Existence of weak solutions for p-Laplacian problem with impulsive effects

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Abstract. By virtue of variational method and critical point theory, we will investigate the existence of weak solutions for a p-Laplacian impulsive differential equation with boundary conditions.

M.S.C. 2010: 35J60; 35B30, 35B40.
Key words: Impulsive differential equation; p-Laplacian; weak solution; Browder theorem.

1 Introduction

Impulsive differential equations arising in real world phenomena describe the dynamics of processes in which sudden, discontinuous jumps occur. These processes are subject to short term perturbations whose duration is negligible in comparison with duration of the process. In fact, it is known, for example, that many biological phenomena involving thresholds, optimal control models in economics, pharmacokinetics and frequency modulated systems, do exhibit impulse effects. The theory of impulsive differential equations has been emerging as an important area of investigation in recent years [4, 23, 31].

Recently, variational methods and critical point theory have been successfully employed to investigate impulsive differential equations, to the best our knowledge; we refer the reader to [30, 32, 33, 34, 38] and references cited therein. Meanwhile, some people begin to study p-Laplacian differential equations with impulsive effects; for example, see [1, 2, 26, 15, 16, 35].

Chen and Tang [15] adopt the least action principle and the saddle point theorem to obtain some existence theorems for second-order p-Laplacian systems with or without impulsive effects under weak sublinear growth conditions. In [16], They also consider that a class of second-order impulsive differential equations with Dirichlet problems has one or infinitely many solutions under more relaxed assumptions on their nonlinearity f, which satisfies a kind of new superquadratic and subquadratic conditions. [8] proved the existence of weak solutions for p-Laplacian Dirichlet boundary value problem with impulsive e ects by topological degree theory and the Fountain theorem under the Cerami condition. In [5], the authors investigated the existence of weak solutions for a p-Laplacian impulsive differential equation with antiperiodic

Applied Sciences, Vol. 22, 2020, pp. 128-145.
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boundary condition by using the Browder theorem and the Fountain theorem under the Cerami condition.

The problem of finding infinitely many large energy solutions is a very classical problem; there is an extensive literature concerning the existence of infinitely many large energy solutions of a plethora of problems via the symmetric mountain pass theorem and fountain theorem; for instance, see [25, 3, 14, 22, 24, 27, 12, 37].

In this paper we consider the following problem

$$\begin{cases} -\left(\rho(x)\left|u^{'}\right|^{p-2}u^{'}\right)^{'}+s(x)\left|u\right|^{p-2}u=f\left(x,\ u\right) \quad \text{in } [0,\ T] \\ \triangle\left(\left|u^{'}\left(x_j\right)\right|^{p-2}u^{'}\left(x_j\right)\right)=I_j\left(u\left(x_j\right)\right), \quad j=1,2,...,n \\ u\left(0\right)=u\left(T\right)=0 \end{cases} \tag{1.1}$$

where $p>1, T>0, \rho(x), s(x)\in L^{\infty}\left([0,\ T]\right)$ satisfy the conditions $\operatorname{essinf}_{t\in[0,\ T]}\rho(x)>0$, $\operatorname{essinf}_{t\in[0,\ T]}s(x)>0$, $0=x_0<x_1<x_2<\cdots<x_n<x_{n+1}=T$, and $I_j:\mathbb{R}\longrightarrow\mathbb{R}$ are continuous for every $j=1,\ 2,\cdots,n$, $f\in\left(C\left([0,\ T]\right)\times\mathbb{R},\ \mathbb{R}\right)$. Moreover $\triangle\left(\left|u^{'}\left(x_j\right)\right|^{p-2}u^{'}\left(x_j\right)\right)=\left|u^{'}\left(x_j^{+}\right)\right|^{p-2}u^{'}\left(x_j^{+}\right)-\left|u^{'}\left(x_j^{-}\right)\right|^{p-2}u^{'}\left(x_j^{-}\right)$, where $u^{'}\left(x_j^{+}\right)$ and $u^{'}\left(x_j^{-}\right)$ denote the right and left limits, respectively, of $u^{'}(x)$ at $x=x_j$, for $j=1,\ 2,\ldots,n$.

2 Variational structure

We first introduce the Banach space

$$X=\left\{u\in W^{1,\ p}\left([0,\ T]\right):\ u\left(0\right)=u\left(T\right)=0\right\},$$

endowed with the norm

$$\|u\|=\left(\int_0^T\rho(x)\left|u^{'}(x)\right|^p+\int_0^T s(x)\left|u(x)\right|^p\right)^{\frac{1}{p}} \tag{2.1}$$

Suppose that $u\in C[0;T]$ satisfies the Dirichlet conditions $u(0)=u(T)=0$. Take $v\in X$ and multiplying

$$-\left(\rho(x)\left|u^{'}\right|^{p-2}u^{'}\right)^{'}+s(x)\left|u\right|^{p-2}u=f\left(x,\ u\right)$$

throughout by v and integrating from 0 to T:

$$-\int_0^T \left(\rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)\right)^{'}v(x)dx+\int_0^T s(x)\left|u(x)\right|^{p-2}u(x)v(x)dx=\int_0^T f\left(x,\ u(x)\right)v(x)dx$$

Now, we consider the first term

$$\begin{aligned}
-\int_0^T \left(\rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)\right)^{'}v(x)dx &= \sum_{j=0}^{n}\int_{x_j^+}^{x_{j+1}^-} -\left(\rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)\right)^{'}v(x)dx \\
&= \int_0^T \rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)v^{'}(x)dx \\
&\quad +\sum_{j=1}^{n}\rho\left(x_j\right)\left[\left|u^{'}\left(x_j^+\right)\right|^{p-2}u^{'}\left(x_j^+\right)-\left|u^{'}\left(x_j^-\right)\right|^{p-2}u^{'}\left(x_j^-\right)\right]v\left(x_j\right) \\
&= \int_0^T \rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)v^{'}(x)dx \\
&\quad +\sum_{j=1}^{n}\rho\left(x_j\right)\Delta\left(\left|u^{'}\left(x_j\right)\right|^{p-2}u^{'}\left(x_j\right)\right)v\left(x_j\right) \\
&= \int_0^T \rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)v^{'}(x)dx+\sum_{j=1}^{n}\rho\left(x_j\right)I_j\left(u\left(x_j\right)\right)v\left(x_j\right),
\end{aligned}$$

and then, we get

$$\begin{aligned}
&\int_0^T \rho(x)\left|u^{'}(x)\right|^{p-2}u^{'}(x)v^{'}(x)dx+\int_0^T s(x)\left|u(x)\right|^{p-2}u(x)v(x)dx+\sum_{j=1}^{n}\rho\left(x_j\right)I_j\left(u\left(x_j\right)\right)v\left(x_j\right) \\
=&\int_0^T f\left(x,\ u(x)\right)v(x)dx,
\end{aligned}$$

Considering the above, we introduce the following concept for the solution for problem (1.1).

Definition 2.1. A function $u \in X$ is said to be a weak solution of (1.1) if u satisfies

$$\begin{aligned} & \int_0^T \rho(x) \left|u^{'}(x)\right|^{p-2} u^{'}(x)v^{'}(x)dx + \int_0^T s(x) \left|u(x)\right|^{p-2} u(x)v(x)dx \\ & + \sum_{j=1}^{n} \rho\left(x_j\right) I_j\left(u\left(x_j\right)\right) v\left(x_j\right) \\ & - \int_0^T f\left(x,\ u(x)\right) v(x)dx \\ = \ & 0, \end{aligned} \tag{2.2}$$

for any $v \in X$.

From (2.2), we can obtain the weak solutions for (1.1) that coincide with critical points of the energy functional

$$\phi(u) = \frac{1}{p} \|u\|^p + \sum_{j=1}^{n} \rho\left(x_j\right) \int_0^{u(x_j)} I_j(x)dx - \int_0^T F\left(x,\ u(x)\right) dx, \tag{2.3}$$

where $F\left(x,\ u\right) = \int_0^u f\left(x,\ t\right) dt$. Clearly, ϕ is class of C^1 and its derivative is

$$\begin{aligned} \left\langle \phi^{'}(u),\ v \right\rangle = & \int_0^T \rho(x) \left|u^{'}(x)\right|^{p-2} u^{'}(x)v^{'}(x)dx + \int_0^T s(x) \left|u(x)\right|^{p-2} u(x)v(x)dx \\ & + \sum_{j=1}^{n} \rho\left(x_j\right) I_j\left(u\left(x_j\right)\right) v\left(x_j\right) \\ & - \int_0^T f\left(x,\ u(x)\right) v(x)dx, \end{aligned} \tag{2.4}$$

Lemma 2.1. (*see* [14, *Proposition* 2.1]) *Let* $u \in X$. *Then*

$$\|u\|^p_{L^p([0,\ T])} \le M_0 \|u\|^p, \tag{2.5}$$

where $M_0 := \max\left\{\operatorname{essinf}_{x\in[0,\ T]} \rho(x);\ \operatorname{essinf}_{x\in[0,\ T]} s(x)\right\}$.

Lemma 2.2. (*see* [14, *Proposition* 2.2]) *Let* $u \in X$. *Then*

$$\|u\|_\infty \le \left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{1}{p}} \|u\|, \tag{2.6}$$

where $\rho_0 := \operatorname{essinf}_{x\in[0,\ T]} \rho(x)$.

Definition 2.2. (see [10, page 303])Let X be a reflexive real Banach space and X^* its dual. The operator $\pounds : X \longrightarrow X^*$ is said to be demicontinuous if $\pounds$ maps strongly convergent sequences in X to weakly convergent sequences in X^*.

Lemma 2.3. *(Browder theorem, see [17, Theorem 5.3.22]) Let X be a reflexive real Banach space. Moreover, Let $\pounds : X \longrightarrow X^*$ be an operator satisfying the following conditions:*

(i) $\pounds$ is bounded and demicontinuous;

(ii) $\pounds$ is coercive, that is, $\lim_{\|x\| \longrightarrow \infty} (\pounds(x),\ x) / \|x\| = +\infty$;

(iii) $\pounds$ is monotone on the space X; that is; for all $x, y \in X$, one has

$$\langle \pounds(x) - \pounds(y),\ x - y \rangle \geq 0. \tag{2.7}$$

Then the equation $\pounds(x) = f^*$ has at least one solution $x \in X$ for every $f^* \in X^*$. If, moreover, the inequality (2.7) is strict for all $x, y \in X$, $x \neq y$, then the equation $\pounds(x) = f^*$ has precisely one solution $x \in X$ for all $f^* \in X^*$.

3 Main results

In this section, we list our assumptions on f and I_j $(j = 1, 2, ..., n)$, which will be used in this paper;

$(H1)$ $f(x,\ u)$ is a decreased function about u, uniformly in $x \in [0,\ T]$, and $I_j(u)$ $(j = 1,\ 2,\ \cdots,\ n)$ are increased functions with u.

$(H2)$ there exist $\alpha_j,\ \beta_j > 0$ and $\gamma_j \in [1,\ p)$ such that $|I_j(u)| \leq \alpha_j + \beta_j |u|^{\gamma_j - 1}$, for all $u \in \mathbb{R}$ and $j = 1,\ 2,\ \cdots,\ n$.

$(H3)$ There exist $c_1,\ c_2 > 0$ such that $f(x,\ u) \leq c_1 + c_2 |u|^{p-1}$, for all $u \in \mathbb{R}$, $x \in [0,\ T]$.

$(H4)$ There is a positive constant $a > 0$ such that $\lim_{|u| \longrightarrow \infty} \frac{-pF(x,\ u) + f(x,\ u)}{|u|} \geq a$, uniformly in $x \in [0,\ T]$.

$(H5)$ $p \int_0^u I_j(s)\, ds - I_j(u)u \geq 0$, $\int_0^u I_j(s)\, ds \geq 0$, for all $u \in \mathbb{R}$, $j = 1,\ 2,\ \ldots,\ n$.

$(H6)$ $\lim_{|u| \longrightarrow \infty} \frac{F(x,\ u)}{|u|^p} = +\infty$, uniformly on $x \in [0,\ T]$.

$(H7)$ $F(x,\ u)$ is an even function about u and $I_j(u)$ $(j = 1,\ 2, \ldots,$ n) are odd functions about u, for all $x \in [0,\ T]$.

Theorem 3.1. *Let $0 < c_2 < \frac{1}{M_0}$, and $(H1) - (H3)$ hold. Then (1.1) has precisely a weak solution.*

Proof.

$$
\begin{aligned}
\langle L_1(u),\ v\rangle &= \int_0^T \rho(x)\left|u'(x)\right|^{p-2}u'(x)v'(x)dx+\int_0^T s(x)\left|u(x)\right|^{p-2}u(x)v(x)dx,\\
\langle L_2(u),\ v\rangle &= \sum_{j=1}^{n}\rho\left(x_j\right)I_j\left(u\left(x_j\right)\right)v\left(x_j\right), \qquad (3.1)\\
\langle L_3(u),\ v\rangle &= \int_0^T f\left(x,\ u(x)\right)v(x)dx.
\end{aligned}
$$

Let

$$\langle \pounds(u),\ v\rangle=\langle L_1(u),\ v\rangle+\langle L_2(u),\ v\rangle-\langle L_3(u),\ v\rangle,\qquad \forall u,\ v\in X.$$

Then, to find a weak solution of (1.1) is equivalent to finding a solution for the operator equation $\pounds(u)=0$. In what follows, we shall sketch the properties of operators $L_i\ (i=1,\ 2,\ \cdots,\ n)$. By Hölder inequality, we have

$$
\begin{aligned}
\left|\langle\langle L_1(u),\ v\rangle\rangle\right| &= \left|\int_0^T \rho(x)\left|u'(x)\right|^{p-2}u'(x)v'(x)dx+\int_0^T s(x)\left|u(x)\right|^{p-2}u(x)v(x)dx\right|\\
&\le \left(\int_0^T \rho(x)\left|u'(x)\right|^{p}dx\right)^{\frac{p-1}{p}}\left(\int_0^T \rho(x)\left|v'(x)\right|^{p}dx\right)^{\frac{1}{p}}\\
&\quad+\left(\int_0^T s(x)\left|u(x)\right|^{p}dx\right)^{\frac{p-1}{p}}\left(\int_0^T s(x)\left|v(x)\right|^{p}dx\right)^{\frac{1}{p}}
\end{aligned}
$$

using the following inequality

$$(a+b)^{\beta}\left(c+d\right)^{1-\beta}\ge a^{\beta}c^{1-\beta}+b^{\beta}d^{1-\beta} \qquad (3.2)$$

which holds for any $\beta\in(0,\ 1)$ and $a>0$, $b>0$, $c>0$, $d>0$. Set $\beta=\frac{p-1}{p}$ and

$$
\begin{aligned}
a &= \int_0^T \rho(x)\left|u'(x)\right|^{p}dx \qquad b=\int_0^T s(x)\left|u(x)\right|^{p}dx,\\
c &= \int_0^T \rho(x)\left|v'(x)\right|^{p}dx \qquad d=\int_0^T s(x)\left|v(x)\right|^{p}dx,
\end{aligned}
$$

then we can deduce that

$$\begin{aligned} |\langle L_1(u),\ v\rangle| &\leq \left[\int_0^T \rho(x)\left|u^{'}(x)\right|^p dx + \int_0^T s(x)\left|u(x)\right|^p dx\right]^{\frac{p-1}{p}} \left[\int_0^T \rho(x)\left|v^{'}(x)\right|^p dx + \int_0^T s(x)\left|v(x)\right|^p dx\right]^{\frac{1}{p}} \\ &\leq \|u\|^{p-1}\|v\| \\ &< \infty \qquad \forall u,\ v \in X. \end{aligned}$$

Consequently, L_1 is bounded.

$$\begin{aligned} & |\langle L_1(u_n) - L_1(u),\ u_n - u\rangle| \\ = & \left|\int_0^T \rho(x)\left[\left|u_n^{'}(x)\right|^{p-2} u_n^{'}(x) - \left|u^{'}(x)\right|^{p-2} u^{'}(x)\right](u_n - u)^{'}(x)dx + \right. \\ & \left. \int_0^T s(x)\left[\left|u_n(x)\right|^{p-2} u_n(x) - \left|u(x)\right|^{p-2} u(x)\right](u_n - u)(x)dx\right| \\ \leq & \left[\int_0^T \rho(x)\left(\left|u_n^{'}(x)\right|^{p-2} u_n^{'}(x) - \left|u^{'}(x)\right|^{p-2} u^{'}(x)\right)^{\frac{p}{p-1}} + \right. \\ & \left. s(x)\left(\left|u_n(x)\right|^{p-2} u_n(x) - \left|u(x)\right|^{p-2} u(x)\right)^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \left[\int_0^T \left(\rho(x)\left|u_n^{'}(x) - u^{'}(x)\right|^p + s(x)\left|u_n - u\right|^p\right)dx\right]^{\frac{1}{p}} \\ \leq & \left[\int_0^T \rho(x)\left(\left|u_n^{'}(x)\right|^{p-2} u_n^{'}(x) - \left|u^{'}(x)\right|^{p-2} u^{'}(x)\right)^{\frac{p}{p-1}} + \right. \\ & \left. s(x)\left(\left|u_n(x)\right|^{p-2} u_n(x) - \left|u(x)\right|^{p-2} u(x)\right)^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \|u_n - u\|. \end{aligned}$$

The last integral tends to zero as $\|u_n - u\| \longrightarrow 0$. Hence L_1 is continuous.

We assume that $u_n \rightharpoonup u$ in X; by Lemma 2, we know that there is a subsequence, still denoted by $\{u_n\}$, which strongly converges to u in $C[0,\ T]$ implies that

$$f(t,\ u_n(t)) \longrightarrow f(t,\ u(t)), \quad \forall t \in [0,\ T]$$

and

$$|f(t,\ u_n(t))| \leq \sup_{y\in[-M,\ M]} |f(t,\ y)| = K(t) \in L^1([0,\ T]),$$

then by the Lebesgue's dominated convergence theorem, we have

$$\int_0^T f(t,\ u_n(t))\,dt \longrightarrow \int_0^T f(t,\ u(t))\,dt \quad \text{as } n \longrightarrow \infty,$$

we have

$$|\langle L_3(u_n) - L_3(u), v\rangle| = \left|\int_0^T [f(t, u_n(t)) - f(t, u(t))]\,v(t)\,dt\right| \to 0 \quad \text{as } n \to \infty.$$

So, L_3 is continuous. In the same methods, we see that L_2 is also continuous.

By (2.6) and (H_2), for all $u,\ v \in X$, we have

$$\begin{aligned}
|\langle L_2(u),\ v\rangle| &= \left|\sum_{j=1}^{n} \rho\left(x_j\right) I_j\left(u\left(x_j\right)\right) v\left(x_j\right)\right| \\
&\leq \sum_{j=1}^{n} \left|\rho\left(x_j\right) I_j\left(u\left(x_j\right)\right)\right| \left|v\left(x_j\right)\right| \\
&\leq \sum_{j=1}^{n} \left|\rho\left(x_j\right)\right| \left(\alpha_j + \beta_j \left|u\left(x_j\right)\right|^{\gamma_j - 1}\right) \left|v\left(x_j\right)\right| \\
&\leq \sum_{j=1}^{n} \|\rho\|_\infty \left(\alpha_j + \beta_j \|u\|_\infty^{\gamma_j - 1}\right) \|v\|_\infty \\
&\leq n\rho_1 \left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{1}{p}} \left(\alpha_j + \beta_j \left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{\gamma_j - 1}{p}} \|u\|^{\gamma_j - 1}\right) \|v\| \\
&< \infty \qquad \forall u,\ v \in X,
\end{aligned}$$

where $\rho_1 = \operatorname{esssup}_{x\in[0,\ T]} \rho(x)$, then by (2.5) and $(H3)$ we have

$$\begin{aligned}
|\langle L_3(u),\ v\rangle| &= \left|\int_0^T f\left(x,\ u(x)\right) v(x) dx\right| \\
&\leq \int_0^T \left|f\left(x,\ u(x)\right)\right| |v(x)|\, dx \\
&\leq \int_0^T \left[c_1 + c_2 |u|^{p-1}\right] |v(x)| \\
&\leq c_1 T^{\frac{p-1}{p}} \|v\|_p + c_2 \|u\|_p^{p-1} \|v\|_p \\
&\leq c_1 T^{\frac{p-1}{p}} M_0^{\frac{1}{p}} \|v\| + c_2 \left(M_0^{\frac{1}{p}} \|u\|\right)^{p-1} M_0^{\frac{1}{p}} \|v\| \\
&\leq M_0^{\frac{1}{p}} \left[c_1 T^{\frac{p-1}{p}} + c_2 M_0^{\frac{p-1}{p}} \|u\|^{p-1}\right] \|v\| \\
&< \infty,
\end{aligned}$$

therefore, L_2 and L_3 are bounded. Up to now, we have proved that $\pounds$ is bounded and continuous, so (i) of Lemma 3 holds.

Now, we prove the monotonicity of $\pounds$. Hence by $(H1)$ for $u,\ v \in X$, we have

$$\begin{aligned}
&\langle \pounds(u) - \pounds(v), \ u - v \rangle \\
= &\int_0^T \rho(x) \left[\left| u'(x) \right|^{p-2} u'(x) - \left| v'(x) \right|^{p-2} v'(x) \right] \left(u'(x) - v'(x) \right) dx \\
&+ \int_0^T s(x) \left[|u(x)|^{p-2} u(x) - |v(x)|^{p-2} v(x) \right] (u(x) - v(x))\, dx \\
&- \int_0^T \left[f(x, \ u(x)) - f(x, \ v(x)) \right] (u(x) - v(x))\, dx \\
&+ \sum_{j=1}^{n} \left[\rho(x_j) I_j(u(x_j)) - \rho(x_j) I_j(v(x_j)) \right] (u(x_j) - v(x_j)) \\
\geq &\int_0^T \rho(x) \left[\left| u'(x) \right|^{p-2} u'(x) - \left| v'(x) \right|^{p-2} v'(x) \right] \left(u'(x) - v'(x) \right) dx \\
&+ \int_0^T s(x) \left[|u(x)|^{p-2} u(x) - |v(x)|^{p-2} v(x) \right] (u(x) - v(x))\, dx
\end{aligned}$$

Then by (3.2) we have

$$\begin{aligned}
\langle \pounds(u) - \pounds(v), \ u - v \rangle \geq &\int_0^T \rho(x) \left[\left| u'(x) \right|^{p-2} u'(x) - \left| v'(x) \right|^{p-2} v'(x) \right] \left(u'(x) - v'(x) \right) dx \\
&+ \int_0^T s(x) \left[|u(x)|^{p-2} u(x) - |v(x)|^{p-2} v(x) \right] (u(x) - v(x))\, dx \\
\geq &\left(\|u\|^{p-1} - \|v\|^{p-1} \right) (\|u\| - \|v\|) \\
> &\ 0,
\end{aligned} \tag{3.3}$$

Consequently, $\pounds$ is monotone.

Finally, we prove $\pounds$ is coercive, by (2.5) and (2.6) for $u, v \in X$, it follows from $(H2)$ and $(H3)$ that

$$
\begin{aligned}
\langle \pounds(u),\ u\rangle &= \int_0^T \rho(x)\left|u'(x)\right|^p dx + \int_0^T s(x)\left|u(x)\right|^p dx + \sum_{j=1}^n \rho\left(x_j\right) I_j\left(u\left(x_j\right)\right) u\left(x_j\right) - \int_0^T f\left(x,\ u(x)\right) u(x)dx \\
&= \|u\|^p + \sum_{j=1}^n \rho\left(x_j\right) I_j\left(u\left(x_j\right)\right) u\left(x_j\right) - \int_0^T f\left(x,\ u(x)\right) u(x)dx \\
&\geq \|u\|^p - \left|\sum_{j=1}^n \rho\left(x_j\right) I_j\left(u\left(x_j\right)\right) u\left(x_j\right)\right| - \int_0^T f\left(x,\ u(x)\right) u(x)dx \\
&\geq \|u\|^p - \sum_{j=1}^n \rho\left(x_j\right)\left(\alpha_j + \beta_j\left|u\left(x_j\right)\right|^{\gamma_j-1}\right) u\left(x_j\right) - \int_0^T \left(c_1 + c_2\left|u\right|^{p-1}\right)\left|u\right| dx \\
&\geq \|u\|^p - \sum_{j=1}^n \|\rho\|_\infty\left(\alpha_j + \beta_j\|u\|_\infty^{\gamma_j-1}\right)\|u\|_\infty - c_1 T^{\frac{p-1}{p}}\|u\|_{L^p} - c_2\|u\|_{L^p}^p \\
&\geq \left(1 - c_2 M_0\right)\|u\|^p - n\rho_1\left(\alpha_j\left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{1}{p}} + c_1 T^{\frac{p-1}{p}} M_0^{\frac{1}{p}}\right)\|u\| - n\beta_j\rho_1\left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{\gamma_j}{p}}\|u\|^{\gamma_j}, \qquad (3.4)
\end{aligned}
$$

Therefore, by the span of $c_2 \in \left(0,\ \frac{1}{M_0}\right)$, we arrive $\lim_{\|u\|\longrightarrow\infty}\langle \pounds(u),\ u\rangle / \|u\| = +\infty$. As a result, (ii) and (iii) of Lemma 3 hold. Hence, Lemma 3 implies that (1.1) has precisely a weak solution. This completes the proof. □

Example 3.1. Let $p = 4$, $x_1 = \frac{1}{\pi}$. Consider the problem

$$
\begin{aligned}
-\left(\left(1+\sqrt{x}\right)\left|u'\right|^2 u'\right)' + \frac{1}{2+x}\left|u\right|^2 u &= f\left(x,\ u\right) \quad \text{in } [0,\ \pi]; \\
\Delta\left(\left|u'\left(x_1\right)\right|^2 u\left(x_1\right)\right) &= \sqrt[3]{u\left(x_1\right)}; \\
u(0) &= u(1) = 0,
\end{aligned} \tag{3.5}
$$

where $f\left(x,\ u\right) = -\sin\left(\frac{\pi}{4}\right) - c_2 u^3$, and $I_j\left(u\left(x_1\right)\right) = \sqrt[3]{u\left(x_1\right)}$, $M_0 = \max\left\{1,\ \frac{1}{2+\pi}\right\}$, $c_2 \in (0,\ 1)$. Clearly, $(H1)-(H3)$ hold true. By Theorem 1, (3.5) has only a weak solution.

4 Infinitely many weak solutions for (1.1)

Definition 4.1. (see [16, Definition 1.1]). Assume that X is a Banach space with norm $\|.\|$, we say that $\phi \in C^1\left(X, \mathbb{R}\right)$, satisfies Cerami condition (C), if for all $d \in \mathbb{R}$:

(i) any bounded sequence $\{u_n\} \subset X$ satisfying $\phi(u_n) \longrightarrow d$, $\phi'(u_n) \longrightarrow 0$, possesses a convergent subsequence;

(ii) there exist $\delta, \zeta, \tau > 0$ such that for any $u \in \phi^{-1}([d-\delta,\ d+\delta])$ with $\|u\| \geq \xi$, $\left\|\phi'(u)\right\| . \|u\| \geq \tau$.

As X is reflexive Banach space, there exist (see [28, Section 17]) $\{e_n\}_{n=1}^{\infty} \subset X$ and $\{f_n\}_{n=1}^{\infty} \subset X^*$ such that $f_n(e_m) = \delta_{n,\ m}$, $X = \overline{\text{span}}\{e_n : n = 1, 2, \ldots\}$ and $X^* = \overline{\text{span}}^{W^*}\{f_n : n = 1,\ 2, \ldots\}$. For $j,\ k \in \mathbb{N}$, denote $X_j := \text{span}\{e_j\}$, $Y_k := \oplus_{j=1}^{k} X_j$, and $Z_k := \overline{\oplus_{j=k}^{\infty} X_j}$. Clearly, $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for all $j \in \mathbb{N}$. Denote $S_\tau := \{u \in X : \ \|u\| = \tau\}$.

Lemma 4.1. *(see [16, Proposition 1.2]) Let X, Y_k, Z_k be defined as above. Assume that $\phi \in C^1(X, \mathbb{R})$ satisfies condition (C), and $\phi(-u) = \phi(u)$. for each $k \in \mathbb{N}$, there exist $\tau_k > r_k > 0$ such that*

(i) $b_k := \inf_{u\in Z_k\cap S_{r_k}} \phi(u) \longrightarrow +\infty$, $k \longrightarrow \infty$,
(ii) $\varsigma_k := \max_{u\in Y_k\cap S_{\tau_k}} \phi(u) \leq 0$.
Then ϕ has a sequence of critical points u_n, such that $\phi(u_n) \longrightarrow +\infty$ as $n \longrightarrow \infty$.

Lemma 4.2. *Let $(H3)-(H5)$ hold. Then ϕ satisfies Cerami condition (C).*

Proof. For all $d \in \mathbb{R}$, we assume that $\{u_n\}_{n=1}^{\infty} \subset X$ is bounded and

$$\phi(u_n) \longrightarrow d, \quad \phi'(u_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{4.1}$$

Going, if necessary, to a subsequence, we can assume that $u_n \rightharpoonup u$ weakly in X, and then

$$\begin{aligned}
\left\langle \phi'(u_n) - \phi'(u),\ u_n - u \right\rangle &= \int_0^T \rho(x)\left(\left|u_n'\right|^{p-2} u_n' - \left|u'\right|^{p-2} u'\right)\left(u_n' - u'\right) dx \\
&\quad + \int_0^T s(x)\left(|u_n|^{p-2} u_n - |u|^{p-2} u\right)(u_n - u)\, dx \\
&\quad + \sum_{j=1}^{n} \rho(x_j)\left(I_j(u_n(x_j)) - I_j(u(x_j))\right)(u_n(x_j) - u(x_j)) \\
&\quad - \int_0 (f(x,\ u_n) - f(x,\ u))(u_n - u)\, dx
\end{aligned} \tag{4.2}$$

by Lemma 2, X$\hookrightarrow\hookrightarrow C[0,\ T]$ enables us to obtain that

$$\begin{aligned}
\sum_{j=1}^{n} \rho(x_j)\left(I_j(u_n(x_j)) - I_j(u(x_j))\right)(u_n(x_j) - u(x_j)) &\longrightarrow 0, \\
\int_0 (f(x,\ u_n) - f(x,\ u))(u_n - u)\, dx &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

It follows from $u_n \rightharpoonup u$ weakly in X and $\left\langle \phi^{'}(u_n) - \phi^{'}(u),\ u_n - u \right\rangle \longrightarrow 0$ that

$$\int_0^T \rho(x)\left(\left|u_n^{'}\right|^{p-2} u_n^{'} - \left|u^{'}\right|^{p-2} u^{'}\right)\left(u_n^{'} - u^{'}\right) dx + \int_0^T s(x)\left(|u_n|^{p-2} u_n - |u|^{p-2} u\right)(u_n - u)\, dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{4.3}$$

Note that (3.3), we have

$$\begin{aligned}\left(\|u_n\|^{p-1} - \|u\|^{p-1}\right)(\|u_n\| - \|u\|) \leq & \int_0^T \rho(x)\left(\left|u_n^{'}\right|^{p-2} u_n^{'} - \left|u^{'}\right|^{p-2} u^{'}\right)\left(u_n^{'} - u^{'}\right) dx \\ & + \int_0^T s(x)\left(|u_n|^{p-2} u_n - |u|^{p-2} u\right)(u_n - u)\end{aligned} \tag{4.4}$$

and thus $\|u_n - u\| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, condition (i) of Definition 3 holds. Next, we prove condition (ii) of Definition 3, if not, there exists a sequence $\{u_n\} \subset X$ such that

$$\phi(u_n) \longrightarrow d, \quad \left\|\phi^{'}(u_n)\right\| . \|u_n\| \longrightarrow 0, \quad n \longrightarrow \infty, \tag{4.5}$$

and

$$\|u_n\| \longrightarrow \infty, \quad n \longrightarrow \infty. \tag{4.6}$$

By (4.6), there exists a constant $\varepsilon_1 > 0$ such that

$$\phi(u_n) - \frac{1}{p}\phi^{'}(u_n)\, u_n \leq \varepsilon_1. \tag{4.7}$$

On the other hand, $(H4)$ implies that there is a $L > 0$ such that $-pF(x,\ u) + f(x,\ u)\, u \geq a\,|u|$, $\forall\,|u| > L$ and $x \in [0,\ T]$. Furthermore, $-pF(x,\ u) + f(x,\ u)\, u$ is bounded for $|u| \leq L$ and $x \in [0,\ T]$.Therefore, there exists $c > 0$ such that -$F(x,\ u) + \frac{1}{p} f(x,\ u)\, u \geq \frac{a}{p}|u| - c$, $\forall u \in \mathbb{R}$, $x \in [0,\ T]$. This, together with $(H5)$, yields

$$\begin{aligned}\phi\left(u_n\right)-\frac{1}{p}\phi^{'}\left(u_n\right)u_n &= \sum_{j=1}^{n}\rho\left(x_j\right)\int\limits_0^{u_n(x_j)} I_j(x)dx-\frac{1}{p}I_j\left(u_n\left(x_j\right)\right)u_n\left(x_j\right)\\ &\quad+\int\limits_0^T\left(-F\left(x,\ u_n\right)+\frac{1}{p}f\left(x,\ u_n\right)u_n\right)dx\\ &\geq \int\limits_0^T\left(-F\left(x,\ u_n\right)+\frac{1}{p}f\left(x,\ u_n\right)u_n\right)dx\\ &\geq \int\limits_0^T\left(\frac{a}{p}\left|u_n\right|-c\right)dx,\end{aligned}\tag{4.8}$$

which implies $\int\limits_0^T \left|u_n\right| dx \leq \frac{p}{a}\left(Tc+\varepsilon_1\right)$. Therefore, there is a $\varepsilon_2>0$ such that $\left\|u_n\right\|_\infty \leq \varepsilon_2$.

It follows from $(H3)$ that there are $c_3,\ c_4>0$ such that

$$F\left(x,\ u\right)\leq c_3\left|u\right|+c_4\left|u\right|^p \quad \forall u\in\mathbb{R},\ \ x\in[0,\ T]\,.\tag{4.9}$$

By this and $(H5)$, we can find

$$\begin{aligned}\phi\left(u_n\right) &= \frac{1}{p}\left\|u_n\right\|^p+\sum_{j=1}^{n}\rho\left(x_j\right)\int\limits_0^{u_n(x_j)} I_j(x)dx-\int\limits_0^T F\left(x,\ u_n\right)dx\\ &\geq \frac{1}{p}\left\|u_n\right\|^p-\int\limits_0^T\left(c_3\left|u_n\right|+c_4\left|u_n\right|^p\right)dx\\ &\geq \frac{1}{p}\left\|u_n\right\|^p-c_3T\left\|u_n\right\|_\infty-c_4T\left\|u_n\right\|_\infty^p\\ &\geq \frac{1}{p}\left\|u_n\right\|^p-c_3T\varepsilon_2-c_4T\varepsilon_2^p,\end{aligned}\tag{4.10}$$

and thus $\phi\left(u_n\right)\longrightarrow\infty$ if (4.7) holds, which contradicts $\phi\left(u_n\right)\longrightarrow d$ in (4.6) . this proves that ϕ satisfies condition (C). □

Theorem 4.3. *Suppose* $(H2)-(H7)$ *hold, then* (1.1) *has infinitely many weak solutions.*

Proof. By $(H7)$, we know that ϕ is even. Denote $\eta_k=\sup_{u\in Z_k\cap S_1}\left\|u\right\|_p$, by the compactness of the embedding $X\hookrightarrow\hookrightarrow L^p\left([0,\ T]\right)$, we know that $\eta_k\longrightarrow 0$ as $k\longrightarrow\infty$ (see [22, Lemma 3.8]) . Note that (4.10) we have by $(H5)$ and Hölder inequality, for any $u\in Z_k$ and $\|u\|=r_k:=\eta_k^{-1}$,

$$
\begin{aligned}
\phi(u) &\geq \frac{1}{p}\|u\|^p - \int_0^T \left(c_3|u| + c_4|u|^p\right)dx \\
&\geq \frac{1}{p}\|u\|^p - c_3 T^{\frac{p-1}{p}}\|u\|_p - c_4\|u\|_p^p \\
&\geq \frac{1}{p}\|u\|^p - c_3 T^{\frac{p-1}{p}}\eta_k\|u\| - c_4\eta_k^p\|u\|^p \\
&\geq \frac{\eta_k^{-p}}{p} - c_3 T^{\frac{p-1}{p}} - c_4.
\end{aligned}
\tag{4.11}
$$

We easily have $r_k \longrightarrow \infty$ as $k \longrightarrow \infty$, then we have

$$
\phi(u) \geq \frac{\eta_k^{-p}}{p} - c_3 T^{\frac{p-1}{p}} - c_4 \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{4.12}
$$

Hence,

$$
b_k := \inf_{u\in Z_k,\ \|u\|=r_k} \phi(u) \longrightarrow \infty \text{ as } k \longrightarrow \infty. \tag{4.13}
$$

On the other hand, by $(H6)$, we find that there are b, $c > 0$ such that

$$
F(x,\ u) \geq b|u|^p - c,\ \forall u \in \mathbb{R},\ x \in [0,\ T]. \tag{4.14}
$$

Since all the norms of a finite dimensional normed space are equivalent, note that $\|.\|_p$ is a norm of Y_k, so there exists a $\zeta > 0$ such that

$$
\|u\|_p^p \geq \zeta\|u\|^p, \quad \forall u \in Y_k. \tag{4.15}
$$

Noting (2.6), we have

$$
\begin{aligned}
\phi(u) &= \frac{1}{p}\|u\|^p + \sum_{j=1}^n \rho(x_j)\int_0^{u(x_j)} I_j(x)dx - \int_0^T F(x,\ u)\,dx \\
&\leq \frac{1}{p}\|u\|^p + \sum_{j=1}^n \rho(x_j)\int_0^{u(x_j)} \left(\alpha_j + \beta_j|u|^{\gamma_j-1}\right)dx - \int_0^T \left(b|u|^p - c\right)dx \\
&\leq \frac{1}{p}\|u\|^p + \sum_{j=1}^n \rho(x_j)\left(\alpha_j|u(x_j)| + \frac{\beta_j}{\gamma_j}|u(x_j)|^{\gamma_j}\right) - b\|u\|_p^p + cT \\
&\leq \frac{1}{p}\|u\|^p - b\zeta\|u\|^p + \sum_{j=1}^n \|\rho\|_\infty\left(\alpha_j\|u\|_\infty + \frac{\beta_j}{\gamma_j}\|u\|_\infty^{\gamma_j}\right) + cT \\
&\leq \left(\frac{1}{p} - b\zeta\right)\|u\|^p + \sum_{j=1}^n \rho_1\left(\alpha_j\left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{1}{p}}\|u\| + \frac{\beta_j}{\gamma_j}\left(\frac{T^{p-1}}{\rho_0}\right)^{\frac{\gamma_j}{p}}\|u\|^{\gamma_j}\right) + cT
\end{aligned}
\tag{4.16}
$$

Note that we can choose a large enough b such that $\frac{1}{p} - b\zeta < 0$ by $(H6)$ and $p > \gamma_j$ by $(H2)$, and then there exists positive constants d_k such that

$$\phi(u) \leq 0, \quad \text{for each } u \in Y_k, \ \|u\| \geq d_k. \tag{4.17}$$

By this and (4.14), we can take $\tau_k := \max\{d_k, \ r_k + 1\}$, and thus $\varsigma_k := \max_{u \in Y_k, \ \|u\| = \tau_k} \phi(u) \leq 0$.Up until now, we have proved the functional ϕ satisfies all the conditions

of Lemma 4, then ϕ has infinitely many solutions. Equivalently, (1.1) has infinitely many weak solutions. This completes the proof. □

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