Transmission problem with $1-D$ mixed type in thermoelasticity and infinite memory

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Abstract. This paper describes a polynomial decay rate of solution for a transmission problem with $1-D$ mixed type I and type II thermoelastic system with infinite memory acting in the first part. The main contributions here are to show that the infinite memory lets our problem still dissipative, and that the system is not exponentially stable, in spite of the kernel in the memory term is sub-exponential. Also we establish that the t^{-1} is the sharp decay rate. We extend the results in [27].

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1 Introduction and position of problem

A qualitative studies for problems described the thermo-mechanical interactions in elastic materials has been increasing interest in recent years. The $1-d$ linear model of the dynamical problems for classical thermoelastic systems is given by:

$$
\left\{\begin{array}{ll} u'' - u_{xx} + l\theta_x = 0, & x \in (0,L), t > 0, \\ \\ \theta' - \theta_{xx} + lu'_x = 0, & x \in (0,L), t > 0, \end{array}\right. \tag{1.1}
$$

where $u(x,t)$ denotes the displacement of the rod at time t and $\theta(x,t)$ is the temperature difference with respect to a fixed reference temperature. This last system is so-called the type I thermoelastic, which is special case when we take $k=0$ from the type III given by:

$$
\left\{\begin{array}{l} \rho u'' - (au_x - l\theta)_x = 0, \\ \\ c\tau'' + lu'_x - (\beta\theta_x + k\tau_x)_x = 0. \end{array}\right. \tag{1.2}
$$

When $\beta = 0$, the following thermoelastic system is named thermoelasticity without dissipation, that is, the energy is conservative (type II):

$$
\left\{\begin{array}{l} \rho u'' - (au_x - l\theta)_x = 0, \\ \\ c\tau'' + lu'_x - k\tau_{xx} = 0. \end{array}\right. \tag{1.3}
$$

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These last three types were introduced by Green an Naghdi [13]-[14].
In the present paper, we consider a transmission problem with $1-D$ mixed type I and type II thermoelastic system and memory term for $t>0$ in the following:

$$(1.4)\quad \begin{cases} \rho_1 u'' - a_1\Big(u_{xx} - \int_{-\infty}^{t}\mu(t-s)u_{xx}(s)ds\Big) + \beta_1\theta_x = 0, & x\in(-L,0),\\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x\in(-L,0),\\ \rho_2 v'' - a_2 v_{xx} + \beta_2 q_x = 0, & x\in(0,L),\\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x\in(0,L),\\ \\ u(0,t) = v(0,t), & \\ \theta(0,t) = q(0,t), & \\ w_1(0,t) = w_2(0,t), & \\ l\theta_x(0,t) = k w_{2,x}(0,t), & \\ a_1 u_x(0,t) - a_2 v_x(0,t) = \beta_1\theta(0,t) + \beta_2 q(0,t), & \end{cases}$$

where u, v are the displacement of the system at time t in $(-L,0)$ and $(0,L)$ and θ, q are respectively the temperature difference with respect to a fixed reference temperature, w_1, w_2 are the so-called thermal displacement, which satisfies

$$w_1(.,t) = \int_0^t \theta(.,s)ds + w_1(.,0)$$

and

$$w_2(.,t) = \int_0^t q(.,s)ds + w_2(.,0).$$

The parameters $a_1, a_2, \rho_1, \rho_2, \beta_1, \beta_2, c_1.c_2, k, l$ and $L<\infty$ are assumed to be positive constants.
The system (1.4) satisfies the Dirichlet boundary conditions:

$$(1.5)\quad \begin{cases} u(-L,t) = v(L,t) = 0, & t>0,\\ \\ w_1(-L,t) = w_2(L,t) = 0, & t>0, \end{cases}$$

and the following initial conditions:

$$(1.6)\quad \begin{cases} u(.,0) = u^0(x), u'(.,0) = u^1(x), w_1(.,0) = w_1^0(x), \theta(.,0) = \theta^0(x), x\in(-L,0)\\ \\ v(.,0) = v^0(x), v'(.,0) = v^1(x), w_2(.,0) = w_2^0(x), q(.,0) = q^0(x), x\in(0,L). \end{cases}$$

We treat the infinite memory as Dafermos [6], adding a new variable η to the system which corresponds to the relative displacement history. Let us define the auxiliary variable

$$\eta = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad (x,s)\in(-L,0)\times\mathbb{R}^+.$$

By differentiation, we have

$$\frac{d}{dt}\eta^t(x,s) = -\frac{d}{ds}\eta^t(x,s) + \frac{d}{dt}u(x,t), \quad (x,s)\in(-L,0)\times\mathbb{R}^+.$$

We can take as initial condition ($t = 0$)

$$\eta^0(x, s) = u^0(x) - u(x, -s), \quad (x, s) \in (-L, 0) \times \mathbb{R}^+.$$

Thus, the original memory term can be rewritten as follows

$$\begin{array}{rcl} \int_{-\infty}^t \mu(t-s) u_{xx}(s) ds & = & \int_0^\infty \mu(s) u_{xx}(t-s) ds \\ & = & \left(\int_0^\infty \mu(t) dt\right) u_{xx} - \int_0^\infty \mu(s) \eta_{xx}^t(s) ds. \end{array}$$

The problem (1.4) is transformed into the system

$$\left\{ \begin{array}{ll} \rho_1 u'' - a_1 \Big(\mu_0 u_{xx} + \int_0^\infty \mu(s) \eta_{xx}^t(s) ds\Big) + \beta_1 \theta_x = 0, & x \in (-L, 0), \\ c_1 w_1'' - l\theta_{xx} + \beta_1 u_x' = 0, & x \in (-L, 0), \\ \rho_2 v'' - a_2 v_{xx} + \beta_2 q_x = 0, & x \in (0, L), \\ c_2 w_2'' - k w_{2,xx} + \beta_2 v_x' = 0, & x \in (0, L), \\ \frac{d}{dt}\eta^t(x, s) + \frac{d}{ds}\eta^t(x, s) - \frac{d}{dt} u(x, t) = 0, & x \in (-L, 0), \\ & \\ u(0, t) = v(0, t), & \\ \theta(0, t) = q(0, t), & \\ w_1(0, t) = w_2(0, t), & \\ l\theta_x(0, t) = k w_{2,x}(0, t), & \\ a_1 u_x(0, t) - a_2 v_x(0, t) = \beta_1 \theta(0, t) + \beta_2 q(0, t), & \\ \eta^0(x, s) = u^0(x, 0) - u^0(x, -s), s > 0, & \end{array} \right. \tag{1.7}$$

where $\mu_0 = 1 - \int_0^\infty \mu(t) dt$.

The stability of various transmission problems on thermoelasticity have been considered [8], [11], [20], [21], [22] and [25]. Without infinite memory, it is proved in [27] that the energy of system (1.4) cannot achieve exponential decay rate. This paper is devoted to show that our system can achieve polynomial decay rate. That is, our main result here is to show that for these types of materials the dissipation produced by the viscoelastic part is not strong enough to produce an exponential decay of the solution despite that the infinite memory satisfies assumptions (3.1) and (3.2).

2 Previous results and stability

The transmission problem to hyperbolic equations was studied by Dautray and Lions [7], where the existence and regularity of solutions for the linear problem have been proved. In [21], the authors considered the transmission problem of viscoelastic waves

$$\left\{ \begin{array}{ll} \rho_1 u'' - \alpha_1 u_{xx} = 0, & x \in (0, L_0), \\ \rho_2 v'' - \alpha_2 v_{xx} + \int_0^t g(t-s) v_{xx}(s) ds = 0, & x \in (L_0, L), \end{array} \right. \tag{2.1}$$

satisfying boundary conditions and initial conditions. The authors studied the wave propagations over materials consisting of elastic and viscoelastic components. They showed that the viscoelastic part produce exponential decay of the solution. In [18], the authors investigated a 1D semi-linear transmission problem in classical thermoelasticity and showed that a combination of the first, second and third energies of the solution decays exponentially to zero. Marzocchi et al [19] studied a multidimensional linear thermoelastic transmission problem. An existence and regularity result has been proved. When the solution is supposed to be spherically symmetric, the authors established an exponential decay result similar to [18]. Next, Rivera and *all* [22], considered a transmission problem in thermoelasticity with memory. As time goes to infinity, they showed the exponential decay of the solution in case of radially symmetric situations. We must mention the pioneer work by Rivera and *all* in [11], where a semilinear transmission problem for a coupling of an elastic and a thermoelastic material is considered. The heat conduction is modeled by Cattaneo's law removing the physical paradox of infinite propagation speed of signals. The damped, totally hyperbolic system is shown to be exponentially stable. In 2009, Mesaoudi and *all* [20] proposed and studied a 1D linear thermoelastic transmission problem, where the heat conduction is described by the theories of Green and Naghdi. By using the energy method, they proved that the thermal effect is strong enough to produce an exponential stability of the solution. The earliest result in this direction was established in [27], where the dynamical behavior of the system is described by

$$\left\{\begin{array}{ll} \rho_1 u_1'' - a_1 u_{1,xx} + \beta_1 \theta_{1,x} = 0, & x \in (-1,0), \\ c_1 \tau_1'' - b\theta_{1,xx} + \beta_1 u_{1,x}' = 0, & x \in (-1,0), \\ \rho_2 u_2'' - a_2 u_{2,xx} + \beta_2 \theta_{2,x} = 0, & x \in (0,1), \\ c_2 \tau_2'' - k\tau_{2,xx} + \beta_2 u_{2,x}' = 0, & x \in (0,1). \end{array}\right. \tag{2.2}$$

The system consists of two kinds of thermoelastic components, one is of type I, another one is of type II. Under certain transmission conditions, these two components are coupled at the interface. The authors proved that the system is lack of exponential decay rate and they obtain the sharp polynomial decay rate.

3 Preliminaries

For simplicity reason denote $u(x,t) = u, v(x,t) = v, w_i(x,t) = w_i, i = 1,2, q(x,t) = q$, when there is no confusion. Here $u' = du(t)/dt, v' = dv(t)/dt$ and $u'' = d^2u(t)/dt^2, v'' = d^2v(t)/dt^2, w_i'' = d^2w_i(t)/dt^2, i = 1,2$.
First we recall and make use the following assumptions on the functions μ:
We assume that the function $\mu : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying:

$$1 - \int_0^\infty \mu(t)dt = \mu_0 > 0, \quad \forall t \in \mathbb{R}^+, \tag{3.1}$$

and that there exists a constants $k_1 > 0$ such that

$$\mu'(t) + k_1\mu(t) \le 0 \quad \forall t \in \mathbb{R}^+. \tag{3.2}$$

We denote by $\mathcal{A}$ the unbounded operator in an appropriate Hilbert state space
Let

$$V^k(0,L) = \{h \in H^k(0,L); h(L) = 0\}.$$

$$V^k(-L,0) = \{h \in H^k(-L,0); h(-L) = 0\},$$

$$\mathcal{H} = V^1(-L,0) \times L^2(-L,0) \times L^2(-L,0) \times V^1(0,L) \times L^2(0,L) \times V^1(0,L) \times L^2(0,L),$$

equipped, for $(u,u^1,\theta,v,v^1,w_2,q),(\tilde{u},\tilde{u^1},\tilde{\theta},\tilde{v},\tilde{v^1},\tilde{w}_2,\tilde{q})\in\mathcal{H}$, with an inner product

$$\Big\langle (u,u^1,\theta,v,v^1,w_2,q),(\tilde{u},\tilde{u^1},\tilde{\theta},\tilde{v},\tilde{v^1},\tilde{w}_2,\tilde{q})\Big\rangle_{\mathcal{H}}$$
$$=$$
$$\int_{-L}^{0}\Big[a_1\Big(\mu_0 u_x+\int_0^t \mu(s)\eta_x^t(s)ds\Big)\overline{\tilde{u}_x}+\rho_1 u^1\overline{\tilde{u^1}}+c_1\theta\overline{\tilde{\theta}}\Big]dx$$
$$+\int_0^L\Big[a_2 v_x\overline{\tilde{v}_x}+\rho_2 v^1\overline{\tilde{v^1}}+kw_{2,x}\overline{\tilde{w_{2,x}}}+c_2 q_x\overline{\tilde{q}_x}\Big]dx.$$

with domain

$$\mathcal{D}(\mathcal{A})=\left\{(u,u^1,\theta,v,v^1,w_2,q)\in\mathcal{H}:\begin{array}{l} u,\theta\in H^2(-L,0),u^1\in H^1(-L,0),\\ v\in H^2(0,L),v^1,q\in H^1(0,L),w_2\in H^2(0,L),\\ \theta(-L)=q(L)=0,l\theta_x(0)=kw_{2,x}(0)\\ a_1\mu_0u_x(0)-\beta_1\theta(0)=a_2v_x(0)-\beta_2q(0)\\ u(0)=v(0),\theta(0)=q(0),\end{array}\right. \tag{3.3}$$

and

$$\mathcal{A}\begin{pmatrix}u\\u^1\\\theta\\v\\v^1\\w_2\\q\end{pmatrix}=\begin{pmatrix}u^1\\ \rho_1^{-1}\Big(a_1\Big(\mu_0u_{xx}+\int_0^\infty\mu(s)\eta_{xx}^t(s)ds\Big)-\beta_1\theta_x\Big)\\ c_1^{-1}\Big(-\beta_1u_x^1+l\theta_{xx}\Big)\\ v^1\\ \rho_2^{-1}\Big(a_2v_{xx}-\beta_2q_x\Big)\\ q\\ c_2^{-1}\Big(-\beta_2v_x^1+kw_{2,xx}\Big)\end{pmatrix} \tag{3.4}$$

For $\mathcal{U}=(u,u^1,\theta,v,v^1,w_2,q)^T$, the problem (1.7) can be reformulated in the abstract from

$$\mathcal{U}'=\mathcal{A}\mathcal{U}, \tag{3.5}$$

where $\mathcal{U}(0)=(u^0,u^1,\theta^0,v^0,v^1,w_2^0,q^0)^T\in\mathcal{H}$ is given.
We will use necessary and sufficient conditions for C_0-semigroups being exponentially stable in a Hilbert space. This result was obtained by Gearhart [12] and Huang [10]

Theorem 3.1. *Let $S(t)=e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A})\supseteq\{i\zeta:\zeta\in\mathbb{R}\}\equiv i\mathbb{R}$$

and

$$\overline{\lim_{|\zeta|\to\infty}}\|(i\zeta I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}<\infty.$$

4 Lack of Exponential Stability

Following the techniques in [2], it is easy to check that $(\mathcal{H}, \|.\|_{\mathcal{H}})$ is a Hilbert space. In this section we prove the lack of exponential decay using Theorem 3.1, that is we show that there exists a sequence of values h_m such that

$$\|(ih_m I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty. \tag{4.1}$$

It is equivalent to prove that there exist a sequence of data $F_m \in \mathcal{H}$ and a sequence of real numbers $h_m \in \mathbb{R}$, with $\|F_m\|_{\mathcal{H}} \leq 1$ such that

$$\|(ih_m I - \mathcal{A})^{-1} F_m\|_{\mathcal{H}} = \|U_m\|^2_{\mathcal{H}} \to \infty. \tag{4.2}$$

Theorem 4.1. *Assume that the kernel is of the form $\mu(s) = e^{-hs}, s \in \mathbb{R}^+$, with $h > 1$. The semi group $S(t)$ on $\mathcal{H}$ is not exponentially stable.*

Proof. As in [1], we will find a sequence of bounded functions

$$F_m = (f_{1,m}, f_{2,m}, f_{3,m}, f_{4,m}, f_{5,m}, f_{6,m}, f_{7,m}, f_{8,m})^T \in \mathcal{H}, h \in \mathbb{R},$$

for which the corresponding solutions of the resolvent equations is not bounded. This will prove that the resolvent operator is not uniformly bounded. We consider the spectral equation

$$ihU_m - \mathcal{A}U_m = F_m.$$

and show that the corresponding solution U_m is not bounded when F_m is bounded in $\mathcal{H}$. Rewriting the spectral equation in term of its components, we get

$$\begin{cases} ihu - u^1 = f_{1m} \\ ih\rho_1 u^1 - \Big(a_1\Big(\mu_0 u_{xx} + \int_0^\infty \mu(s)\eta^t_{xx}(s)ds\Big) - \beta_1\theta_x\Big) = \rho_1 f_{2m} \\ ihc_1\theta - \Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = c_1 f_{3m} \\ ihv - v^1 = f_{4m} \\ ih\rho_2 v^1 - \Big(a_2 v_{xx} - \beta_2 q_x\Big) = \rho_2 f_{5m} \\ ihw_2 - q = f_{6m} \\ ihc_2 q - \Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = c_2 f_{7m} \\ ih\eta^t - u^1 + \eta^t_s = f_{8m}. \end{cases} \tag{4.3}$$

We prove that there exists a sequence of real numbers h_m so that (4.3) verified. Let us consider $f_{1m} = f_{4m} = f_{6m} = f_{8m} = 0$. We eliminate the terms u^1, v^1. We can choose $f_{2m} = f_{3m} = f_{5m} = f_{6m} = \lambda_m$ and we obtain $u^1 = ihu$, $v^1 = ihv$ and $q = ihw_2$. Then, the system (4.3) takes the form

$$\begin{cases} -h^2 u - \rho_1^{-1}\Big(a_1\Big(\mu_0 u_{xx} + \int_0^\infty \mu(s)\eta^t_{xx}(s)ds\Big) - \beta_1\theta_x\Big) = \lambda_m \\ ih\theta - c_1^{-1}\Big(-\beta_1 u^1_x + l\theta_{xx}\Big) = \lambda_m \\ -h^2 v - \rho_2^{-1}\Big(a_2 v_{xx} - \beta_2 ihw_{2,x}\Big) = \lambda_m \\ -h^2 w_2 - c_2^{-1}\Big(-\beta_2 v^1_x + kw_{2,xx}\Big) = \lambda_m \\ ih\eta^t - ihu + \eta^t_s = 0 \end{cases} \tag{4.4}$$

We look for solutions of the form

$$u = a\lambda_m, v = b\lambda_m, \theta = c\lambda_m, w_2 = d\lambda_m, u^1 = e\lambda_m, v^1 = f\lambda_m, \eta^t(x,s) = \gamma(s)\lambda_m$$

with $a, b, c, d, e, f \in \mathbb{C}$ and $\gamma(s)$ depend on h and will be determined explicitly in what follows. From (4.4), we get a, b, c, d, e and f satisfy

$$\begin{cases} -h^2 a - \rho_1^{-1}\Big(a_1 h_m\Big(\mu_0 a + \int_0^\infty \mu(s)\gamma(s)ds\Big) - \beta_1 ch\Big) = 1, \\ ihc - c_1^{-1}\Big(-\beta_1 e + lh_m c\Big) = 1, \\ -h^2 b - \rho_2^{-1}\Big(a_2 h_m b - \beta_2 ihd\Big) = 1, \\ ihd - c_2^{-1}\Big(-\beta_2 f + kh_m d\Big) = 1, \\ \gamma_s + ih\gamma - iha = 0. \end{cases} \tag{4.5}$$

From $(4.5)_5$ we get

$$\gamma(s) = a - ae^{-ihs}. \tag{4.6}$$

Then, from (4.6) we have

$$\begin{aligned} \int_0^\infty \mu(s)\gamma(s)\,ds &= \int_0^\infty \mu(s)(a - ae^{-ihs})ds \\ &= a\int_0^\infty \mu(s)ds - a\int_0^\infty \mu(s)ae^{-ihs}ds \\ &= a(1-\mu_0) - a\int_0^\infty \mu(s)e^{-ihs}\,ds. \end{aligned} \tag{4.7}$$

Now, we would like to find the parameters constants. To this end, we choose

$$c_1 ih = h_m l, \qquad c_2 ih = kh_m, \tag{4.8}$$

and using the equations $(4.5)_2$ and $(4.5)_4$, we obtain

$$e = \frac{c_1}{\beta_1}, \tag{4.9}$$

$$f = \frac{c_2}{\beta_2}. \tag{4.10}$$

We choose $-h^2\rho_2 = a_2 h_m$. By equations $(4.5)_1$ and $(4.5)_3$, we have

$$c = \frac{1}{(-h^2 - \rho_1^{-1}h_m a_1\mu_0)}\Big(1 + \rho_1^{-1}h_m a_1\int_0^\infty \mu(s)\gamma(s)ds - \rho_1^{-1}h_m\beta_1 c\Big),$$

$$d = \frac{\rho_2}{\beta_2 ih}.$$

Since $c_2 l = c_1 k$, recalling from (4.9), (4.10) that

$$\begin{aligned} u^1 + v^1 &= e\lambda_m + f\lambda_m \\ &= \frac{c_1}{\beta_1}\lambda_m + \frac{c_2}{\beta_2}\lambda_m, \end{aligned}$$

we get

$$\|u^1\|_2^2 + \|v^1\|_2^2 = \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big] h_m^2.$$

Therefore we have

$$\begin{aligned}\lim_{m\to\infty} \|U_m\|_{\mathcal{H}}^2 &\geq \lim_{m\to\infty} [\|u^1\|_2^2 + \|v^1\|_2^2] \\ &= \lim_{m\to\infty} \Big[\Big(\frac{c_1}{\beta_1}\Big)^2 + \Big(\frac{c_2}{\beta_2}\Big)^2\Big] h_m^2 \\ &= +\infty\end{aligned}$$

which completes the proof. □

5 Polynomial Stability

Our main result reads as follows.

Theorem 5.1. *Assume that (3.1) and (3.2) hold. Then t^{-1} is the sharp decay rate. Therefore there exists positive constant C such that the solution of our system satisfies*

$$E(t) \leq \frac{C}{t}, \qquad \forall t \in \mathbb{R}^+. \tag{5.1}$$

Proof. We will follow the idea for the proof of the corresponding results in [27]. We would show that

$$\lim_{\zeta\to\infty} \|(i\zeta I - \mathcal{A})^{-1}\| < \infty \tag{5.2}$$

We prove that there exist a sequence

$$V_n = (u_n, u_n^1, \theta_n, v_n, v_n^1, w_{2,n}, q_n) \in \mathcal{D}(\mathcal{A}),$$

with $\|V_n\|_{\mathcal{H}} = 1$, and a sequence $\zeta_n \in \mathbb{R}$ with $\zeta_n \to \infty$ such that

$$\lim_{n\to\infty} \zeta_n \|(i\zeta_n I - \mathcal{A})V_n\|_{\mathcal{H}} = 0$$

or

$$\zeta_n(i\zeta_n u_n - u_n^1) \to 0, \quad in \quad H^1(-L,0), \tag{5.3}$$

$$\zeta_n\Big(i\zeta_n u_n^1 - \rho_1^{-1}\Big(\mu_1\Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds\Big) - \beta_1\theta_{n,x}\Big)\Big) \to 0, \quad in \quad L^2(-L,0), \tag{5.4}$$

$$\zeta_n\Big(i\zeta_n\theta_n - c_1^{-1}\Big(-\beta_1 u^1_{n,x} + l\theta_{n,xx}\Big)\Big) \to 0, \quad in \quad L^2(-L,0), \tag{5.5}$$

$$\zeta_n(i\zeta_n v_n - v_n^1) \to 0, \quad in \quad H^1(0,L), \tag{5.6}$$

$$\zeta_n\Big(i\zeta_n v_n^1 - \rho_2^{-1}\Big(a_2 v_{n,xx} - \beta_2 q_{n,x}\Big)\Big) \to 0, \quad in \quad L^2(0,L), \tag{5.7}$$

$$\zeta_n(i\zeta_n w_{2,n} - q_n) \to 0, \quad in \quad H^1(0,L), \tag{5.8}$$

$$\zeta_n\Big(i\zeta_n q_n - c_2^{-1}\Big(-\beta_2 v^1_{n,x} + k w_{2,n,xx}\Big)\Big) \to 0, \quad in \quad L^2(0,L), \tag{5.9}$$

$$ih\eta^t - u^1_{1,n} + \eta^t_s = 0 \tag{5.10}$$

Note that

$$Re\langle\zeta_n(i\zeta_n - \mathcal{A})V_n, V_n\rangle_{\mathcal{H}} = \zeta_n\|\sqrt{l}\quad\theta_{n,x}\|^2_{L^2} \to 0.$$

Then

$$\sqrt{\zeta_n}\quad\theta_{n,x} \to 0, \quad in \quad L^2(-L,0). \tag{5.11}$$

By Poincaré's inequality, we get

$$\sqrt{\zeta_n}\quad\theta_n \to 0, \quad in \quad L^2(-L,0). \tag{5.12}$$

Thanks to the Gagliardo-Nirenberg inequality, we have

$$\|\sqrt{\zeta_n}\quad\theta_n\|_{L^\infty} \le C_1\sqrt{\|\sqrt{\zeta_n}\quad\theta_{n,x}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n}\quad\theta_n\|_{L^2}} + C_2\|\sqrt{\zeta_n}\quad\theta_n\|_{L^2}. \tag{5.13}$$

Thus,

$$\sqrt{\zeta_n}\quad\theta_n(0) \to 0. \tag{5.14}$$

From (5.3), we have $\beta_1(i\zeta_n)^{-1}u^1_{n,x}$ is bounded in $L^2(-L,0)$. By (5.5) we have the boundedness of $(i\zeta_n)^{-1}\theta_{n,xx}$ in $L^2(-L,0)$.
Using again the Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}\|\Big(\sqrt{\sqrt{\zeta_n}}\Big)^{-1}\theta_{n,x}\|_{L^\infty} &\le d_1\sqrt{\|(\zeta_n)^{-1}\quad\theta_{n,xx}\|_{L^2}}\sqrt{\|\sqrt{\zeta_n}\quad\theta_{n,x}\|_{L^2}} + d_2\|\Big(\sqrt{\sqrt{\zeta_n}}\Big)^{-1}\theta_{n,x}\|_{L^2} \\ &\to 0.\end{aligned}$$

which gives

$$\left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}(-L)\to 0,\qquad \left(\sqrt{\sqrt{\zeta_n}}\right)^{-1}\theta_{n,x}(0)\to 0. \tag{5.15}$$

Multiplying (5.4) by $p(x)u_{n,x}$ in L^2-norm for $p(x)\in C^1[-L,0]$, we get

$$\begin{aligned}&-\zeta_n^2\langle u_n,p(x)u_{n,x}\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\Big\langle\mu_0u_{n,xx},p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\&-\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds,p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}\\&+\rho_1^{-1}\beta_1\langle\theta_{n,x},p(x)u_{n,x}\rangle_{L^2(-L,0)}\to 0.\end{aligned}\tag{5.16}$$

Integration by parts gives

$$-\zeta_n^2\langle u_n,p(x)u_{n,x}\rangle_{L^2(-L,0)}=\zeta_n^2p(-L)|u_n(-L)|^2-\zeta_n^2p(0)|u_n(0)|^2+\zeta_n^2\Big\langle p_x(x)u_n,u_n\Big\rangle_{L^2(-L,0)}$$

$$\begin{aligned}-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx},p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}&=-\rho_1^{-1}a_1\mu_0p(0)|u_{n,x}(0)|^2+\rho_1^{-1}a_1\mu_0p(-L)|u_{n,x}(-L)|^2\\&+\rho_1^{-1}a_1\mu_0\Big\langle p_x(x)u_{n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\end{aligned}$$

and

$$\begin{aligned}-\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds,p(x)u_{n,x}\Big\rangle_{L^2(-L,0)}&=-\rho_1^{-1}a_1p(0)\int_0^\infty\mu(s)\eta^t_{n,x}(0,s)dsu_{n,x}(0)\\&+\rho_1^{-1}a_1p(-L)\int_0^\infty\mu(s)\eta^t_{n,x}(-L,s)dsu_{n,x}(-L)\\&+\rho_1^{-1}a_1\Big\langle p_x(x)\int_0^\infty\mu(s)\eta^t_{n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}.\end{aligned}$$

Since

$$\rho_1^{-1}\beta_1\langle\theta_{n,x},p(x)u_{n,x}\rangle_{L^2(-L,0)}\to 0,$$

then by the above integrations, for $p(x)=x\in C^1[-L,0]$, (5.16) takes the form

$$\begin{aligned}&-\zeta_n^2|u_n(-L)|^2+\zeta_n^2\Big\langle u_n,u_n\Big\rangle_{L^2(-L,0)}\\&-\rho_1^{-1}a_1\mu_0|u_{n,x}(-L)|^2+\rho_1^{-1}a_1\mu_0\Big\langle u_{n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\\&-\rho_1^{-1}a_1\int_0^\infty\mu(s)\eta^t_{n,x}(-L,s)dsu_{n,x}(-L)\\&+\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0,\end{aligned}\tag{5.17}$$

and hence, $u_{n,x}(-L)$ and $\zeta_n u_n(-L)$ are bounded.
Similarly, taking $p(x) = x + L \in C^1[-L, 0]$, (5.16) takes the form

$$
\begin{aligned}
&-\zeta_n^2 |u_n(0)|^2 + \zeta_n^2 \Big\langle u_n, u_n \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \mu_0 |u_{n,x}(0)|^2 + \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta_{n,x}^t(0, s) ds u_{n,x}(0) \\
+ \quad &\rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta_{n,x}^t(s) ds, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned} \tag{5.18}
$$

Then, we get boundedness of $\zeta_n u_n(0)$ and $u_{n,x}(0)$.
Multiplying (5.5) by $u_{n,x}$ and taking the integration, we get

$$
i\zeta_n \Big\langle \theta_n, u_{n,x} \Big\rangle_{L^2(-L,0)} + c_1^{-1} \beta_1 \Big\langle u_{1,n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} - c_1^{-1} l \Big\langle \theta_{n,xx}, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0.
$$

By (5.12), after dividing by $i\sqrt{\zeta_n}$, we have, where we have used $\zeta_n > 0$

$$
i\zeta_n \Big\langle \theta_n, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0
$$

Integrating by parts, we get

$$
\begin{aligned}
&l(i\sqrt{\zeta_n})^{-1} \Big(\theta_{n,x}(-L) \overline{u_{n,x}(-L)} - \theta_{n,x}(0) \overline{u_{n,x}(0)} \Big) + l \Big\langle \sqrt{\zeta_n} \theta_{n,x}, (i\zeta_n)^{-1} u_{n,xx} \Big\rangle_{L^2(-L,0)} \\
&+ \beta_1 \sqrt{\zeta_n} \Big\langle u_{1,n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0
\end{aligned} \tag{5.19}
$$

By (5.15) and the boundedness of $u_{n,x}(-L)$ and $u_{n,x}(0)$, we have

$$
l(i\sqrt{\zeta_n})^{-1} \Big(\theta_{n,x}(-L) \overline{u_{n,x}(-L)} - \theta_{n,x}(0) \overline{u_{n,x}(0)} \Big) \to 0
$$

Moreover, from (5.4), we obtain that $(i\zeta_n)^{-1} u_{n,xx}$ is bounded in $L^2(-L, 0)$. Thus

$$
l(\sqrt{\zeta_n} \theta_{n,x}, (i\zeta_n)^{-1} u_{n,xx}) \to 0
$$

Hence by (5.19), we get

$$
\sqrt{\sqrt{\zeta_n}} \quad u_{n,x} \to 0, \qquad in \qquad L^2(-L, 0). \tag{5.20}
$$

Thanks to the Poincaré inequality, we have

$$
\sqrt{\sqrt{\zeta_n}} \quad u_n \to 0, \quad in \qquad L^2(-L, 0) \tag{5.21}
$$

By (5.20), (5.21) and Galiardo-Nirenberg inequality, we get

$$
\sqrt{\sqrt{\zeta_n}} \quad u_n(0) \to 0. \tag{5.22}
$$

From (5.4) and (5.11), using $\zeta_n > 0$, we have

$$i\zeta_n u_{1,n} - \rho_1^{-1} a_1 \Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s) \eta^t_{n,xx}(s) ds \Big) \to 0, \qquad in \quad L^2(-L,0), \tag{5.23}$$

Multiplying the above by u_n, we get

$$i\zeta_n \Big\langle u_{1,n}, u_n \Big\rangle_{L^2(-L,0)} - \rho_1^{-1} a_1 \Big\langle \Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s) \eta^t_{n,xx}(s) ds \Big), u_n \Big\rangle_{L^2(-L,0)} \to 0.$$

Integrating by parts, we get

$$\begin{aligned}
&-\Big\langle u_{1,n}, u_{1,n} \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \mu_0 u_{n,x}(0) \overline{u_n(0)} + \rho_1^{-1} a_1 \mu_0 u_{n,x}(-L) \overline{u_n(-L)} - \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} \\
&+\rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta^t_{n,x}(0,s) ds \overline{u_n(0)} - \rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta^t_{n,x}(-L,s) ds \overline{u_n(-L)} \\
&+\rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta^t_{n,x}(s) ds, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned}$$

Since $u_{n,x}(0), u_{n,x}(-L)$ are bounded, by (5.20) and $u_n(-L) \to 0, u_n(0) \to 0$, we have

$$u_{1,n}, \zeta_n u_n \to 0, \quad in \quad L^2(-L,0). \tag{5.24}$$

Multiplying (5.4) by $(x+L)u_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re \Big[-\Big\langle \zeta_n^2 u_{1,n}, (x+L) u_{n,x} \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \Big\langle \Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s) \eta^t_{n,xx}(s) ds \Big), (x+L) u_{n,x} \Big\rangle_{L^2(-L,0)} \Big] \\
&= -\zeta_n^2 |u_n(0)|^2 + \zeta_n^2 \Big\langle u_n, u_n \Big\rangle_{L^2(-L,0)} - \rho_1^{-1} a_1 \mu_0 |u_{n,x}(0)|^2 + \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta^t_{n,x}(0,s) ds u_{n,x}(0) + \rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta^t_{n,x}(s) ds, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned}$$

Hence, by (5.20) and (5.24), we get

$$\zeta_n u_n(0), u_{n,x}(0) \to 0 \tag{5.25}$$

Now, multiplying (5.4) by $xu_{n,x}$, we get the real part as follows

$$\begin{aligned}
&2\Re \Big[-\Big\langle \zeta_n^2 u_n, x u_{n,x} \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \Big\langle \Big(\mu_0 u_{n,xx} + \int_0^\infty \mu(s) \eta^t_{n,xx}(s) ds \Big), x u_{n,x} \Big\rangle_{L^2(-L,0)} \Big] \\
&= -\zeta_n^2 |u_n(-L)|^2 + \zeta_n^2 \Big\langle u_n, u_n \Big\rangle_{L^2(-L,0)} - \rho_1^{-1} a_1 \mu_0 |u_{n,x}(-L)|^2 + \rho_1^{-1} a_1 \mu_0 \Big\langle u_{n,x}, u_{n,x} \Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1} a_1 \int_0^\infty \mu(s) \eta^t_{n,x}(-L,s) ds u_{n,x}(-L) + \rho_1^{-1} a_1 \Big\langle \int_0^\infty \mu(s) \eta^t_{n,x}(s) ds, u_{n,x} \Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned} \tag{5.26}$$

Then

$$\zeta_n u_n(-L), u_{n,x}(-L) \to 0. \tag{5.27}$$

Taking again (5.4), multiplying by u_n, we have

$$\begin{aligned}
&\sqrt{\zeta_n}\Big\langle i\zeta_n u^1 1, n, u_n\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle \theta_{n,x}, u_n\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_0\Big\langle u_{n,xx}, u_n\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, u_n\Big\rangle_{L^2(-L,0)} \to 0. \qquad (5.28)
\end{aligned}$$

By (5.20) and (5.25), we have

$$\begin{aligned}
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_0\Big\langle u_{n,xx}, u_n\Big\rangle_{L^2(-L,0)} \\
&= -\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}u_{n,x}(0)\overline{u_n(0)} + \rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}u_{n,x}(-L)\overline{u_n(-L)} \\
&+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\Big\langle u_{n,x}, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0
\end{aligned}$$

and

$$\begin{aligned}
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, u_n\Big\rangle_{L^2(-L,0)} \\
&= -\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds\overline{u_n(0)} \\
&+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds\overline{u_n(-L)} \\
&+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds, u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \qquad (5.29)
\end{aligned}$$

Thus, by(5.29) and (5.11), we go to

$$\sqrt{\sqrt{\zeta_n}}u_n^1 \to 0, \quad in \qquad L^2(-L,0). \tag{5.30}$$

Multiplying (5.4) by $(x+L)u_{n,x}$, we have

$$\begin{aligned}
&\Big\langle i\sqrt{\zeta_n}\zeta_n u_n^1, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} + \rho_1^{-1}\sqrt{\zeta_n}\beta_1\Big\langle \theta_{n,x}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\mu_0\Big\langle u_{n,xx}, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \\
&-\rho_1^{-1}\sqrt{\zeta_n}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds, (x+L)u_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \qquad (5.31)
\end{aligned}$$

Integrating by parts and using (5.11) and the boundedness of $u_{n,x}$ in $L^2(-L,0)$, we get

$$\begin{aligned}
&-\sqrt{\zeta_n}|u_n^1(0)|^2+\sqrt{\zeta_n}\Big\langle u_n^1,u_n^1\Big\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}|u_{n,x}(0)|^2\\
&+\rho_1^{-1}a_1\mu_0\sqrt{\zeta_n}\Big\langle u_{n,x},u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_0^\infty\mu(s)\eta^t_{n,x}(0,s)ds u_{n,x}(0)\\
&-\rho_1^{-1}a_1\sqrt{\zeta_n}\int_{-L}^0\Big\langle\int_0^\infty\mu(s)\eta^t_{n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0.
\end{aligned}\tag{5.32}$$

Thus by (5.20) and (5.30), we go to

$$\sqrt{\sqrt{\zeta_n}}u_n^1(0),\sqrt{\sqrt{\zeta_n}}u_{n,x}(0)\to 0\tag{5.33}$$

Multiplication of (5.23) by $u_{n,x}$ yields

$$\begin{aligned}
&i\zeta_n\Big\langle u_n^1,u_{n,x}\Big\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx},u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0.
\end{aligned}\tag{5.34}$$

Due to (5.25)and (5.27), we get

$$\begin{aligned}
&-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx},u_{n,x}\Big\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\\
&=\frac{1}{2}(-\rho_1^{-1}a_1\mu_0)|u_{n,x}(0)|^2+\rho_1^{-1}a_1\mu_0)|u_{n,x}(-L)|^2\\
&-\rho_1^{-1}a_1\int_0^\infty\mu(s)\eta^t_{n,x}(0,s)ds u_{n,x}(0)+\rho_1^{-1}a_1\int_0^\infty\mu(s)\eta^t_{n,x}(-L,s)ds u_{n,x}(-L)\\
&+\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,x}(s)ds,u_{n,x}\Big\rangle_{L^2(-L,0)}\to 0.
\end{aligned}\tag{5.35}$$

Thus, it follows from (5.34) that

$$(i\zeta_n u_n^1,u_{n,x})\to 0.\tag{5.36}$$

Taking the product of (5.23) with θ_n, yields

$$\begin{aligned}
&i\zeta_n\Big\langle u_{1,n},\theta_n\Big\rangle_{L^2(-L,0)}-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx},\theta_n\Big\rangle_{L^2(-L,0)}\\
&-\rho_1^{-1}a_1\Big\langle\int_0^\infty\mu(s)\eta^t_{n,xx}(s)ds,\theta_n\Big\rangle_{L^2(-L,0)}\to 0\qquad in\quad L^2(-L,0).
\end{aligned}\tag{5.37}$$

Due to (5.11),(5.14) and (5.25), we have

$$\begin{aligned}
&-\rho_1^{-1}a_1\mu_0\Big\langle u_{n,xx},\theta_n\Big\rangle_{L^2(-L,0)}\\
&=-\rho_1^{-1}a_1\mu_0u_{n,x}(0)\overline{\theta_n(0)}+\rho_1^{-1}a_1\mu_0u_{n,x}(-L)\overline{\theta_n(-L)}\\
&+\rho_1^{-1}a_1\mu_0\Big\langle u_{n,x},\theta_{n,x}\Big\rangle_{L^2(-L,0)}\to 0
\end{aligned}\tag{5.38}$$

and

$$
\begin{aligned}
&-\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,xx}(s)ds,\theta_n\Big\rangle_{L^2(-L,0)}\\
&= -\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(0,s)ds\overline{\theta_n(0)}\\
&+\rho_1^{-1}a_1\int_0^\infty \mu(s)\eta^t_{n,x}(-L,s)ds\overline{\theta_n(-L)}\\
&+\rho_1^{-1}a_1\Big\langle \int_0^\infty \mu(s)\eta^t_{n,x}(s)ds,\theta_{n,x}\Big\rangle_{L^2(-L,0)} \to 0.
\end{aligned}
\tag{5.39}
$$

Then from (5.37), we obtain

$$
i\zeta_n\Big\langle u_n^1,\theta_n\Big\rangle_{L^2(-L,0)} \to 0 \tag{5.40}
$$

Multiplying (5.5) by u_n^1, we have

$$
\Big\langle i\zeta_n\theta_n,u_n^1\Big\rangle_{L^2(-L,0)} - c_1^{-1}l\Big\langle \theta_{n,xx},u_n^1\Big\rangle_{L^2(-L,0)} + c_1^{-1}\beta_1\Big\langle u^1_{n,x},u_n^1\Big\rangle_{L^2(-L,0)} \to 0. \tag{5.41}
$$

By (5.36), (5.40), we have

$$
\Big\langle \theta_{n,xx},u_n^1\Big\rangle_{L^2(-L,0)} \to 0. \tag{5.42}
$$

Integrating by parts

$$
\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} - \Big\langle \theta_{n,x},u^1_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{5.43}
$$

Due to (5.15)and (5.33), we get

$$
\theta_{n,x}(0)\overline{u_n^1(0)} - \theta_{n,x}(-L)\overline{u_n^1(-L)} \to 0. \tag{5.44}
$$

From (5.43) we have

$$
\Big\langle \theta_{n,x},u^1_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{5.45}
$$

Multiplying (5.5) by $(x+L)\theta_{n,x}$ and integrating, we get

$$
\Re\Big[\Big\langle i\zeta_n\theta_n,(x+L)\theta_{n,x}\Big\rangle_{L^2(-L,0)} - c_1^{-1}\Big\langle (l\theta_{n,xx}-\beta_1u^1_{n,x}),(x+L)\theta_{n,x}\Big\rangle_{L^2(-L,0)}\Big] \to 0 \tag{5.46}
$$

By (5.11) and (5.12), we obtain

$$
\Big\langle i\zeta_n\theta_n,(x+L)\theta_{n,x}\Big\rangle_{L^2(-L,0)} \to 0. \tag{5.47}
$$

Thus by (5.46) and (5.11), we have

$$
-c_1^{-1}l\theta_{n,x}(0)\overline{\theta_{n,x}(0)} + 2\Re[c_1^{-1}\beta_1(u^1_{n,x},(x+L)\theta_{n,x})] \to 0. \tag{5.48}
$$

Then, by (5.45), we get

$$\theta_{n,x}(0) \to 0 \tag{5.49}$$

Hence, by (5.35),(5.25),(5.14)and (5.49), we have

$$u_{n,x}(0), u_n(0), \theta_n(0), \theta_{n,x}(0) \to 0. \tag{5.50}$$

Taking the product of (5.9) with $(x-L)w_{2,n,x}$, yields

$$\begin{aligned} &\Re\Big[i\zeta_n\Big\langle q_n, (x-L)w_{2,n,x}\Big\rangle_{L^2(0,L)} + c_2^{-1}\beta_2\Big\langle v^1_{n,x}, (x-L)w_{2,n,x}\Big\rangle_{L^2(0,L)} \\ &-c_2^{-1}k\Big\langle w_{2,n,xx}, (x-L)w_{2,n,x}\Big\rangle_{L^2(0,L)}\Big] \to 0. \end{aligned} \tag{5.51}$$

Using the transmission conditions in (1.4), we get

$$(q_n, q_n) + c_2^{-1}k(w_{2,n,x}, w_{2,n,x}) - 2\Re\Big[c_2^{-1}\beta_2\Big\langle v_{n,x}, (x-L)q_{n,x}\Big\rangle_{L^2(0,L)}\Big] \to 0. \tag{5.52}$$

Taking the product of (5.7) with $(x-L)v_{n,x}$, we obtain

$$\begin{aligned} &i\zeta_n\Big\langle v^1_n, (x-L)v_{n,x}\Big\rangle_{L^2(0,L)} - \rho_2^{-1}a_2\Big\langle v_{n,xx}, (x-L)v_{n,x}\Big\rangle_{L^2(0,L)} \\ &+\rho_2^{-1}\beta_2\Big\langle q_{n,x}, (x-L)v_{n,x}\Big\rangle_{L^2(0,L)} \to 0. \end{aligned} \tag{5.53}$$

Integrating (5.53) by parts we have

$$\begin{aligned} &\Big\langle v^1_n, v^1_n\Big\rangle_{L^2(0,L)} + \rho_2^{-1}a_2\Big\langle v_{n,x}, v_{n,x}\Big\rangle_{L^2(0,L)} \\ &+2\Re\Big[\rho_2^{-1}\beta_2\Big\langle q_{n,x}, (x-L)q_{n,x}\Big\rangle_{L^2(0,L)}\Big] \to 0. \end{aligned} \tag{5.54}$$

Thus by (5.52) and (5.54), we obtain

$$\begin{aligned} &a_2\Big\langle v_{n,x}, v_{n,x}\Big\rangle_{L^2(0,L)} + \Big\langle \rho_2 v^1_n, v^1_n\Big\rangle_{L^2(0,L)} + k\Big\langle w_{2,n,x}, w_{2,n,x}\Big\rangle_{L^2(0,L)} \\ &+c_2\Big\langle q_n, q_n\Big\rangle_{L^2(0,L)} \to 0. \end{aligned} \tag{5.55}$$

Then

$$v_{n,x}, v^1_n, w_{2,n,x}, q_n \to 0, \qquad in \qquad L^2(0,L). \tag{5.56}$$

Thus (5.56) together with (5.12), (5.24) and (5.56), we give

$$V_n = (u_n, u^1_n, \theta_n, v_n, v^1_n, w_{2,n}, q_n)^T \to 0, \tag{5.57}$$

which contradicts $\|V_n\| = 1$. Therefore, (5.2) holds. This completes the proof. $\square$

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