Lie symmetry analysis and conservation laws of ZDE

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Abstract. In this paper, we get the set of symmetry of Zoomeron Differential Equation (ZDE). Using Lie symmetry method the classical symmetry operators are obtained. Also, we will find the one-dimensional optimal system of the ZDE. Furthermore, the reduction Lie invariants corresponding to infinitesimal symmetries are obtained. Along them we will study the conservation law for ZDE.

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1 Introduction

One of Sophus Lie's most important discoveries in the case of differential equations is that he was able to show it is possible to locally transform the complex non-linear condition in one system of equation by infinitesimal invariants corresponding to the symmetry group generator of the system to solvable linear condition [5]. This task is of utmost important in physics. In this article, our aim is to obtain a set of symmetries of ZDE [4, 13]:

$$\text{ZDE} \ : \ \left(\frac{u_{xt}}{u}\right)_{tt} - \left(\frac{u_{xt}}{u}\right)_{xx} + 2(u^2)_{xt} = 0.$$

The classic Lie symmetries are obtain using the Lie symmetry method. This requires the utilization of computer softwares because working with continuous groups has computations that follow from the algorithmic process. Having the symmetry group of a system of equations has a lot of advantages one of which would be the classification of the solutions of the system. This classification is in this way that we consider both of the solutions in one class on the condition that they can be converted to each other by one element of the symmetry group. If we work with an ordinary system of equation, the symmetry group will help us to obtain the exact solutions by integrating once through the reduction of the order of the equation to one. And if the given equation is of order one type, it is also possible to obtain its general solution, but such a thing is not the case for the PDE (partial differential equation); that is, it is not possible to obtain the general solution of one PDE necessarily by having the symmetry group unless in a case that the system is convertible to a linear system. Also, in this condition, the solutions that are invariant whit respect to some of the subgroups

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of the symmetry group, are obtained. Another application of the symmetry group is that the symmetry group for one PDE is probably able to reduce the number of independent variables and in the ideal condition convert to one ODE (ordinary differential equation). Also, another application of symmetry group is to calculate the conservation laws in physics. In a theorem, Noether showed how symmetry groups lead to the production of the conservation laws for Euler-Lagrange equation. For instance, the conservation laws of energy are a matter of invariance under the motion symmetry in relation to time, whereas the conservation law of angular and linear movement is a matter of invariance under the transformations of movement and circulation.

2 Method of Lie Symmetry of ZDE

In this part of the article, a general method for the determination of the symmetries of a system of PDE has been given based on [8] and [3]. Let's suppose that in the general case we have a non-linear PDE system:

$$\Xi_\nu(x, u^{(n)}) = 0, \quad \nu = 1, ..., l, \tag{2.1}$$

that has l equations of the order of n, each of which involving p independent variables and q dependent variables. In it $x = (x^1, ..., x^p)$, $u = (u^1, ..., u^q)$ and involving derivation of u in relation to x to the order of n which we show with $u^{(n)}$. Now, let's suppose that we have a one-parametric Lie group of infinitesimal transformations that act on independent and dependent variables $(x, t, u) \in M = J^0_{x,t,u} \cong \mathbb{R}^3$ as follows:

$$(\tilde{x}, \tilde{t}, \tilde{u}) = (x, t, u) + s(\xi^1, \xi^2, \phi)(x, t, u) + O(s^2). \tag{2.2}$$

In it s is the group parameter and ξ^1, ξ^2, and ϕ are the infinitesimals parts of transformations. To calculate the Lie symmetry group for ZDE, let's suppose in the general case $\mathbf{v} = \xi^1(x, t, u)\,\partial_x + \xi^2(x, t, u)\,\partial_t + \phi\,(x, t, u)\,\partial_u$, is the same infinitesimal transformation groups. Now, we prolong the vector field $\mathbf{v}$ to order four using the following formula:

$$\mathrm{Pr}^{(4)}\mathbf{v} = \mathbf{v} + \phi^x\,\partial_{u_x} + \phi^t\,\partial_{u_t} + \phi^{xx}\,\partial_{u_{xx}} + \cdots + \phi^{tttt}\,\partial_{u_{tttt}}, \tag{2.3}$$

with coefficients

$$\phi^J = D_J Q + \sum_{i=1}^{2} \xi^i\, u_{J,i}, \tag{2.4}$$

in which $Q = \phi - \sum_{i=1}^{2} \xi^i\, u_i^\alpha$ and $J = (j_1, \cdots, j_k)$, $1 \le j_k \le 4$, $1 \le k \le 4$ and the sum is all over J's of order $0 < \#J \le n$ and $u_i^\alpha := \partial u^\alpha / x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / x^i$(Theorem (2.36) in [8]). The invariant condition, based on the theorem (6.5) from the [9], for the ZDE is the system consisting of $\mathrm{Pr}^{(4)}\mathbf{v}\,(\Delta) = 0$ and ZDE itself, where Δ is the left hand side of ZDE. The solution of which yields the system of PDE from the functions ξ^1, ξ^2, and ϕ. In it, the ZDE is a manifold in the jet space $J^4_{x,t;u} \cong \mathbb{R}^{17}$ and $\mathrm{Pr}^{(4)}\mathbf{v}$ is prolongation to the order four from $\mathbf{v}$. As a result, we have the PDE system:

$$\begin{array}{lllll} \xi^1_x = -\phi/u, & \xi^2_x = 0, & \xi^1_t = 0, & \phi_x = 0, & \phi_t = 0, \\ \xi^2_t = -\phi/u, & \phi_u = \phi/u, & \xi^2_u = 0, & \xi^1_u = 0. & \end{array} \tag{2.5}$$

We will have the following theorem by solving the system of above PDE's.

Theorem 2.1. *The Lie group of point symmetries of the ZDE has a Lie algebra generator in the form of the vector field* $\mathbf{v}$ *with the following functional coefficients.*

$$\xi^1(x,t,u) = c_1 x + c_3, \quad \xi^2(x,t,u) = c_1 t + c_2, \quad \phi(x,t,u) = -c_1 u. \tag{2.6}$$

In it, the constant amounts c_i, $i = 1,2,3$ *are arbitrary.*

Theorem 2.2. *The infinitesimal generators from the Lie one-parameter group of the symmetries of the ZDE are* $\mathbf{v}_1 = \partial_x$, $\mathbf{v}_2 = \partial_t$, $\mathbf{v}_3 = x\,\partial_x + t\,\partial_t - u\,\partial_u$. *These vector fields produce one Lie algebra space* $\mathcal{G}$ *with the following commutator table:*

[,]	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	$\mathbf{v}_2$
$\mathbf{v}_3$	- $\mathbf{v}_1$	- $\mathbf{v}_2$	0

3 Group Invariant Solutions of ZDE

To obtain the group of transformations which are generated by infinitesimal generators $\mathbf{v}_i$ for $i = 1,2,3$, we should solve the first order system involving first order equations in correspondence to each of the generators simultaneously.

By solving this system, the one parameter group of $g_k(s) : M \to M$ generated by $\mathbf{v}_i$ for $i = 1,2,3$ involved in theorem (2) is obtained in the following way;

$$\begin{array}{rcccl} g_1 & : & (x,t,u) & \longmapsto & (x - s, t, u), \\ g_2 & : & (x,t,u) & \longmapsto & (x, t - s, u), \\ g_3 & : & (x,t,u) & \longmapsto & (xe^{-s}, te^{-s}, ue^{s}). \end{array} \tag{3.1}$$

therefore, we will have the following theorem:

Theorem 3.1. *If* $u = f(x,t)$ *is one solution of ZDE, then the following functions that have been produced through acting* g_s^k *on* $u = f(x,t)$ *will also be the solution of ZDE.*

$$g_s^1 f(x,t) = f(x - s, t), \quad g_s^2 f(x,t) = f(x, t - s), \quad g_s^3 f(x,t) = f(xe^{-s}, te^{-s})e^{-s}.$$

4 Optimal System of One-Dimensional Subalgebras of ZDE

In this part of the article, we want to obtain the one-dimensional optimal system of the ZDE using its symmetry group. The optimal system is in fact a standard method for the classification of one-dimensional sub-algebras in which each class involves conjugate equivalent members [10]. Also, they involve the group adjoint representation which establishes an equivalent relation among all conjugate sub-algebra elements. In fact, the classification problem for one-dimensional sub-algebra is the same as the problem of the classification of the representation of its adjoint orbits. In this way, the optimal system is constructed. The set of invariant solutions corresponding to a one-dimensional sub-algebra is a list of minimal solutions by which all the other invariant solutions can be obtained by the use of transformations [6]. To calculate the adjoint representation, we consider the following Lie series

$$\mathrm{Ad}(\exp(\mathrm{s}\mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - \mathrm{s}\ \mathrm{ad}_{\mathbf{v}_j}\mathbf{v}_j + \frac{s^2}{2}\mathrm{ad}^2_{\mathbf{v}_j}\mathbf{v}_j - \cdots, \tag{4.1}$$

for the favorite vector fields $\mathbf{v}_i$, $\mathbf{v}_j$ in which $\mathrm{ad}_{\mathbf{v}_j}\mathbf{v}_j = [\mathbf{v}_i, \mathbf{v}_j]$ is the Lie algebra communicator and s is the group parameter; and $i, j = 1, 2, 3$ ([8],page 199). Therefore,we will have the following table.

$\mathrm{Ad}(\exp(\mathrm{s})\mathbf{v}_i)\mathbf{v}_j)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - s\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - s\mathbf{v}_2$
$\mathbf{v}_3$	$e^s\mathbf{v}_1$	$e^s\mathbf{v}_2$	$\mathbf{v}_3$

First, we consider a favorite member (an optional member) from $\mathcal{G}$ in the form of

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3, \tag{4.2}$$

and for the simplicity of calculations, each $\mathcal{G}$ element in the form of (4.2) can be corresponded to a vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$; therefor, the adjoint action can be considered the same as a type of linear transformation group of vectors, so we can have the following theorem:

Theorem 4.1. *The one-dimensional optimal system of Lie algebra $\mathcal{G}$ for the ZDE is (i): Scaling: $\mathbf{v}_3$, and wave traveling solutons: (ii) $\mathbf{v}_1 - c\mathbf{v}_2$, where $c \in \mathbb{R}$ is arbitrary constant.*

Proof: We define the map $F_i^s : \mathcal{G} \to \mathcal{G}$ by $\mathbf{v} \mapsto Ad(exp(s\mathbf{v}_i)\mathbf{v})$ as a linear map, for $i = 1, 2, 3$. So the matrices M_i^s corresponding to each of the F_i^s, $i = 1, 2, 3$, in relation to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be as follows:

$$M_1^s = \mathtt{I}_3 - s\mathtt{E}_{13}, \quad M_2^s = \mathtt{I}_3 - s\mathtt{E}_{23}, \quad M_3^s = e^s(\mathtt{E}_{11} + \mathtt{E}_{22}) + \mathtt{E}_{33}.$$

In it, $\mathtt{E}_{ij}$s are 3×3-elementary matrixes, for $i, j = 1, 2, 3$; on the condition, that the $(i; j)$-entry of $\mathtt{E}_{ij}$ is 1, and others are zero. Suppose $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$, in this case, we will have the map combinations as follows:

$$F_3^s \circ F_2^s \circ F_1^s : \mathbf{v} \mapsto [e^s a_1 - sa_3]\mathbf{v}_1 + [e^s a_2 - sa_3]\mathbf{v}_2 + a_3\mathbf{v}_3.$$

We can simplify the $\mathbf{v}$ as follows: If $a_3 \neq 0$ then we can vanish the coefficient of $\mathbf{v}_1$ and $\mathbf{v}_2$ using F_1^s, and F_2^s by substitution $s = \frac{a_1}{a_3}$, and $s = \frac{a_2}{a_3}$. And if necessary, we can suppose $a_3 = 1$ through the Scaling of $\mathbf{v}$. In this case $\mathbf{v}$ is reduced to form (i), and if $a_3 = 0$, then $\mathbf{v}$ is reduced to form (ii).

5 Similarity Reduction of ZDE

The ZDE has been stated with the $(x, t; u)$ coordinate, but we are looking for a new coordinate that the equation will reduce if we write it in new coordinate.This new coordinate is obtained through $(y; v)$ dependent invariant corresponding to the infinitesimal symmetry generator. If we state the ZDE with the new coordinate, using the chain rule, a reduced equation will result.Now we calculate the invariants corresponding to the symmetry generators existing in the optimal system. The first status for the first element of the optimal system is $\mathbf{v}_3$. It has the determining equation in the form: $dx/x = dt/t = -du/u$. Solving this equation will result in the two invariants of $y = x/t$, $v = tu$. Now, if we consider $u(x,t) = v(y)/t$ as a function of $y = x/t$, we can state the derivatives of u with respect to x and t, in the form of v and y and the derivatives of v with respect to y, and substituting it in the ZDE, turns the ZDE to an ordinary equation one as follows:

$$\begin{aligned}
&-(2y(y^2-1)v' + 2v(y^2+2) + yv)vv''' - y(y^2-1)vv''^2\\
&\quad +(2y(y^2-1)v'^2 + 2(4-7y^2)vv' - 4v^2y(v^2+3))v''\\
&\quad +4v'((y^2-1)v'^2 - yv(v^2+3)v' - 3v^2(v^2+1) = 0.
\end{aligned}$$

The second status for the second element of the optimal system is $\mathbf{v}_1 - c\mathbf{v}_2$. It has the determining equation in the form $dx/1 = dt/1 = du/0$. Solving this equation will result in the two invariants of $y = cx + t$, $v = u$. Now, if we consider $u(x,t) = v(y)$ as a function of $y = cx + t$, we can state the derivatives of u with respect to x and t, in the form of v and y and the derivatives of v with respect to y, and substituting it in the ZDE, turns the ZDE to an ordinary equation $-2vv'v''' - vv''^2 + 2v'^2v'' + v''''v^2 - 2c^2v'^2v'' + 2c^2vv'v''' + c^2vv''^2 + 4v^3v'^2 + 4v^4v'' = 0$. Let $c = 1$, then we have the following ODE $v''' + 4v(v'^2 + vv'') = 0$ and $v = 0$.

6 Characterization of differential invariants

Let's suppose G that acts on the manifold $M \subset X \times U$ is a local group of transformation. A differential invariant of order n from group G is defined as a smooth function having the form, $I : J^n_{x,t,u} \longrightarrow \mathbb{R}$. It is dependent on x, u and derivatives of u up to order n. If I is a differential invariant of order n, then $I(\mathrm{Pr}^{(n)}g.(x, u^{(n)}) = I(x, u^{(n)})$ for $(x, u^{(n)}) \in J^n_{x,t,u}$ and $g \in G$, [8]. To obtain the differential invariant of the ZDE, up to order two, we solve the system $I_x = 0$, $I_t = 0$, $-uI_u = 0$, where, I is a smooth function of (x,t,u). And

$$(I_1)_x = 0, \quad (I_1)_t = 0, \quad u(I_1)_u + 2u_x(I_1)_{u_x} + 2u_t(I_1)_{u_t} = 0, \tag{6.1}$$

where I_1 is a smooth function of (x,t,u,u_x,u_t),

$$(I_2)_x = 0, \quad (I_2)_t = 0, \quad u(I_2)_u + \cdots + 3u_{xt}(I_2)_{u_{xt}} + 3u_{tt}(I_2)_{u_{tt}} = 0, \tag{6.2}$$

where I_2 is a smooth function of $(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt})$. The solution of these systems are listed in order in table:

Vector field	Ordinary invariant	1st order	2nd order
$\mathbf{X}_1$	t, u	$*$, u_x, u_t	$*$, $**$, u_{xx}, u_{xt}, u_{tt}
$\mathbf{X}_2$	x, u	$*$, u_x, u_t	$*$, $**$, u_{xx}, u_{xt}, u_{tt}
$\mathbf{X}_3$	t/x, xu	$*$, x^2u_x, x^2u_t	$*$, $**$, x^3u_{xx}, x^3u_{xt}, x^3u_{tt}

In it, $*$ and $**$ refer back to ordinary invariants and order two of the columns before them respectively.

7 Conservation laws for ZDE

There are various methods to calculate the conservation laws, some of these methods are: Noether's basic method, multiplier method, direct method, etc.[8, 2, 1, 11, 7]. In this part of the article, we will obtain the conservation laws of the ZDE using the multiplier method. A conservation law for system (2.1) is defined as a divergence expression: $D_i\Phi^i[u] = D_1\Phi^1[u] + \cdots + D_n\Phi^n[u]$, that is true for all of the solutions of system (2.1). In it, $\Phi^i[u] = \Phi^i(x, u^{(r)})$, $i = 1, \cdots, n$, are called the fluxes of the conservation law, and the highest-order derivative (r) in the fluxes statement $\Phi^i[u]$ is called the order of the conservation law [2].

A set of multipliers $\{\Lambda_\nu[U]\}_{\nu=1}^l = \{\Lambda_\nu(x, U^{(r)})\}_{\nu=1}^l$ results in the existence of the divergence expression for the system $\Xi_\nu[u] = \Xi\nu(x, u^{(n)})$, on the condition that if the identity $\Lambda_\nu[U]\Xi_\nu[U] \equiv D_i\Phi^i[U]$ holds for the arbitrary functions $U(x)$. Then there will be one conservation law as $\Lambda_\nu[u]\Xi_\nu[u] = D_i\Phi^i[u] = 0$ for the solution of system (2.1), that is written as $U(x) = u(x)$; providing $\Lambda_\nu[U]$ is non-singular. The Euler operator in respect to U^j is defined as $E_{U^j} = \partial_{U^j} - D_j\partial_{U^j} + \cdots + (-1)^s D_{i_1} \cdots D_{i_n}\partial_{U^j_{i_1\cdots i_s}} + \cdots$, for $j = 1, \cdots, q$ [2]. The equations $E_{U^j}F(x, U^{(s)}) \equiv 0, \quad j = 1, \cdots, q$ hold for every function $U(x)$ if and only if the relation $F(x, U^{(s)}) \equiv D_i\Psi^i(x, U^{(s-1)})$ holds for some of functions $\Psi^i(x, U^{(s-1)}), \quad i = 1, \cdots, q$ (Theorem 1.3.2, [2]).

A set of non-singular local multipliers $\{\Lambda_\mu(x, U^{(r)})\}_{\nu=1}^l$ results in the production of the locally conservation law for the system $\Xi_\nu(x, u^{(n)})$ if and only if the set of identities

$$E_{U^j}(\Lambda_\nu(x, U^{(r)})\Xi_\nu(x, u^{(n)})) \equiv 0, \qquad j = 1, \cdots, q, \tag{7.1}$$

hold for every optional function $U(x)$ (Theorem 1.3.3, [2]). The set of equations (7.1) results in the linear determining equations, from the solution of which a set of locally conservation law multipliers for system $\Xi_\nu(x, u^{(n)})$ is produced. Now, we want to obtain the local multipliers of the conservation law in the form $\Lambda = \xi(x, t, u)$ for the ZDE. The determining equations (7.1) for the ZDE is as follows

$$E_U[\xi(x, t, U)\Delta] \equiv 0, \tag{7.2}$$

where Δ is the left hand side of ZDE and $U(x, t)$ is an arbitrary function.

The calculation of equation (7.2) yields the PDE system. Through solving the determining equation produced from (7.2), we will have the following solution $\xi = c_1x + c_3t + c_4(t^2 + x^2) + c_2$, where $c_1, \cdots, c_4$ are arbitrary constants, so local multipliers are obtained as (i) $\xi = 1$, (ii) $\xi = x$, (iii) : $\xi = t$, (iv) $\xi = (t^2 + x^2)/2$. Each of the local multipliers ξ determine a non-trivial local conservation law $D_t\Psi + D_x\Phi = 0$ with a determining form of $D_t\Psi + D_x\Phi = \xi(x, t, U)(\Delta)$. To calculate the $\Psi$and Φ we should invert the operator and this involves getting multi-dimensions integral from the statement involving the optional function and its derivatives and this is practically difficult in direct manner. Here, we use the homotopy operators to achieve this end [12]. The homotpy operator is powerful algorithmic device originated from homological algebra and variational bi-complexes. The two-dimensional homotpy operator is a vector operator with two components in the form of $(\mathcal{H}_u^x f, \mathcal{H}_u^t f)$, defined in the following form

$$\text{(7.3)}\qquad \mathcal{H}^x_u f = \int_0^1 \frac{1}{\lambda}\Big(\sum_{j=1}^{q} \mathcal{I}^x_{u^j} f\Big)[\lambda u]\, d\lambda, \qquad \mathcal{H}^t_u f = \int_0^1 \frac{1}{\lambda}\Big(\sum_{j=1}^{q} \mathcal{I}^t_{u^j} f\Big)[\lambda u]\, d\lambda.$$

Which $\mathcal{I}^x_{u^j} f$, is obtained in the following way by getting integral from it

$$\text{(7.4)}\qquad \mathcal{I}^x_{u^j} f = \sum_{k_1=1}^{M_1^j}\sum_{k_2=0}^{M_2^j}\Big(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2} B^x u^j_{x^{i_1}t^{i_2}}(-D_x)^{k_1-i_1-1}(-D_t)^{k_2-i_2}\Big)\frac{\partial f}{\partial u^j_{x^{k_1}t^{k_2}}}$$

In it M_1^j, M_2^j are the order of f in u^j to x and t respectively, which in ZDE, $j = 1$, $M_1^j = M_2^j = 3$, and combinatorial coefficient $B^x = B(i_1, i_2, k_1, k_2) = \binom{i_1+i_2}{i_1}.\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}/\binom{k_1+k_2}{k_1}$. Similarly, the t-integrand, $\mathcal{I}^t_{u^j} f$, is defined as

$$\mathcal{I}^t_{u^j} f = \sum_{k_1=0}^{M_1^j}\sum_{k_2=1}^{M_2^j}\left(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1} B^t u^j_{x^{i_1}t^{i_2}}(-D_x)^{k_1-i_1}(-D_t)^{k_2-i_2-1}\right)\frac{\partial f}{\partial u^j_{x^{k_1}t^{k_2}}},$$

where $B^t = B(i_2, i_1, k_2, k_1)$. We apply homotopy operator to find conserved quantities Ψ and Φ which yield of multiplier $\xi = 1$. Now f is the left hand side of ZDE. The integrands $\mathcal{I}^x_{u^j} f$ and $\mathcal{I}^t_{u^j} f$ are

$$\text{(7.5)}\qquad \begin{aligned} &\frac{1}{12}\Big(2u^2(u_{xxt}+3u_{ttt}) + 12uu_{tt}(u_t-1) + 9uu_xu_{xt} \\ &\qquad +12u^4u_t - 8u_t^3(2u_t+5) + 57u_tu_x^2\Big), \\ &\frac{1}{12}\Big(48u_xu_t + 48u^4u_x + uu_{xt}(48-13u_t) - 62u_xu_t^2 \\ &\qquad -4u^2u_{xtt} + 4uu_xu_{tt} - 24uu_xu_{xx} + 72u_x^3\Big). \end{aligned}$$

Apply (7.4) to the integrands (7.5), therefore $\Psi := \mathcal{H}^x_{u^j} f$ is

$$\frac{1}{9}u^2(3u_{xxt}+u_{ttt}) - \frac{2}{9}u_t^2(5u_t-9) + \frac{19}{6}u_tu_x^2 + \frac{1}{5}u^4u_t + \frac{1}{6}uu_{tt}(2u_t-3) + \frac{3}{4}uu_xu_{xt},$$

and $\Phi := \mathcal{H}^t_{u^j} f$ is

$$-\frac{1}{9}u^2u_{xtt} - \frac{13}{36}uu_{xt}(u_t-132) + \frac{1}{9}uu_x(u_{tt}-6u_{xx}) - \frac{31}{18}u_xu_t(u_t-36) + \frac{1}{5}u^4u_x + 2u_x^3.$$

So, we have the first conservation low of the ZDE respect to multiplier $\xi = 1$ leads

$$\begin{aligned} &D_x\Big(\frac{1}{9}u^2(3u_{xxt}+u_{ttt}) - \frac{2}{9}u_t^2(5u_t-9) + \frac{19}{6}u_tu_x^2 + \frac{1}{5}u^4u_t + \frac{1}{6}uu_{tt}(2u_t-3) \\ &\quad +\frac{3}{4}uu_xu_{xt}\Big) \ + \ D_t\Big(-\frac{1}{9}u^2u_{xtt} - \frac{13}{36}uu_{xt}(u_t-132) + \frac{1}{9}uu_x(u_{tt}-6u_{xx}) \\ &\quad -\frac{31}{18}u_xu_t(u_t-36) + \frac{1}{5}u^4u_x + 2u_x^3\Big) = 0. \end{aligned}$$

Now we find conservation law respect to multiplier $\xi = x$, in this cases we have:

$$D_x\Big(\frac{1}{5}xu_tu^4 - \frac{10}{6}xu_t^3 - \frac{15}{12}uu_xu_t + \frac{3}{4}xu_xu_{xt}u + \frac{1}{3}xuu_tu_tt + \frac{19}{6}xu_tu_x^2 + 2xu_t^2$$
$$-\frac{1}{2}xuu_{tt} + \frac{1}{3}xu_{xxt}u^2 + \frac{1}{9}xu_{ttt}u^2\Big) + D_t\Big(- \frac{1}{9}xu_{xtt}u^2 + \frac{1}{5}xu_xu^4 + 2xu_x^3$$
$$-\frac{1}{2}uu_t + \frac{1}{9}xuu_xu_{tt} - \frac{2}{3}xuu_xu_{xx} - \frac{31}{18}xu_xu_t^2 - \frac{1}{9}u_{tt}u^2 - uu_x^2 + \frac{19}{36}uu_t^2$$
$$-\frac{2}{5}u^5 + 2xu_tu_x - \frac{13}{36}xuu_tu_{xt} + \frac{1}{2}xuu_{xt}\Big) = 0.$$

And to multiplier $\xi = t$ we have:

$$D_x\Big(- \frac{2}{5}u^5 - \frac{1}{9}u_{tt}u^2 - \frac{11}{12}uu_x^2 + \frac{4}{9}uu_t^2 - \frac{1}{2}uu_t + 2tu_t^2 - \frac{1}{2}tuu_{tt} + \frac{1}{3}tu_{xxt}u^2$$
$$+\frac{1}{9}tu_{ttt}u^2 + \frac{1}{3}tu_tu^2 - \frac{10}{9}tu_t^3 + \frac{3}{4}tuu_xu_{xt} + \frac{1}{3}tuu_tu_{tt} + \frac{19}{6}tu_tu_x^2\Big)$$
$$+D_t\Big(- \frac{13}{36}tuu_tu_{xt} + \frac{2}{9}u_{xt}u^2 + \frac{1}{9}tuu_xu_{tt} - \frac{2}{3}tuu_xu_{xx} - \frac{31}{18}tu_xu_t^2$$
$$-\frac{1}{9}tu_{xtt}u^2 + \frac{1}{5}u_xu^4 + 2tu_x^3 - \frac{5}{36}uu_xu_t + \frac{1}{2}tuu_{xt} + 2tu_xu_t\Big) = 0.$$

And to multiplier $\xi = (t^2 + x^2)/2$ we have:

$$D_x\Big(- \frac{1}{2}tuu_t - \frac{1}{9}tu^2u_{tt} - \frac{11}{12}tuu_x^2 - \frac{1}{4}(t^2 + x^2)uu_{tt} + \frac{19}{12}(t^2 + x^2)u_tu_x^2 + \frac{2}{3}u_tu^2$$
$$+\frac{1}{10}(t^2 + x^2)u_tu^4 + \frac{1}{6}(t^2 + x^2)u_{xxt}u^2 + \frac{1}{18}(t^2 + x^2)u_{ttt}u^2 - \frac{2}{5}tu^5 + (t^2 + x^2)u_t^2$$
$$-\frac{5}{6}(t^2 + x^2)u_t^3 - \frac{5}{4}xuu_tu_x + \frac{1}{6}(t^2 + x^2)uu_tu_{tt} + \frac{3}{8}(t^2 + x^2)uu_xu_{xt} + \frac{4}{9}tuu_t^2\Big)$$
$$+D_t\Big(- \frac{2}{5}xu^5 + (t^2 + x^2)u_x^3 + \frac{2}{3}u_xu^2 - \frac{1}{9}xu^2u_{tt} - xuu_x^2 + \frac{1}{10}(t^2 + x^2)u_xu^4$$
$$-\frac{1}{18}(t^2 + x^2)u_{xtt}u^2 + \frac{19}{36}xuu_t^2 - \frac{31}{36}(t^2 + x^2)u_xu_t^2 - \frac{1}{2}xuu_t + (t^2 + x^2)u_tu_x$$
$$+\frac{1}{4}(t^2 + x^2)uu_{xt} + \frac{1}{18}(t^2 + x^2)uu_xu_{tt} - \frac{1}{3}(t^2 + x^2)uu_xu_{xx} + \frac{2}{9}tu_{xt}u^2$$
$$-\frac{13}{72}(t^2 + x^2)uu_tu_{xt} - \frac{5}{36}tuu_tu_x\Big) = 0.$$

Conclusion

In this paper we obtained the Lie point symmetries of the Zoomeron equation by using the Lie symmetry method. Also computed the one dimensional optimal system. This led to reducing the Zoomeron equation to ODE's and computing the invariants and conservation law of Zoomeron equation.

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