, we concern ourselves with direct summands and direct sums of self-p-injective modules. We provide examples which show that, in general, direct summands of a self-p-injective module need not to be self-p-injective. Amongst some affirmative answers for the former closure property we also prove that any direct sum of self-p-injective modules is again self-p-injective.

Lemma 2.1. *Let M be an indecomposable module. Then the following statements are equivalent.*

(i) M is quasi-injective.

(ii) *M is extending.*
(iii) *M is PI-extending.*
(iv) *M is self-p-injective.*

Proof. $(i) \Rightarrow (ii)$ Obvious.

$(ii) \Rightarrow (iii)$ Clear from [3, Proposition 3.7].

$(iii) \Rightarrow (iv)$ Let X be a projection invariant submodule of M and $\varphi : X \to M$ be a homomorphism. Then there exists a direct summand D of M such that X is essential in D where $M = D \oplus D'$. Let π be projection map on X in D. Then define $\alpha : M \to M$ such that $\alpha = \varphi\pi$. It can be easily seen that α lifts φ. Hence M is self-p-injective.

$(iv) \Rightarrow (i)$ Since M is an indecomposable module, every submodule of M is projection invariant. Then it is clear that self-p-injectivity implies quasi-injectivity. □

Observe that every quasi-injective module is self-p-injective. However there are self-p-injective modules which are not quasi-injective. For example let $M_{\mathbb{Z}} = (\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ where p is any prime integer. Then $M_{\mathbb{Z}}$ is not quasi-injective but it is self-p-injective by Theorem 2.8. Note that every PI-extending module is self-p-injective. But the converse of this result is not true, in general. For instance, let M be the Specker group, $M_{\mathbb{Z}} = \prod_{i=1}^{\infty} A_i$ with $A_i = \mathbb{Z}$ for any positive integer i. Then it can be checked that $M_{\mathbb{Z}}$ is not PI-extending by [5], but it is self-p-injective by Theorem 2.14.

The next example shows that direct summands of self-p-injective modules need not to be self-p-injective, in general.

Example 2.2. *(See, [3, Example 5.5] or [17, Example 4]) Let $\mathbb{R}$ be the real field and n be any odd integer with $n \geq 3$. Let S be the polynomial ring $\mathbb{R}[x_1, ..., x_n]$ over $\mathbb{R}$ in indeterminates $x_1, ..., x_n$. Let R be the ring S/Ss, where $s = x_1^2 + ... + x_n^2 - 1$. Then the free R- module, the countable direct sum $M = R^{(\mathbb{N})}$ of copies of R is self-p-injective which contains a direct summand K_R which is not self-p-injective.*

Surprisingly, we may provide more examples in the next result which is based on certain hypersurfaces in projective spaces, $\mathbb{P}_{\mathbb{C}}^{\mathrm{n}+1}$ over complex numbers.

Theorem 2.3. *Let X be the hypersurface in $\mathbb{P}_{\mathbb{C}}^{n+1}$, $n \geq 2$, defined by the equation $x_0^m + x_1^m + ... + x_{n+1}^m = 0$. Let $R = \mathbb{C}[x_1, ..., x_{n+1}]/(\sum_{i=1}^{n+1} x_i^m + 1)$ be the coordinate ring of X. There exist self-p-injective R-modules but contain direct summands which are not self-p-injective for $m \geq n+2$.*

Proof. By [12], there are indecomposable projective R-modules of rank n over R. It follows that $F_R = K \oplus W$ where F_R is a free module, K is indecomposable and projective R-module of rank n. From [3, Corollary 4.11], F_R is PI-extending and hence it is self-p-injective. Now K_R is not uniform. Thus K_R is not PI-extending so it is not self-p-injective by Lemma 2.1. □

However, we deal with some special cases when the self-p-injectivity is inherited by direct summands in the following results.

Proposition 2.4. *Let $M = M_1 \oplus M_2$ where M_1 and M_2 are projection invariant submodules of M. If M is self-p-injective then M_1, M_2 are also self-p-injective.*

Proof. Let N_1 be a projection invariant submodule of M_1 and $\varphi : N_1 \to M_1$ be a homomorphism. Since N_1 is projection invariant submodule of M_1 and M_1 is projection invariant submodule of M, then N_1 is projection invariant submodule of M. Observe that $\iota\varphi : N_1 \to M$ where ι is inclusion map. Then there exists $\theta : M \to M$ such that θ lifts to $\iota\varphi$. Define $\gamma : M_1 \to M_1$ by $\gamma(m_1) = \theta(m_1)$. It is clear that φ can be extended to γ. Then M_1 is self-p-injective. Similarly, it can be shown that M_2 is also self-p-injective. □

Corollary 2.5. *Let $M = M_1 \oplus M_2$ for submodules M_1 and M_2 of M with $S = End(M_R)$ an Abelian ring. If M is self-p-injective then any direct summand of M is also self-p-injective.*

Proof. Let $\pi : M \to M_2$ be projection map with $\ker(\pi) = M_1$. Let $e = e^2 \in S$. Since S is Abelian, $e(\ker(\pi)) \subseteq \ker(\pi)$. Hence M_1 is projection invariant in M. Now, apply Proposition 2.4 which yields the corollary. □

Proposition 2.6. *Let $M = M_1 \oplus M_2$ where $M_1, M_2 \leq M$ such that M_2 is a projection invariant submodule of M. If M is self-p-injective then M_1 is self-p-injective.*

Proof. Let N be projection invariant submodule of M_1 and $\varphi : N \to M_1$ be a homomorphism. Then $N \oplus M_2$ is projection invariant in M by [3, Lemma 4.13]. Now consider $\theta = \iota\varphi\pi_1$ where $\pi_1 : N \oplus M_2 \to N$ is projection and $\iota : M_1 \to M$ is inclusion. Thus there exists $\gamma : M \to M$ such that γ lifts to θ. Hence

$$\gamma(n + m_2) = \theta(n + m_2) = \iota\varphi\pi_1(n + m_1) = \varphi(n).$$

Define $\tau : M_1 \to M_1$ by $\tau = \pi\beta$ where β is a restriction of θ to M_1 and $\pi : M \to M_1$. Then let $n \in N$. $\tau(n) = \pi\beta(n) = \pi\theta(n) = \varphi(n)$. Thus φ can be extended to τ so M_1 is self-p-injective. □

Proposition 2.7. *Let $M = M_1 \oplus M_2$ for submodules M_1, M_2 of M. If M_1 is self-p-injective then $\varphi : N \to M$ can be lifted to $\theta : M \to M$ for all projection invariant submodule N of M_1.*

Proof. Let N be projection invariant submodule of M_1 and $\varphi : N \to M$ be a homomorphism. Then $\pi_1\varphi \in Hom(N, M_1)$ where $\pi_1 : M \to M_1$. Then there exists $\theta : M_1 \to M_1$ such that $\pi_1\varphi$ can be lifted to θ. Define $\gamma : M \to M$ by $\gamma = \iota\theta\pi_1$. It is easy to check that γ lifts to φ. □

Theorem 2.8. *Any direct sum of self-p-injective modules is self-p-injective.*

Proof. Let M_λ $(\lambda \in \Lambda)$ be a nonempty collection of self-p-injective modules. Let $M = \underset{\lambda\in\Lambda}{\oplus} M_\lambda$ and $\varphi : N \longrightarrow M$ be homomorphism where N is a projection invariant submodule of M. Let $\Lambda^{'}$ be a nonempty subset of Λ. Consider the set

$$\begin{aligned}\mathcal{H} = \{(\Lambda^{'}, K', \alpha', \theta') \mid \Lambda^{'} \subseteq \Lambda, K' \trianglelefteq_p M' = \underset{\lambda\in\Lambda'}{\oplus} M_\lambda \text{ and } \alpha' : K' \to M' \\ \text{homomorphism with } \theta' : M' \to M' \text{ such that } \theta' \text{ lifts to } \alpha'\}\end{aligned}$$

which becomes a partially ordered set by the componentwise order $\leq$ defined by

$$\begin{aligned}(\Lambda_1, K_1, \alpha_1, \theta_1) \leq (\Lambda_2, K_2\ \alpha_2, \theta_2) \Leftrightarrow \Lambda_1 \subseteq \Lambda_2,\ K_1 \subseteq K_2, \\ \alpha_2 \mid_{K_1} = \alpha_1 \text{ and } \theta_1 = \pi\theta_2\iota\end{aligned}$$

where π is canonical projection from $\underset{\lambda\in\Lambda_2}{\oplus} M_\lambda$ to $\underset{\lambda\in\Lambda_1}{\oplus} M_\lambda$ and ι is natural inclusion from $\underset{\lambda\in\Lambda_1}{\oplus} M_\lambda$ to $\underset{\lambda\in\Lambda_2}{\oplus} M_\lambda$.

Since M_λ is self-p-injective for all $\lambda \in \Lambda$, the identity map ι can extend to $\theta : M_\lambda \to M_\lambda$. Hence $(\{\lambda\}, M_\lambda, \iota, \theta) \in \mathcal{H}$ so $\mathcal{H} \neq \varnothing$. Applying Zorn's Lemma, we can find a maximal element $(\Lambda_1, K_1, \alpha_1, \theta_1)$ in $\mathcal{H}$.

We claim that $\Lambda = \Lambda_1$. Suppose not, then there exists $\mu \in \Lambda$, $\mu \notin \Lambda_1$. Let $\Lambda_2 = \Lambda_1 \cup \{\mu\}$ and $M^{''} = \underset{\lambda\in\Lambda_2}{\oplus} M_\lambda = \underset{\lambda\in\Lambda_1}{\oplus} M_\lambda \oplus M_\mu = M' \oplus M_\mu$. Since M_μ is self-p-injective, then for any projection invariant submodule N_μ of M_μ and $\alpha_\mu : K_\mu \to M_\mu$ homomorphism, there exists $\theta_\mu : M_\mu \to M_\mu$ such that θ_μ extends to φ_μ. Observe that $K_1 \oplus K_\mu$ is projection invariant in $M^{''}$. Consider the homomorphism $\gamma : K_1 \oplus K_\mu \to M^{''}$ such that $\gamma(k_1 + k_\mu) = \alpha_1(k_1) + \alpha_\mu(k_\mu)$. It is clear that γ extends to α_1. Define $\theta : M^{''} \to M^{''}$ by $\theta = \theta_1\pi_1 + \theta_\mu\pi_2$ where $\pi_1 : M^{''} \to M'$ and $\pi_2 : M^{''} \to M_\mu$. Let $m' \in M'$. Then $\pi\theta\iota(m') = \pi(\theta_1\pi_1(m') + \theta_\mu\pi_2(m')) = \pi\theta_1(m') = \theta_1(m')$. Hence $\theta_1 = \pi\theta\iota$. Now $(\Lambda_2, K_1 \oplus K_\mu, \gamma, \theta) \in \mathcal{H}$. Note that $(\Lambda_1, K_1, \alpha_1, \theta_1) \leq (\Lambda_2, K_1 \oplus K_\mu, \gamma, \theta)$

which contradicts the maximalitiy of $(\Lambda_1, K_1, \alpha_1, \theta_1)$ in $\mathcal{H}$. Therefore $\Lambda = \Lambda_1$, so $M = \underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is self-p-injective. □

Corollary 2.9. *Any direct sum of modules which are PI-extending (resp., quasi-injective, extending or uniform) is self-p-injective.*

Proof. Immediate by Theorem 2.8. □

Corollary 2.10. *Let M be a right R-module and $M = U \oplus V$ where U and V are uniform submodules of M. Then every direct summand of M is self-p-injective.*

Proof. Let $0 \neq K$ be a direct summand of M. If $K = M$ then K is self-p-injective from Corollary 2.9. If $K \neq M$ then K is uniform. Hence K is self-p-injective. □

Theorem 2.11. *Let M be a $\mathbb{Z}$-module such that M is a direct sum of uniform modules. Then every direct summand of M is self-p-injective.*

Proof. Let N be a direct summand of M. Then N is also a direct sum of uniform modules by [15, Theorem 5.5]. Now Corollary 2.9 yields that N is also self-p-injective. □

One might expect that an essential extension of a self-p-injective module is self-p-injective. However, the next example eliminates this situation.

Example 2.12. *Let R be a principial ideal domain. If R is not a complete discrete valuation ring then there exists an indecomposable torsion-free R-module M of rank 2 by [8, Theorem 19]. Hence there exist uniform U_1, U_2 submodules of M such that $U_1 \oplus U_2$ is essential in M. Then $U_1 \oplus U_2$ is self-p-injective by Corollary 2.9. However M is not self-p-injective by Lemma 2.1.*

Lemma 2.13. *Let $X = \prod_{i\in I} X_i$ be a direct product of modules of X_i for each $i \in I$. If N is a projection invariant submodule of X then $N = \prod_{i\in I}(N \cap X_i)$.*

Proof. It is straightforward to check. □

Theorem 2.14. *Let $X = \prod_{i\in I} X_i$. Then X is self-p-injective if and only if X_i is self-p-injective for all $i \in I$.*

Proof. Let N_i be projection invariant in X_i and $\varphi : N_i \to X_i$ be a homomorphism. Then $T = \prod_{i\in I} N_i$ is projection invariant in X. Let $\alpha_i : T \to N_i$ be a projection map. Consider $\iota\varphi\alpha_i : T \to X$ where ι is inclusion. Then there exists $\theta : X \to X$ such that θ lifts $\iota\varphi\alpha_i$. Define $\gamma : X_i \to X_i$ by $\gamma = \pi_i\theta\iota$ where π_i is projection map from X to X_i. It is clear that γ lifts to φ, so X_i is self-p-injective.

Conversely, let N be a projection invariant submodule of X and $\varphi : N \to X$ be a homomorphism. Then $N \cap X_i$ is projection invariant submodule of X_i. Let θ be the restriction of φ to $N \cap X_i$. Then $\pi_i\theta : N \cap X_i \to X_i$ for all $i \in I$. Hence there exists $\gamma : X_i \to X_i$ such that γ lifts to $\pi_i\theta$ for all $i \in I$. Define $\alpha : X \to X$ by $\alpha = \iota\gamma\pi_i$. Let $n \in N$. By Lemma 2.13, $n = (\pi_i(n))_{i\in I}$. Thus $\alpha(n) = \alpha((\pi_i(n))_{i\in I}) = \iota\gamma\pi_i((\pi_i(n))_{i\in I}) = \iota\gamma((\pi_i(n))_{i\in I}) = \iota\pi_i\theta((\pi_i(n))_{i\in I}) = \theta((\pi_i(n))_{i\in I}) = \varphi((\pi_i(n))_{i\in I}) = \varphi(n)$. Hence α lifts to φ so X is self-p-injective. □

The conditions C_2 and C_3 can be characterized by the lifting homomorphisms from certain submodules to the module itself, as was shown in [16]. We obtain relations between the class of modules which is self-p-injective and the class of modules which has C_3 condition. From [16, Lemma 1], we have the following implication for a module M.

$$C_3 \Rightarrow \text{self-p-injective}$$

Note that this implication is not reversible. For example, let $M_{\mathbb{Z}} = \oplus_{i=1}^{\infty}\mathbb{Z}$. Then $M_{\mathbb{Z}}$ is self-p-injective by Theorem 2.8, but $M_{\mathbb{Z}}$ does not satisfy C_3 by [16, Example 9].

The next few results, which generalize [9, Theorem 2.12], concern the endomorphism ring of self-p-injective π-duo modules. We call a module M is π-duo if every submodule is projection invariant in M. We will use S and $J(S)$ to denote the endomorphism ring of a module M and the Jacobson radical of S, respectively. Further Δ will stand for the ideal $\{\alpha \in S \mid \ker(\alpha) \text{ is essential in } M\}$.

Theorem 2.15. *Let M_R be a self-p-injective module and S an Abelian ring. Then S/Δ is a (von Neumann) regular ring and $\Delta = J(S)$.*

Proof. Let $f \in S$ and $K = \ker f$. Since $ker(f) \cap ker(1-f) = 0$, there exists an isomorphism $\alpha : K \to (1-f)K$. Consider the inverse map of α. Since M is π-duo module, $(1-f)K$ is projection invariant in M. By hypothesis, there exists $g \in S$ such that g lifts inverse map of α. Then $g(1-f)(k) = k$ for all $k \in K$. Let B be a complement of $\ker(f)$ in M. Note that f restricts to an isomorphism of B onto $f(B)$, since $B \cap \ker(f) = 0$. Observe that $f(B)$ is also projection invariant submodule of M as S is Abelian. By self-p-injectivity of M, extend the inverse isomorphism $f(B) \to B$ to some $\gamma \in S$. Now, $\gamma(f(b)) = b$ for all $b \in B$ and hence $(f\gamma f - f)(B) = 0$. Moreover, $B \oplus \ker(f) \leq \ker(f\gamma f - f) \leq M$ which gives that $(f\gamma f - f) \in \Delta$. Hence S/Δ is a (von Neumann) regular ring. It is well known that, regular rings have zero radical, hence $J(S/\Delta) = 0$. Since $J(S)/\Delta \subseteq J(S/\Delta)$, then $J(S) = \Delta$. □

Corollary 2.16. *Let M be a nonsingular π-duo right R-module. If M is a self-p-injective module , then S is a regular ring.*

Proof. Let $g \in \Delta$ and $N = \ker(g)$. Then for any $x \in M$, build up the following set

$$L = \{r \in R \mid xr \in N\}$$

Then clearly L is a right ideal of R and also L is essential in R. Now, $g(x)L = 0$. Since M is nonsingular then $g(x) = 0$, and since x is arbitrary $g = 0$. Therefore $\Delta = 0$ ([13, Lemma 2.3]). Hence the result follows from Theorem 2.16. □

The following example shows that the converse of Corollary 2.16 does not hold. Moreover this example explains that endomorphism ring of a PI-extending (and hence self-p-injective) module need not to be Abelian.

Example 2.17.
(i) *Let $T = M_2(R)$ be the ring in [19, Example 4.77]. Note that T_T is nonsingular self-p-injective which is not CS-module. Since T is regular, so does $S = End(T_T)$. However T is not π-duo. Because if it were π-duo, then it would be a CS-module, a contradiction.*
(ii) *Let V be a countably infinite dimensional vector space over a division ring D and let $S = End(V_D)$. Let $\{x_1, x_2, ...\}$ be a basis of V. It is clear that V_D is PI-extending so it is self-p-injective. Since $\Delta = 0$, S is regular ring. However S is not an Abelian ring. In fact, define $\sigma : V \to V$ by $\sigma(x_i) = x_{i+1}$ for all $i \geq 1$ and $\pi : V \to x_iD$ by $\pi(x_i) = x_i$ and $\pi(x_j) = 0$ for $i \neq j$. Now $\sigma\pi(x_i) = \sigma(x_i) = x_{i+1}$ but $\pi\sigma(x_i) = \pi(x_{i+1}) = 0$.*

3 Relatively p-injective Modules

In this section we introduce the concept of a relative p-injective module and investigate some properties of these modules. Let us begin with the definition.

Definition 3.1. *Let M_1 and M_2 be modules. The module M_2 is M_1-p-injective if every homomorphism $\alpha : N \to M_2$, where N is a projection invariant submodule of M_1, can be extended to a homomorphism $\beta : M_1 \to M_2$.*

It is clear that relative p-injectivity is more general than relative injectivity. Next result provides equivalent conditions to be *PI*-extending in terms of relative p-injectivitiy.

Proposition 3.2. *The following statements are equivalent for a module M.*
(i) *M is PI-extending.*
(ii) *Every module is M-p-injective.*
(iii) *Every projection invariant submodule of M is M-p-injective.*

Proof. $(i) \Rightarrow (ii)$ Let X be a module and N be a projection invariant submodule of M with $\varphi : N \to X$ homomorphism. Since M is PI-extending, there exists a direct summand D of M such that N is essential in D. Then $M = D \oplus D'$ for some D' submodule of M. Let π be the projection map on N in D. Define $\alpha : M \to X$ by $\alpha = \varphi\pi$. Clearly, α lifts to $\varphi\pi$ so X is M-p-injective.

$(ii) \Rightarrow (iii)$ It is obvious.

$(iii) \Rightarrow (i)$ Let N be a projection invariant submodule of M. By hypothesis, N is M-p-injective so the identity map $\iota : N \to N$ can be extended to $\alpha : M \to N$. It is easy to check that $M = N \oplus \ker\alpha$. Thus M is PI-extending. □

Next result, which generalizes [4, 7.5], concerns relative p-injective direct summands of a module.

Theorem 3.3. *Let M_1, M_2 be modules and $M = M_1 \oplus M_2$. Then M_2 is M_1-p-injective if and only if every submodule N of M such that $N \cap M_2 = 0$; and $\pi_1(N)$ is a projection invariant submodule of M_1, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$.*

Proof. Let $N \leq M$ such that $N \cap M_2 = 0$ and $\pi_1(N)$ is projection invariant submodule of M_1. Let $\pi_1 : M \to M_1$ be the projection and consider the restriction of π_1 to N. Then $\pi_1|_N$ is an isomorphism between N to $\pi_1(N)$, since $N \cap M_2 = 0$. Consider the homomorphism $\alpha : \pi_1(N) \to M_2$ by $\alpha(x) = \pi_2(\pi_1|_N)^{-1}(x)$. Since $\pi_1(N)$ is projection invariant submodule of M_1 and M_2 is M_1-p-injective, the map α can be extended to a homomorphism $\beta : M_1 \to M_2$. Define $N' = \{x + \beta(x) \mid x \in M_1\}$. Clearly, N' is a submodule of M and $M = N' \oplus M_2$. Let $n \in N$. $\beta\pi_1(x) = \alpha\pi_1(x) = \pi_2(x)$ and hence $x = \pi_1(x) + \pi_2(x) = \pi_1(x) + \beta\pi_1(x) \in N'$. Then $N \leq N'$.

Conversely, let K be a projection invariant submodule of M_1 and $\alpha : K \to M_2$ be a homomorphism. Define $N = \{x - \alpha(x) \mid x \in K\}$. N is a submodule of M and $N \cap M_2 = 0$. Moreover, it can be easily seen that $\pi_1(N) = K$. Hence $\pi_1(N)$ is projection invariant submodule of M_1. By hypothesis, there exists a submodule N' of M such that $N \leq N'$ and $M = N' \oplus M_2$. Let $\pi : M \to M_2$ be projection with kernel N' and let $\beta : M_1 \to M_2$ be the restriction of π to M_1. Let $x \in K$. $\beta(x) = \pi(x) = \pi(x - \alpha(x) + \alpha(x)) = \alpha(x)$. It follows that M_2 is M_1-p-injective. □

Proposition 3.4. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of R-modules. Then X is M_λ-p-injective for all $\lambda \in \Lambda$ if and only if X is $\bigoplus_{\lambda\in\Lambda} M_\lambda$-p-injective.*

Proof. Let $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ and $\alpha : N \to A$ be homomorphism with a projection invariant submodule N of M. Then $N = \bigoplus_{\lambda\in\Lambda}(N \cap M_\lambda)$ where $N \cap M_\lambda$ is projection invariant in M_λ. Consider the restriction of α on $N \cap M_\lambda$. Then there exists a homomorphism $\theta : M_\lambda \to X$ such that θ lifts to $\alpha|_{N\cap M_\lambda}$. Define $\gamma : M \to X$ by $\gamma = \theta\pi_\lambda$ where $\pi_\lambda : M \to M_\lambda$ canonical projection. Then it can be easily seen that γ lifts to α, hence X is M-p-injective.

Conversely, let K_λ be projection invariant in M_λ for any $\lambda \in \Lambda$ and $\alpha : K_\lambda \to X$ be homomorphism. Then $K = \bigoplus_{\lambda\in\Lambda} K_\lambda$ is projection invariant in $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. Now, there exists a homomorphism $\theta : M \to X$ such that θ lifts to $\alpha\pi_\lambda$ where π_λ is a projection map from K onto K_λ. Define $\gamma : M_\lambda \to X$ by $\gamma = \theta\iota$ where ι is inclusion map. Then, it is clear that γ lifts to α. Therefore X is M_λ-p-injective. $\square$

Our next two results give conditions for a module X and its quotient being relative p-injective. To this end, we refer to [2, 16.8] for the corresponding relative injectivity results.

Theorem 3.5. *Let K be any projection invariant submodule of an R-module M. Then an R-module X is M-p-injective if and only if*

(i) X is K-p-injective.

(ii) X is (M/K)-p-injective.

(iii) any homomorphism $\varphi : K \to X$ can be lifted to a homomorphism $\theta : M \to X$.

Proof. Suppose that X is M-p-injective. Then (i) and (iii) clearly hold. Now suppose that N/K is projection invariant in M/K for $K \subseteq N \leq M$ and $\alpha : N/K \to X$ is a homomorphism. Since N/K is projection invariant in M/K, then N is projection invariant in M. Let $\pi_1 : M \to M/K$ and $\pi_2 : N \to N/K$ be the canonical epimorphisms. Since X is M-p-injective, the homomorphism $\alpha\pi_2 : N \to X$ can be extended to the homomorphism $\gamma : M \to X$. Since $N \leq \ker\gamma$, there exists a homomorphism $\beta : M/K \to X$ such that $\beta\pi_1 = \gamma$. Let $n \in N$. $\beta(n+K) = \beta(\pi_1(n)) = \gamma(n) = \alpha\pi_2(n) = \alpha(n+K)$. Hence β lifts to α so X is (M/K)-p-injective.

Conversely, suppose that X satisfies (i), (ii) and (iii). Let N be projection invariant submodule of M and $\varphi : N \to X$ be homomorphism. It is clear that $N \cap K$ is also projection invariant in K. Let φ' be the restriction of φ to $N \cap K$. By (i), there exists $\alpha : K \to X$ such that φ' can be lifted to α. By (iii), there exists $\beta : M \to X$ such that β lifts to α. Thus

$$\beta(k) = \alpha(k) = \varphi'(k) = \varphi(k) \text{ for all } k \in N \cap K.$$

Let $\gamma = \varphi - \beta$. It is clear that $\gamma : N \to X$ and $\gamma(N \cap K) = 0$. Define $\varphi'' : N + K/K \to X$ by $\varphi''(n+K) = \gamma(n+K)$ for all $n \in N$. φ'' is well defined, since $\gamma(N \cap K) = 0$. Note that $N + K$ and $N + K/K$ are both projection invariant in M. Thus by (ii), there exists $\theta' : M/K \to X$ such that θ' lifts to φ''. Define $\theta'' : M \to X$ by $\theta''(m) = \theta'(m+K)$ for all $m \in M$. Let $\theta = \beta + \theta''$ where $\theta : M \to X$. Let $n \in N$. Then

$$\theta(n) = \beta(n) + \theta''(n) = \varphi(n) - \varphi''(n+K) + \theta'(n+K) = \varphi(n).$$

Thus θ lifts φ so X is M-p-injective. $\square$

Proposition 3.6. *Let $K \subseteq N$ be submodules of an R-module M. Then the following statements are equivalent.*

(i) N/K is M-p-injective.

(ii) For all $K \leq N \leq M$ with projection invariant submodule N/K of M/K, N/K is a direct summand of M/K.

Proof. $(i) \Rightarrow (ii)$ N/K is M/K-p-injective by Theorem 3.5. Thus the identity map $\iota : N/K \to N/K$ can be lifted to a homorphism $\theta : M/K \to N/K$. It is easy to check that $M/K = \ker\theta \oplus (N/K)$.

$(ii) \Rightarrow (i)$ Let X be a projection invariant submodule of M and $\varphi : X \to N/K$ be a homomorphism with $K = \ker\varphi$. Since X/K is projection invariant in M/K and $K = \ker\varphi \leq X \leq A$, then X/K is a direct summand of M/K by (ii). Thus there exists $L \leq M$ such that $K \subseteq L$, $M/K = X/K \oplus L/K$. Define $\theta : M \to N/K$ by $\theta(x+l) = \varphi(x)$ where $x \in X$ and $l \in L$. Note that if $x \in X$, $l \in L$ and $x+l = 0$, then $x = -l \in X \cap L = K$. Thus θ is well defined. Clearly θ is an R-homomorphism and θ lifts φ. Thus N/K is M-p-injective. $\square$

4 Examples

We provide examples which show that self-p-injective and tight are different notions. Recall that 2-by-2 upper triangular matrix ring over a field is a right CS ring by [18, Theorem 3.4]. To this end, the following example corrects [6, Example 2.11].

Example 4.1. *Let $S = \mathbb{Z}[x]$ and let R be the 2-by-2 full matrix ring over S. Then R is not right CS by [4, Lemma 12.8]. By [10, Corollary 11.18 and Corollary 11.19], R is a semiprime right Goldie ring. Then R is R-tight by [7].*

There is a self-p-injective module which is not tight (see, [1, Example 3.1]). However the following example (see, [6, Example 2.13]) is R-tight but it is not self-p-injective.

Example 4.2. *Let $R = \{(m,n) \mid m \equiv n (mod\ 2)\} \subseteq \mathbb{Z} \times \mathbb{Z}$. Then R_R is tight by [6, Example 2.13]. Since R is indecomposable, Lemma 2.1 yields that R is not self-p-injective.*

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