Insertion of a contra-continuous function between two comparable contra-precontinuous real-valued functions

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Abstract. A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that kernel of sets are open.

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1 Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [20], while the concept of a locally dense set was introduced by H.H. Corson and E. Michael [4].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [17]. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [19].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [23] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had

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focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [6] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 12, 13, 22].

Hence, a real-valued function f defined on a topological space X is called *contra-continuous* (resp. *contra-semi−continuous* , *contra-precontinuous*) if the preimage of every open subset of $\mathbb{R}$ is closed (resp. *semi*−closed , preclosed) in X[6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets or kernel of sets are open [19].

If g and f are real-valued functions defined on a space X, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a *cc−property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are $cc-$properties, the following terminology is used:(i) A space X has the *weak cc−insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$.(ii) A space X has the *cc−insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g < h < f$.(iii) A space X has the *weakly cc−insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 , f has property P_2 and $f - g$ has property P_2, then there exists a contra-continuous function h such that $g < h < f$.

In this paper, for a topological space whose $\Lambda-$sets or kernel of sets are open, is given a sufficient condition for the weak $cc-$insertion property. Also for a space with the weak $cc-$insertion property, we give a necessary and sufficient condition for the space to have the $cc-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2 The main result

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:

$A^\Lambda = \cap\{O : O \supseteq A, O \in (X, \tau)\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in (X, \tau)\}$.

In [7, 18, 21], A^Λ is called the *kernel* of A.

The family of all preopen, preclosed, *semi*−open and *semi*−closed will be denoted by $pO(X, \tau)$, $pC(X, \tau)$, $sO(X, \tau)$ and $sC(X, \tau)$, respectively.

We define the subsets $p(A^{\Lambda}), p(A^{V}), s(A^{\Lambda})$ and $s(A^{V})$ as follows:
$p(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in pO(X, \tau)\}$,
$p(A^{V}) = \cup\{F : F \subseteq A, F \in pC(X, \tau)\}$,
$s(A^{\Lambda}) = \cap\{O : O \supseteq A, O \in sO(X, \tau)\}$ and
$s(A^{V}) = \cup\{F : F \subseteq A, F \in sC(X, \tau)\}$.
$p(A^{\Lambda})$ (resp. $s(A^{\Lambda})$) is called the *prekernel* (resp. $semi-kernel$) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \; \bar{\rho} \; y$ if and only if $y \; \rho \; v$ implies $x \; \rho \; v$ and $u \; \rho \; x$ implies $u \; \rho \; y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i \; \rho \; B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \; \rho \; C$ and $C \; \rho \; B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.

2) If $A \subseteq B$, then $A \; \bar{\rho} \; B$.

3) If $A \; \rho \; B$, then $A^{\Lambda} \subseteq B$ and $A \subseteq B^{V}$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which kernel sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \; \rho \; A(g, t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f, t)$ and $A(g, t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f, t_1) \; \rho \; A(g, t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f, t)$ and $G(t) = A(g, t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1) \; \bar{\rho} \; F(t_2), G(t_1) \; \bar{\rho} \; G(t_2)$, and $F(t_1) \; \rho \; G(t_2)$. By Lemmas 1 and 2 of [15] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1) \; \rho \; H(t_2), H(t_1) \; \rho \; H(t_2)$ and $H(t_1) \; \rho \; G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g, t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f, t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1, t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1, t_2)$ is closed in X, i.e., h is a contra-continuous function on X. □

The above proof used the technique of theorem 1 in [14].

Theorem 2.2. Let P_1 and P_2 be $cc-$property and X be a space that satisfies the weak $cc-$insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2.The space X has the $cc-$insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by contra-continuous functions.

Proof. Assume that X has the weak $cc-$insertion property for (P_1, P_2). Let g and f be functions such that $g < f, g$ has property P_1 and f has property P_2. By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by contra-continuous functions. Let k_n be a contra-continuous function such that $k_n = 0$ on $A(f - g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the properties contra-continuous functions, the function k is a contra-continuous function. Since $\bigcap_{n=1}^{\infty} D_n = \varnothing$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f - g, 3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If x is any point in X, then $x \notin A(f - g, 1)$ or for some n,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are E-properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak $cc-$insertion property for (P_1, P_2), then there exists a contra-continuous function h such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that X satisfies the $cc-$insertion property for (P_1, P_2). (The technique of this proof is by Katětov[14]).

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a contra-continuous function h such that $g < h < f$. We follow an idea contained in Lane [16]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the $cc-$insertion property for (P_1, P_2), then there exists a contra-continuous function k such that $0 < k < f - h$. Let $A(f - g, 3^{-n+1})$ be any lower cut set for $f - g$ and let

$D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \varnothing$. Since

$$A(f-g, 3^{-n+1}) \subseteq \{x \in X : (f-g)(x) \leq 3^{-n+1}\} \subseteq \{x \in X : k(x) \leq 3^{-n+1}\}$$

and since $\{x \in X : k(x) \leq 3^{-n+1}\}$ and $\{x \in X : k(x) \geq 3^{-n+2}\} = X \setminus D_n$ are completely separated by contra-continuous functions $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$, it follows that for each $n, A(f-g, 3^{-n+1})$ and $X \setminus D_n$ are completely separated by contra-continuous functions. □

3 Applications

The abbreviations *cpc* and *csc* are used for contra-precontinuous and contra-$semi-$continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2, we suppose that X is a topological space whose kernel sets are open.

Corollary 3.1. If for each pair of disjoint preopen (resp. $semi-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $cc-$insertion property for (cpc, cpc) (resp. (csc, csc)).

Proof. Let g and f be real-valued functions defined on X, such that f and g are *cpc* (resp. *csc*), and $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $p(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $s(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preopen (resp. $semi-$open) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. $semi-$closed) set, it follows that $p(A(f,t_1)^{\Lambda}) \subseteq p(A(g,t_2)^{V})$ (resp. $s(A(f,t_1)^{\Lambda}) \subseteq s(A(g,t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. □

Corollary 3.2. If for each pair of disjoint preopen (resp. $semi-$open) sets G_1, G_2, there exist closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-precontinuous (resp. contra-$semi-$continuous) function is contra-continuous.

Proof. Let f be a real-valued contra-precontinuous (resp. contra-$semi-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-continuous function h such that $g = h = f$. □

Corollary 3.3. If for each pair of disjoint preopen (resp. $semi-$open) sets G_1, G_2 of X , there exist closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the $cc-$insertion property for (cpc, cpc) (resp. (csc, csc)).

Proof. Let g and f be real-valued functions defined on the X, such that f and g are *cpc* (resp. *csc*), and $g < f$. Set $h = (f+g)/2$, thus $g < h < f$, and by Corollary 3.2, since g and f are contra-continuous functions hence h is a contra-continuous function.

□

Corollary 3.4. If for each pair of disjoint subsets G_1, G_2 of X , such that G_1 is preopen and G_2 is *semi*−open, there exist closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $cc-$insertion property for (cpc, csc) and (csc, cpc).

Proof. Let g and f be real-valued functions defined on X, such that g is cpc (resp. csc) and f is csc (resp. cpc), with $g \leq f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $s(A^\Lambda) \subseteq p(B^V)$ (resp. $p(A^\Lambda) \subseteq s(B^V)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a *semi*−open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. *semi*−closed) set, it follows that $s(A(f,t_1)^\Lambda) \subseteq p(A(g,t_2)^V)$ (resp. $p(A(f,t_1)^\Lambda) \subseteq s(A(g,t_2)^V)$). Hence $t_1 < t_2$ implies that $A(f,t_1)\ \rho\ A(g,t_2)$. The proof follows from Theorem 2.1. □

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets G_1, G_2 of X, such that G_1 is preopen and G_2 is *semi*−open, there exist closed subsets F_1, F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$.

(ii) If G is a *semi*−open (resp. preopen) subset of X which is contained in a preclosed (resp. *semi*−closed) subset F of X, then there exists a closed subset H of X such that $G \subseteq H \subseteq H^\Lambda \subseteq F$.

Proof. (i) ⇒ (ii) Suppose that $G \subseteq F$, where G and F are *semi*−open (resp. preopen) and preclosed (resp. *semi*−closed) subsets of X, respectively. Hence, F^c is a preopen (resp. *semi*−open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is an open subset containing F_1, we conclude that $F_1^\Lambda \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^\Lambda \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) ⇒ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is preopen and G_2 is *semi*−open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of X. Hence by (ii) there exists a closed set H such that $G_2 \subseteq H \subseteq H^\Lambda \subseteq G_1^c$.
But

$$H \subseteq H^\Lambda \Rightarrow H \cap (H^\Lambda)^c = \varnothing$$

and

$$H^\Lambda \subseteq G_1^c \Rightarrow G_1 \subseteq (H^\Lambda)^c.$$

Furthermore, $(H^\Lambda)^c$ is a closed subset of X. Hence $G_2 \subseteq H, G_1 \subseteq (H^\Lambda)^c$ and $H \cap (H^\Lambda)^c = \varnothing$. This means that condition (i) holds. □

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is preopen and G_2 is *semi*−open, can be separated by closed subsets of X then there exists a contra-continuous function $h : X \to [0,1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is *semi*−open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a preclosed subset of X containing the *semi*−open subset G_2 of X,by Lemma 3.1, there exists a closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a preclosed subset of X and contains G_2, and G_1^c is a preclosed subset of X and contains the *semi*−open subset $H_{1/2}^\Lambda$ of X. Hence, by Lemma 3.1, there exists closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq H_{1/4}^\Lambda \subseteq H_{1/2} \subseteq H_{1/2}^\Lambda \subseteq H_{3/4} \subseteq H_{3/4}^\Lambda \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^\Lambda)^c : t > \alpha\}$ hence, every of them is a closed subset. Consequently h is a contra-continuous function. □

Lemma 3.3. Suppose that X is a topological space such that every two disjoint *semi*−open and preopen subsets of X can be separated by closed subsets of X. The following conditions are equivalent:

(i) Every countable convering of *semi*−closed (resp. preclosed) subsets of X has a refinement consisting of preclosed (resp. *semi*−closed) subsets of X such that for every $x \in X$, there exists a closed subset of X containing x such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{G_n\}$ of $semi-$open (resp. preopen) subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of preclosed (resp. $semi-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of $semi-$open (resp. preopen) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of $semi-$closed (resp. preclosed) subsets of X . By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a closed subset of X and $V_n^{\Lambda} \subseteq G_n^c$. By setting $F_n = (V_n^{\Lambda})^c$, we obtain a decreasing sequence of closed subsets of X with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $semi-$closed (resp. preclosed) subsets of X, we set for $n \in \mathbb{N}, G_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{G_n\}$ is a decreasing sequence of $semi-$open (resp. preopen) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of preclosed (resp. $semi-$closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$.Now we define the subsets W_n of X in the following manner:

W_1 is a closed subset of X such that $F_1^c \subseteq W_1$ and $W_1^{\Lambda} \cap G_1 = \varnothing$.

W_2 is a closed subset of X such that $W_1^{\Lambda} \cup F_2^c \subseteq W_2$ and $W_2^{\Lambda} \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of closed sets. Moreover, we have

(i) $W_n^{\Lambda} \subseteq W_{n+1}$

(ii) $F_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus W_{n-1}^{\Lambda}$.

Then since $W_{n-1}^{\Lambda} \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of closed sets and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i-j| \leq 1$. Finally, consider the following sets:

$$
\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \cdots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}
$$

These sets are closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are closed sets, and for every point in X we can find a closed set containing the point that intersects only finitely many elements of that refinement. □

Corollary 3.5. If every two disjoint *semi*−open and preopen subsets of X can be separated by closed subsets of X, and in addition, every countable covering of *semi*−closed (resp. preclosed) subsets of X has a refinement that consists of preclosed (resp. *semi*−closed) subsets of X such that for every point of X we can find a closed subset containing that point such that it intersects only a finite number of refining members then X has the weakly $cc-$insertion property for (cpc, csc) (resp. (csc, cpc)).

Proof. Since every two disjoint *semi*−open and preopen sets can be separated by closed subsets of X, therefore by Corollary 3.4, X has the weak $cc-$insertion property for (cpc, csc) and (csc, cpc). Now suppose that f and g are real-valued functions on X with $g < f$, such that g is cpc (resp. csc), f is csc (resp. cpc) and $f - g$ is csc (resp. cpc). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is csc (resp. cpc), hence $A(f - g, 3^{-n+1})$ is a *semi*−open (resp. preopen) subset of X. Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of *semi*−open (resp. preopen) subsets of X and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preclosed (resp. *semi*−closed) subsets of X such that $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.2, the pair $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of *semi*−open (resp. preopen) and preopen (resp. *semi*−open) subsets of X can be completely separated by contra-continuous functions. Hence by Theorem 2.2, there exists a contra-continuous function h defined on X such that $g < h < f$, i.e., X has the weakly $cc-$insertion property for (cpc, csc) (resp. (csc, cpc)). □

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