Riemann-Lagrange geometry for starfish/coral dynamical system

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Abstract. In this paper we develop the Riemann-Lagrange geometry, in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy, associated with the dynamical system concerning social interaction in colonial organisms. Some possible trophodynamic interpretations are derived.

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Key words: tangent spaces; least squares Lagrangian functions; Riemann-Lagrange geometry; starfish/coral dynamics.

1 Social interactions in colonial organisms

Let $m \geq 2$ be an integer. We introduce social interactions for starfish/coral dynamics as follows (see Antonelli et al. [1]):

$$
(1.1)\qquad \begin{cases}
\dfrac{dN^1}{dt} &= \lambda_1 N^1 - \alpha_1 \left(N^1\right)^2 - \alpha_2 \left(\dfrac{m}{m-1}\right) \cdot N^1 N^2 + \\
&+ \dfrac{\alpha_1}{m-1} \left(\dfrac{N^2}{N^1}\right)^{m-2} \cdot \left(N^2\right)^2 - \delta_1 F N^1 \\
\dfrac{dN^2}{dt} &= \lambda_2 N^2 - \alpha_2 \left(N^2\right)^2 - \alpha_1 \left(\dfrac{m}{m-1}\right) \cdot N^1 N^2 + \\
&+ \dfrac{\alpha_2}{m-1} \left(\dfrac{N^1}{N^2}\right)^{m-2} \cdot \left(N^1\right)^2 - \delta_2 F N^2 \\
\dfrac{dF}{dt} &= \beta F \left(N^1 + N^2\right) + \gamma F^2 - \rho F,
\end{cases}
$$

where

- α_1, α_2, λ_1, λ_2, δ_1, δ_2, β, γ, ρ are positive coefficients;
- N^1, N^2 are coral densities;

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- F is the starfish density;
- λ_1 and λ_2 are growth rates;
- λ_1/α_1 and λ_2/α_2 are single species carrying capacities;
- β, δ_1 and δ_2 are the interaction coefficients for starfish preying on corals;
- γ is the coefficient of starfish aggregation.

Note that m is the effect of increasing the social parameter. If we set $m = 2$, we obtain the (2 corals/1 starfish)-model of Antonelli and Kazarinoff [2], in which every term of degree greater than one is quadratic. It is $m \geq 3$ which forces the social interaction terms to be nonquadratic.

By differentiation, the dynamical system (1.1) can be extended to a dynamical system of order two coming from a first order Lagrangian of least squares type. This extension is called in the literature in the field as geometric dynamical system (see Udrişte [7]).

2 The Riemann-Lagrange geometry

The system (1.1) can be regarded on the tangent space $T\mathbb{R}^3$, whose coordinates are

$$\left(x^1 = N^1,\ x^2 = N^2,\ x^3 = F,\ y^1 = \frac{dN^1}{dt},\ y^2 = \frac{dN^2}{dt},\ y^3 = \frac{dF}{dt}\right).$$

Remark 2.1. We recall that the transformations of coordinates on the tangent space $T\mathbb{R}^3$ are given by

$$\widetilde{x}^i = \widetilde{x}^i(x^j), \quad \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j, \tag{2.1}$$

where $i, j = \overline{1,3}$.

In this context, the solutions of class C^2 of the system (1.1) are the global minimum points of the least squares Lagrangian function (see [7], [6])

$$L = \left(y^1 - X^1\left(N^1, N^2, F\right)\right)^2 + \left(y^2 - X^2\left(N^1, N^2, F\right)\right)^2 + \tag{2.2}$$

$$+ \left(y^3 - X^3\left(N^1, N^2, F\right)\right)^2 \geq 0,$$

where

$$\begin{aligned} X^1\left(N^1, N^2, F\right) &= \lambda_1 N^1 - \alpha_1 \left(N^1\right)^2 - \alpha_2 \left(\frac{m}{m-1}\right) \cdot N^1 N^2 + \\ &+ \frac{\alpha_1}{m-1} \left(\frac{N^2}{N^1}\right)^{m-2} \cdot \left(N^2\right)^2 - \delta_1 F N^1, \end{aligned}$$

$$\begin{aligned} X^2\left(N^1,N^2,F\right) &= \lambda_2 N^2-\alpha_2\left(N^2\right)^2-\alpha_1\left(\frac{m}{m-1}\right)\cdot N^1N^2+ \\ &+ \frac{\alpha_2}{m-1}\left(\frac{N^1}{N^2}\right)^{m-2}\cdot\left(N^1\right)^2-\delta_2 FN^2, \\ X^3\left(N^1,N^2,F\right) &= \beta F\left(N^1+N^2\right)+\gamma F^2-\rho F, \end{aligned}$$

Remark 2.2. The solutions of class C^2 of the system (1.1) are solutions of the Euler-Lagrange equations attached to the least squares Lagrangian (2.2), namely (geometric dynamics, in Udrişte's terminology)

$$\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial y_1^i}\right)=0,\ y^i=\frac{dx^i}{dt},\quad \forall\ i=\overline{1,3},\Leftrightarrow \tag{2.3}$$

$$\frac{d^2x^i}{dt^2}+2G^i(x^k,y^k)=0\Leftrightarrow\frac{d^2x^i}{dt^2}+\frac{1}{2}\left(\frac{\partial^2 L}{\partial y^i\partial x^k}y^k-\frac{\partial L}{\partial x^i}\right)=0\Leftrightarrow$$

$$\frac{d^2x^i}{dt^2}=\left(\frac{\partial X^i}{\partial x^k}-\frac{\partial X^k}{\partial x^i}\right)y^k+\frac{\partial X^k}{\partial x^i}X^k,$$

where

$$\begin{aligned} G^i(x^k,y^k) &= \frac{1}{4}\left(\frac{\partial^2 L}{\partial y^i\partial x^k}y^k-\frac{\partial L}{\partial x^i}\right)= \\ &= -\frac{1}{2}\left[\left(\frac{\partial X^i}{\partial x^k}-\frac{\partial X^k}{\partial x^i}\right)y^k+\frac{\partial X^k}{\partial x^i}X^k\right] \end{aligned} \tag{2.4}$$

is endowed with the geometrical meaning of **semispray** of L (for more geometrical details, see Miron and Anastasiei book [5] and Udrişte's book [7]).

But, the least squares Lagrangian (2.2), together with its Euler-Lagrange equations (2.3), provide us with an entire Riemann-Lagrange geometry on the tangent space $T\mathbb{R}^3$, in the sense of nonlinear connection, d-torsions, d-curvatures and Yang-Mills-like energy. These geometrical objects are naturally associated with the tropho-dynamical system (1.1).

Let us recall the main geometrical ideas developed in the Miron and Anastasiei book [5]. The canonical nonlinear connection $N=\left(N_j^i\right)_{i,j=\overline{1,2}}$ produced by the semispray (2.4) is given by the components

$$N_j^i=\frac{\partial G^i}{\partial y^j}=-\frac{1}{2}\left(\frac{\partial X^i}{\partial x^j}-\frac{\partial X^j}{\partial x^i}\right).$$

Remark 2.3. We recall that, under a transformation of coordinates (2.1), the local components of the nonlinear connection obey the rules [4], [5]

$$\widetilde{N}_l^k=N_i^j\frac{\partial x^i}{\partial\widetilde{x}^l}\frac{\partial\widetilde{x}^k}{\partial x^j}-\frac{\partial x^i}{\partial\widetilde{x}^l}\frac{\partial\widetilde{y}^k}{\partial x^i}.$$

From a geometrical point of view, we point out that the coefficients N_j^i of the above nonlinear connection have not a global character on $T\mathbb{R}^3$.

Remark 2.4. Using the well-known Cartan-Kosambi-Chern (KCC) theory, used also in the paper of Böhmer, Harko and Sabău [3], we can remark that the **deviation curvature tensor** associated with the dynamical system (1.1) is given by the formula

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l\frac{\partial N^i_j}{\partial y^l} + \frac{\partial N^i_j}{\partial x^l}y^l + N^i_l N^l_j.$$

It is important to note that the solutions of the Euler-Lagrange equations (2.3) are Jacobi stable iff the real parts of the eigenvalues of the deviation tensor P^i_j are strictly negative everywhere, and Jacobi unstable, otherwise. For more details, see [3] and references therein.

The canonical nonlinear connection defines the adapted bases of vector fields and covector fields on the tangent space $T\mathbb{R}^3$, namely

$$\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j},\ \frac{\partial}{\partial y^i}\right\} \subset \mathcal{X}(T\mathbb{R}^3),$$

$$\left\{dx^i, \delta y^i = dy^i + N^i_j dx^j\right\} \subset \mathcal{X}^*(T\mathbb{R}^3).$$

The adapted local components of the Cartan N-linear connection $C\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$ are given by the formulas

$$L^i_{jk} = \frac{g^{ir}}{2}\left(\frac{\delta g_{rk}}{\delta x^j} + \frac{\delta g_{rj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^r}\right),\quad C^i_{jk} = \frac{g^{ir}}{2}\left(\frac{\partial g_{rk}}{\partial y^j} + \frac{\partial g_{rj}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^r}\right),$$

where

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = \delta_{ij}.$$

The only non-vanishing d-torsion adapted component associated with the Cartan N-linear connection $C\Gamma(N)$ is given by the coefficient

$$R^r_{ij} = \frac{\delta N^r_i}{\delta x^j} - \frac{\delta N^r_j}{\delta x^i} = \frac{\partial N^r_i}{\partial x^j} - \frac{\partial N^r_j}{\partial x^i}.$$

At the same time, all the adapted components of the curvature attached to the Cartan N-linear connection $C\Gamma(N)$ are zero (for all curvature formulas, see [5]).

The electromagnetic-like distinguished 2-form attached to the Lagrangian L, defined via its deflection d-tensors (for more details, see Miron and Anastasiei book [5]), is given by $\mathbb{F} = F_{ij}\delta y^i \wedge dx^j$, where

$$F_{ij} = \frac{1}{2}\left(g_{ir}N^r_j - g_{jr}N^r_i\right) = \frac{1}{2}\left(N^i_j - N^j_i\right) = N^i_j.$$

In this context, let us use the notation

$$J(X) = \left(\frac{\partial X^i}{\partial x^j}\right)_{i,j=\overline{1,3}} = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{pmatrix},$$

where

$$J_{11} = \lambda_1 - 2\alpha_1 N^1 - \alpha_2 \left(\frac{m}{m-1}\right) \cdot N^2 - \alpha_1 \left(\frac{m-2}{m-1}\right) \frac{\left(N^2\right)^m}{\left(N^1\right)^{m-1}} - \delta_1 F,$$

$$J_{12} = -\alpha_2 \left(\frac{m}{m-1}\right) \cdot N^1 + \alpha_1 \left(\frac{m}{m-1}\right) \frac{\left(N^2\right)^{m-1}}{\left(N^1\right)^{m-2}},$$

$$J_{13} = -\delta_1 N^1, \quad J_{21} = -\alpha_1 \left(\frac{m}{m-1}\right) \cdot N^2 + \alpha_2 \left(\frac{m}{m-1}\right) \frac{\left(N^1\right)^{m-1}}{\left(N^2\right)^{m-2}},$$

$$J_{22} = \lambda_2 - 2\alpha_2 N^2 - \alpha_1 \left(\frac{m}{m-1}\right) \cdot N^1 - \alpha_2 \left(\frac{m-2}{m-1}\right) \frac{\left(N^1\right)^m}{\left(N^2\right)^{m-1}} - \delta_2 F,$$

$$J_{23} = -\delta_2 N^2, \quad J_{31} = \beta F, \quad J_{32} = \beta F, \quad J_{33} = \beta \left(N^1 + N^2\right) + 2\gamma F - \rho.$$

Following the above Miron and Anastasiei's geometrical ideas, we obtain the following geometrical results:

Theorem 2.1. (i) *The canonical nonlinear connection on $T\mathbb{R}^3$, produced by the system (1.1), has the local components $N = \left(N_j^i\right)_{i,j=\overline{1,3}}$, where N_j^i are the entries of the skew-symmetric matrix*

$$N = -\frac{1}{2}\left[J(X) - {}^{T}J(X)\right] = \begin{pmatrix} N_1^1 & N_2^1 & N_3^1 \\ N_1^2 & N_2^2 & N_3^2 \\ N_1^3 & N_2^3 & N_3^3 \end{pmatrix},$$

where

$$N_1^1 = N_2^2 = N_3^3 = 0,$$

$$N_2^1 = -N_1^2 = -\frac{1}{2}\left\{\left(\frac{m}{m-1}\right)\left(\alpha_1 N^2 - \alpha_2 N^1\right) + \right.$$
$$\left. + \left(\frac{m}{m-1}\right)\left[\alpha_2 \frac{\left(N^1\right)^{m-1}}{\left(N^2\right)^{m-2}} - \alpha_1 \frac{\left(N^2\right)^{m-1}}{\left(N^1\right)^{m-2}}\right]\right\},$$

$$N_3^1 = -N_1^3 = \frac{1}{2}\left(\beta F + \delta_1 N^1\right), \quad N_3^2 = -N_2^3 = \frac{1}{2}\left(\beta F + \delta_2 N^2\right).$$

(ii) *All adapted components of the canonical Cartan connection $C\Gamma(N)$, produced by the system (1.1), are zero.*

(iii) *The effective adapted components R_{jk}^i of the torsion d-tensor* $\mathbf{T}$ *of the canonical Cartan connection $C\Gamma(N)$, produced by the system (1.1), are the entries of the following skew-symmetric matrices:*

$$R_1 = \left(R_{j1}^i\right)_{i,j=\overline{1,3}} = \frac{\partial N}{\partial N^1} = \begin{pmatrix} 0 & \dfrac{\partial N_2^1}{\partial N^1} & \dfrac{\delta_1}{2} \\ -\dfrac{\partial N_2^1}{\partial N^1} & 0 & 0 \\ -\dfrac{\delta_1}{2} & 0 & 0 \end{pmatrix},$$

where

$$\frac{\partial N_2^1}{\partial N^1}=\frac{1}{2}\left(\frac{m}{m-1}\right)\left[\alpha_2-\alpha_2\left(m-1\right)\left(\frac{N^1}{N^2}\right)^{m-2}-\alpha_1\left(m-2\right)\left(\frac{N^2}{N^1}\right)^{m-1}\right];$$

$$R_2=\left(R_{j2}^i\right)_{i,j=\overline{1,3}}=\frac{\partial N}{\partial N^2}=\begin{pmatrix} 0 & \dfrac{\partial N_2^1}{\partial N^2} & 0 \\ -\dfrac{\partial N_2^1}{\partial N^2} & 0 & \dfrac{\delta_2}{2} \\ 0 & -\dfrac{\delta_2}{2} & 0 \end{pmatrix},$$

where

$$\frac{\partial N_2^1}{\partial N^2}=\frac{1}{2}\left(\frac{m}{m-1}\right)\left[-\alpha_1+\alpha_2\left(m-2\right)\left(\frac{N^1}{N^2}\right)^{m-1}+\alpha_1\left(m-1\right)\left(\frac{N^2}{N^1}\right)^{m-2}\right];$$

$$R_3=\left(R_{j3}^i\right)_{i,j=\overline{1,3}}=\frac{\partial N}{\partial F}=\begin{pmatrix} 0 & 0 & \dfrac{\beta}{2} \\ 0 & 0 & \dfrac{\beta}{2} \\ -\dfrac{\beta}{2} & -\dfrac{\beta}{2} & 0 \end{pmatrix}.$$

(iv) *All adapted components of the curvature d-tensor* $\mathbf{R}$ *of the canonical Cartan connection* $C\Gamma(N)$*, produced by the system (1.1), vanish.*

(v) *The geometric electromagnetic-like distinguished* 2*-form, produced by the system (1.1), is given by* $\mathbb{F}=F_{ij}\delta y^i\wedge dx^j$*, where the adapted components* F_{ij} *are the entries of the skew-symmetric matrix* $F=\left(F_{ij}\right)_{i,j=\overline{1,3}}=N$.

(vi) *The geometric Yang-Mills-like energy, produced by the system (1.1), is given by the formula*

$$\mathcal{EYM}(t)=F_{12}^2+F_{13}^2+F_{23}^2=$$

$$=\frac{1}{4}\left(\frac{m}{m-1}\right)^2\left[\alpha_1N^2-\alpha_2N^1+\alpha_2\frac{\left(N^1\right)^{m-1}}{\left(N^2\right)^{m-2}}-\alpha_1\frac{\left(N^2\right)^{m-1}}{\left(N^1\right)^{m-2}}\right]^2+$$

$$+\frac{1}{4}\left(\beta F+\delta_1N^1\right)^2+\frac{1}{4}\left(\beta F+\delta_2N^2\right)^2.$$

Remark 2.5. In the author's opinion, from a trophodynamic point of view the zero level of the jet geometric Yang-Mills energy produced by the system (1.1) is important. The jet geometric Yang-Mills trophodynamical energy produced by the system (1.1) is zero iff

$$\beta F+\delta_1N^1=0,\quad \beta F+\delta_2N^2=0,$$

$$\left(\alpha_1N^2-\alpha_2N^1\right)+\left[\alpha_2\frac{\left(N^1\right)^{m-1}}{\left(N^2\right)^{m-2}}-\alpha_1\frac{\left(N^2\right)^{m-1}}{\left(N^1\right)^{m-2}}\right]=0.$$

If $\delta_1 \neq \delta_2$, these conditions imply the impossible fact that $F = N^1 = N^2 = 0$, and if $\delta_1 = \delta_2 = \delta$, then we obtain $N^1 = N^2 = -\beta F/\delta$. In this last case, we find a Bernoulli differential equation as the last equation of the system (1.1), namely

$$\frac{dF}{dt} = -\rho F + \left(\gamma - 2\frac{\beta^2}{\delta}\right) F^2.$$

This equation can be integrated by using the changing of variable $z = F^{-1}$. The solution of the above Bernoulli differential equation is

$$F(t) = \frac{1}{a \exp(\rho t) + b},$$

where $a \in \mathbb{R}$ is an arbitrary constant, and we have

$$b = \frac{1}{\rho}\left(\gamma - 2\frac{\beta^2}{\delta}\right).$$

At the same time, we consider that the constant level surfaces of the jet geometric Yang-Mills trophodynamical energy $\mathcal{EYM}(t) = C$, $C > 0$, could contain important trophodynamic connotations. Consequently, the graphical representation of these surfaces in the system of axes OFN^1N^2 could be a fruitful and open problem in trophodynamics.

Remark 2.6. The deviation curvature tensor components P^i_j can be obtained by contracting with y^k the nonzero components of the torsion tensor R^i_{jk}, that is $P^i_j = R^i_{jk}y^k = \left(\partial N^i_j/\partial x^k\right) y^k$. Consequently, the matrix of the deviation curvature tensor is given by

$$P = R_k y^k = \begin{pmatrix} 0 & \dfrac{\partial N^1_2}{\partial N^1} & \dfrac{\delta_1}{2} \\ -\dfrac{\partial N^1_2}{\partial N^1} & 0 & 0 \\ -\dfrac{\delta_1}{2} & 0 & 0 \end{pmatrix} y^1 + \begin{pmatrix} 0 & \dfrac{\partial N^1_2}{\partial N^2} & 0 \\ -\dfrac{\partial N^1_2}{\partial N^2} & 0 & \dfrac{\delta_2}{2} \\ 0 & -\dfrac{\delta_2}{2} & 0 \end{pmatrix} y^2 +$$

$$+ \begin{pmatrix} 0 & 0 & \dfrac{\beta}{2} \\ 0 & 0 & \dfrac{\beta}{2} \\ -\dfrac{\beta}{2} & -\dfrac{\beta}{2} & 0 \end{pmatrix} y^3 = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

where

$$a = \frac{\partial N^1_2}{\partial N^1} y^1 + \frac{\partial N^1_2}{\partial N^2} y^2, \quad b = \frac{\delta_1}{2} y^1 + \frac{\beta}{2} y^3, \quad c = \frac{\delta_2}{2} y^2 + \frac{\beta}{2} y^3.$$

The eigenvalues of the matrix P are the real values

$$\lambda_1 = 0,\ \lambda_{2,3} = \pm\sqrt{a^2 + b^2 + c^2}.$$

In conclusion, the behavior of neighboring solutions of the Euler-Lagrange equations (2.3) is Jacobi unstable.

Open problem. The trophodynamic interpretations associated with the geometrical objects constructed in this paper still represent an open problem.

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References

[1] P. Antonelli, P. Auger, R. Bradbury, *Corals and starfish waves on the Great Barrier Reef: Analytical trophodynamics and 2-patch aggregation methods*, Mathl. Comput. Modelling, 27, 4 (1998), 121-135.

[2] P. Antonelli, N. Kazarinoff, *Starfish predation of a growing coral reef community*, J. Theor. Biol. 107 (1984), 667-684.

[3] C. G. Böhmer, T. Harko, S. V. Sabău, *Jacobi stability analysis of dynamical systems - applications in gravitation and cosmology*, Adv. Theor. Math. Phys. 16 (2012), 1145-1196.

[4] I. Bucătaru, R. Miron, *Finsler-Lagrange Geometry. Applications to Dynamical Systems*, Romanian Academy Eds., Bucharest, 2007.

[5] R. Miron, M. Anastasiei, *The Geometry of Lagrange spaces: Theory and Applications*, Kluwer Academic Publishers, 1994.

[6] M. Neagu, C. Udrişte, *From PDEs systems and metrics to multi-time field theories and geometric dynamics*, Seminarul de Mecanică 79, Timişoara, Romania, 2001, 1-33.

[7] C. Udrişte, *Geometric Dynamics*, Kluwer Academic Publishers, 2000.

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