A theorem on the auxiliary zeta function

B. F. F. Flora

Abstract. In this work, the analysis of the Riemann's zeta function in the critical strip, Σ, has been performed, formulating a new equation of the zeros of the $\tilde{\xi}(s)$ auxiliary function, by means of the Schwarz's reflection principle and an extension Caccioppoli's theorem on the unit elements of the functional transformations. We demonstrate that, in the strip Σ, the Riemann's $\xi(t)$ function has only real zeros whose map in the complex plane s correspond to the zeros of the Riemann zeta function.

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Key words: Zeta function; non-trivial zeros; principle of reflection; functional equation in the complex domain; upper incomplete Gamma function.

Notations used throughout the article:

- $j = \sqrt{-1}$;
- $\mathbb{R}$ set of real numbers;
- $\mathbb{C}$ set of complex numbers ;
- $\mathbb{Z}$ set of relative numbers;
- $log(x)$ is the natural logarithm of x;
- $\Re(\cdot)$ is the real part;
- $\Im(\cdot)$ is the imaginary part;
- $\sigma = \Re(s)$ and $\rho = \Im(s) : s = \sigma + j\rho \in \Sigma = (0, 1) \times (-j\infty, 0) \cup (0, +j\infty)$;
- $\tau = \Re(t)$ and $\mu = \Im(t) : t = \tau + j\mu \in \Sigma' = (-\infty, 0) \cup (0, +\infty) \times \left(-\frac{j}{2}, +\frac{j}{2}\right)$;
- $\psi(x) = \sum\limits_{n=1}^{\infty} e^{-\pi n^2 x}$ is the Poisson's elliptic function;
- $\tilde{\xi}(s)$ is the auxiliary zeta function;
- $\xi(t)$ is the Riemann's auxiliary zeta function;
- $\zeta(s)$ is the Riemann's zeta function;
- $\Omega_0 = \{\forall s \in \Sigma : \tilde{\xi}(s) = 0\}$;
- $\Omega_0^{'} = \{\forall t \in \Sigma^{'} : \tilde{\xi}\left(\frac{1}{2} + jt\right) = 0\}$;
- $(\cdot)^*$ is the conjugate of a number or function.

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1 Introduction

In the XIX-th century, Riemann (1859, [16]) obtained the functional equation:

$$2(2\pi)^{s-1}\Gamma(1-s)\sin(s\pi/2)\zeta(1-s)=\zeta(s) \tag{1.1}$$

by means of the connection between the gamma function Γ(s) and the prime numbers, it found by Euler. The (1.1) characterizes a function ζ(s), known as zeta function of Riemann, defined in the complex plane ([5], [8], [12], [18]). Using a new analytic approach, Riemann stated that the zeros of the function $\zeta(s)$ are all on the straight line of the equation $\Re(s)=1/2$, known as critical straight line of the complex plane. This statement represents the Riemann Hypothesis (RH) [3]. The proof of RH was suspended by the Riemann who determined the best approximation of the prime counting function in a given interval. In order to proof the Riemann hypothesis, many mathematicians and theoretic physicians used different methods that can be summarized as follows:

a) Trying to demonstrate, through proof by contradiction, that in the strip Σ of the complex plane it is not possible to find a zero outside of the straight line of equation $\Re(s)=1/2$;

b) Formulating the Riemann hypothesis in a larger work of the generalized Dirichlet series;

c) Using non commutative algebra techniques, used for the study of some phenomena in quantistic physics.

In the note [11], by evidencing a contradiction present in the article of M. H. Bohr and E. Landau [2], G.H. Hardy confirmed the presence of an infinite number zeros on the critical straight line, in his proof which was published previously in Proc. London Math. Soc. Ser. march 1914, but he hadn't proved the total absence of zeros outside of it. I prove it by means of the associated functional equation obtained from an integral transformation of the auxiliary zeta function. The purpose of this work is to extend the analysis of the Riemann's zeta function in the critical strip, Σ, formulating a new equation of the zeros of the $\tilde{\xi}(s)$ auxiliary function, by means of the Schwarz's reflection principle and an extension Caccioppoli's theorem on the unit elements of the functional transformations, it recalled in appendix C. We demonstrate that, in the strip Σ, the Riemann's $\xi(t)$ function has only real zeros whose map in the complex plane s correspond to the zeros of the Riemann zeta function.

2 The equation of the zeros of $\tilde{\xi}(s)$

Riemann (1859, [16]) showed that the zeta function of Riemann ζ(s) is linked to Poisson's elliptic function ψ(x) and Euler's gamma function Γ(s), by means of the following formula:

$$\Gamma(s/2)(\pi)^{-s/2}\zeta(s)=\frac{1}{s(s-1)}+\int_1^{+\infty}\psi(x)[x^{(s/2-1)}+x^{-(s+1)/2}]dx \tag{2.1}$$

using the functional equation of Jacobi:

$$2\psi(x) + 1 = x^{-1/2}[2\psi(1/x) + 1], \tag{2.2}$$

where: s= σ+j ρ, with s $\in \mathbb{C}$ *and* $x \in [1, \infty)$. By multiplying both the members of (2.1) for s(s - 1)/2 , we defined the function $\tilde{\xi}$(s). Consider the auxiliary function:

$$\tilde{\xi}(s) = \frac{1}{2}s(s-1)\Gamma(s/2)(\pi)^{-s/2}\zeta(s). \tag{2.3}$$

In (2.3), it is noteworthy that in the strip Σ, all the zeros of the Riemann zeta function are the zeros of the function $\tilde{\xi}(s)$. Therefore, if all the zeros of the function $\tilde{\xi}(s)$ have real part equal to 1/2, then even the real part of the zeta function zeros is 1/2. So, the RH is proved. By replacing (2.1) in (2.3) and imposing the zero condition, the equation of function $\tilde{\xi}(s)$ is obtained by:

$$\frac{1}{2} + \frac{s(s-1)}{2}\int_1^{+\infty} \psi(x)[x^{(s/2-1)} + x^{-(s+1)/2}]\, dx = 0. \tag{2.4}$$

3 Theorem on the auxiliary zeta function associated to the RH

The (2.4) can be decomposed in two equations (see Appendix A), ([6]), with a real part and an imaginary part, respectively:

$$\Phi_R(\sigma_0, \rho_0) = \int_1^{+\infty} \psi(x)x^{-3/4}cos[\rho_0 log(\sqrt{x})]cosh[(1/2 - \sigma_0)log(\sqrt{x})]\, dx, \tag{3.1}$$

$$\Phi_I(\sigma_0, \rho_0) = \int_1^{+\infty} \psi(x)x^{-3/4}sin[\rho_0 log(\sqrt{x})]sinh[(\sigma_0 - 1/2)log(\sqrt{x})]\, dx. \tag{3.2}$$

with $\sigma_0 \in (0, 1) \cap \Re$ and $\rho_0 \in \Re - \{0\}$, we obtain:

$$\Phi_R(\sigma_0, \rho_0) = \frac{\sigma_0(1-\sigma_0) + \rho_0^2}{2\left\{[\sigma_0(1-\sigma_0) + \rho_0^2]^2 + (2\sigma_0 - 1)^2\rho_0^2\right\}} \tag{3.3}$$

$$\Phi_I(\sigma_0, \rho_0) = \frac{(2\sigma_0 - 1)\rho_0}{2\left\{[\sigma_0(1-\sigma_0) + \rho_0^2]^2 + (2\sigma_0 - 1)^2\rho_0^2\right\}} \tag{3.4}$$

Recomposing the two equations in a complex shape, and assuming:

$$\Delta(\sigma_0, \rho_0) = 2\left\{[\sigma_0(1-\sigma_0) + \rho_0^2]^2 + (2\sigma_0 - 1)^2\rho_0^2\right\},$$

we obtain:

$$\int_1^{+\infty} \psi(x)x^{-3/4}cos[\alpha(x;\rho_0) + j\beta(x;\sigma_0)]\, dx = \Phi_R(\sigma_0, \rho_0) + j\Phi_I(\sigma_0, \rho_0), \tag{3.5}$$

where (see Appendix A, argument of the $cos(\cdot)$ function, at the second member of the (A.7)):

$$\alpha(x;\rho_0) = \rho_0 \log(\sqrt{x}) \tag{3.6}$$

$$\beta(x;\sigma_0) = (\sigma_0 - 1/2) log(\sqrt{x}) \tag{3.7}$$

By denoting:

$$\phi(x;s_0) = \alpha(x;\rho_0) + j\beta(x;\sigma_0), \tag{3.8}$$

the relation (3.5) can be written as:

$$\int_1^{+\infty} \psi(x) x^{-3/4} cos[\phi(x;s_0)]\, dx = \frac{\sigma_0(1-\sigma_0) + \rho_0^2 + j(2\sigma_0 - 1)\rho_0}{\Delta(\sigma_0,\rho_0)}. \tag{3.9}$$

By detailing the function $\psi(x)$, under the integral at the first member of the (3.9), we have:

$$\int_1^{+\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{-3/4} cos[\phi(x;s_0)]\, dx = \frac{\sigma_0(1-\sigma_0) + \rho_0^2 + j(2\sigma_0 - 1)\rho_0}{\Delta(\sigma_0,\rho_0)}, \tag{3.10}$$

and by replacing the integration for the sum, we obtain:

$$\sum_{n=1}^{\infty} \int_1^{+\infty} e^{-\pi n^2 x} x^{-3/4} cos[\phi(x;s_0)]\, dx = \frac{\sigma_0(1-\sigma_0) + \rho_0^2 + j(2\sigma_0 - 1)\rho_0}{\Delta(\sigma_0,\rho_0)}. \tag{3.11}$$

By making the change of variable with s_0 fixed parameter:

$$\nu = \pi n^2 x \tag{3.12}$$

we obtain (see Appendix B):

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\{ (1/\pi n^2)^{s_0/2} \int_{\pi n^2}^{+\infty} \nu^{s_0/2-1} e^{-\nu}\, d\nu \right\} + \\ &+ \sum_{n=1}^{\infty} \left\{ (1/\pi n^2)^{(1-s_0)/2} \int_{\pi n^2}^{+\infty} \nu^{(1-s_0)/2-1} e^{-\nu}\, d\nu \right\} = \tfrac{1}{s(1-s)}; \end{aligned} \tag{3.13}$$

by denoting:

$$\Psi_1(s) = S\left[\frac{1}{s_0}\right] - \frac{1}{s_0} = \sum_{n=1}^{\infty} \left\{ (1/\pi n^2)^{s_0/2} \int_{\pi n^2}^{+\infty} \nu^{s_0/2-1} e^{-\nu}\, d\nu \right\} - \frac{1}{s_0}, \tag{3.14}$$

$$\Psi_2(s) = S\left[\tfrac{1}{1-s_0}\right] - \tfrac{1}{1-s_0} =$$

(3.15)
$$= \sum_{n=1}^{\infty}\left\{(1/\pi n^2)^{(1-s_0)/2}\int\limits_{\pi n^2}^{+\infty}\nu^{(1-s_0)/2-1}e^{-\nu}\,d\nu\right\} - \tfrac{1}{1-s_0},$$

and by following the notation in [4], with $\frac{1}{s}$ at the place of φ, in Σ, S: $\Sigma \to \Sigma$, the (3.13) can be written as:

(3.16)
$$\left[\Psi_1(s_0) + \Psi_2(s_0) + \frac{1}{s(1-s_0)}\right] = \frac{1}{s_0(1-s_0)},$$

and (2.3) as:

(3.17)
$$\tilde{\xi}(s_0) = \frac{1}{2} - \frac{1}{2}s_0(1-s_0)\left[\Psi_1(s_0) + \Psi_2(s_0) + \frac{1}{s_0(1-s_0)}\right]$$

We notice that:

(3.18)
$$\tilde{\xi}(1-s_0) = \frac{1}{2} - \frac{1}{2}(1-s_0)s_0\left[\Psi_1(1-s_0) + \Psi_2(1-s_0) + \frac{1}{(1-s_0)s_0}\right]$$

and that

(3.19)
$$\Psi_1(1-s_0) = \Psi_2(s_0)$$

(3.20)
$$\Psi_2(1-s_0) = \Psi_1(s_0),$$

whence we obtain:

(3.21)
$$\tilde{\xi}(1-s_0) = \frac{1}{2} - \frac{1}{2}(1-s_0)s_0\left[\Psi_2(s_0) + \Psi_1(s_0) + \frac{1}{(1-s_0)s_0}\right],$$

i.e.,

(3.22)
$$\tilde{\xi}(s_0) = \tilde{\xi}(1-s_0).$$

Then, in Σ we have

(3.23)
$$\tilde{\xi}(s_0) = 0,$$

if:

(3.24)
$$\Psi_1(s_0) + \Psi_2(s_0) = 0 \Rightarrow \Psi_1(s_0) + \Psi_1(1-s_0) = 0,$$

i.e.,:

(3.25)
$$S\left[\frac{1}{s_0}\right] - \frac{1}{s_0} + S\left[\frac{1}{1-s_0}\right] - \frac{1}{1-s_0} = 0.$$

Let $s = \frac{1}{2} + jt$, with $t \in \Sigma'$ then the (3.24) is: $\Psi_1\left(\frac{1}{2}+jt\right) + \Psi_1\left(\frac{1}{2}-jt\right) = 0.$

Lemma 3.1 (Principle of Reflection on the zeros). *$\tilde{\xi}\left(\frac{1}{2}+jt\right)$ is an analytic function in Σ' respect to $\Re \Leftrightarrow \forall t_0 \in \Omega_0^{'} \subset \Sigma' \Rightarrow t_0^* \in \Omega_0^{'}$.*

Proof. Let $\tilde{\xi}\left(\frac{1}{2}+jt\right)$ an analytic function and $t_0 \in \Omega_0^{'}$. Suppose that $t_0^* \notin \Omega_0^{'} \Rightarrow \tilde{\xi}\left(\frac{1}{2}+jt_0\right) = 0$ and $\tilde{\xi}\left(\frac{1}{2}+jt_0^*\right)^* \neq 0 \Rightarrow \tilde{\xi}\left(\frac{1}{2}+jt_0\right) \neq \tilde{\xi}\left(\frac{1}{2}+jt_0^*\right)^*$ is not analytic function against the hypothesis. Therefore $t_0^* \in \Omega_0^{'}$. □

Lemma 3.2. *Let $t \in \Sigma'$, $\tilde{\xi}\left(\frac{1}{2}+jt\right) = A(t) \times \left[\Psi_1\left(\frac{1}{2}+jt\right) + \Psi_1\left(\frac{1}{2}-jt\right)\right]$, with: $A(t) \neq 0$, $\Psi_1\left(\frac{1}{2}+jt\right)$ and $\Psi_1\left(\frac{1}{2}-jt\right)$ analytic functions in $\Sigma^{'}$.*
$\forall t_0 \in \Omega_0^{'}$, $\tilde{\xi}\left(\frac{1}{2}+jt_0\right) = 0 \Leftrightarrow \Psi_1\left(\frac{1}{2}+jt_0\right) = 0$, $\Psi_1\left(\frac{1}{2}-jt_0\right) = 0$.

Proof. Let $t_0 \in \Omega_0^{'}$. Being $\tilde{\xi}\left(\frac{1}{2}+jt\right)$, in Σ', an analytic function, for Lemma 1, it is satisfaction the Schwarz's reflection principle. $\tilde{\xi}\left(\frac{1}{2}+jt_0\right) = 0$ iff:
$\Psi_1\left(\frac{1}{2}+jt_0\right) + \Psi_1\left(\frac{1}{2}-jt_0\right) = 0$ and $\Psi_1\left(\frac{1}{2}+jt_0^*\right) + \Psi_1\left(\frac{1}{2}-jt_0^*\right) = 0$,
i.e.:

$$\Psi_1\left(\frac{1}{2}+jt_0\right) + \Psi_1\left(\frac{1}{2}+jt_0^*\right) + \Psi_1\left(\frac{1}{2}-jt_0\right) + \Psi_1\left(\frac{1}{2}-jt_0^*\right) = 0.$$

This complex equation has two component in the two variables real and immaginary part of t_0. The imaginary component of the complex equation is identically null while its real component it is indeterminate. Therefore, $\tilde{\xi}\left(\frac{1}{2}+jt_0\right) = 0$ iff:

$$\Psi_1\left(\frac{1}{2}+jt_0\right) = 0, \tag{3.26}$$

and

$$\Psi_1\left(\frac{1}{2}-jt_0\right) = 0. \tag{3.27}$$

□

As seen in Figures 1 and 2 obtained by means of the software Mathematica, the reflection principle is satisfied.

Theorem 3.3 (The new equation). *Let s_0 be a zero of the $\tilde{\xi}(s)$ in the strip $\Sigma = (0,1) \times ((-j\infty, 0) \cup (0, +j\infty))$ of the complex plane. T then the real part of s_0, denoted with σ_0, is equal to 1/2, whereas the imaginary part, denoted with ρ_0, satisfies the new equation:*

$$\sum_{n=1}^{\infty} \frac{\Gamma_R(1/4 + j\rho_0/2, \pi n^2) cos[\alpha_n(\rho_0)] + \Gamma_I(1/4 + j\rho_0/2, \pi n^2) sin[\alpha_n(\rho_0)]}{(\pi n^2)^{1/4}} = \frac{2}{1+4\rho_0^2}$$

with:

$$\alpha_n(\rho_0) = \rho_0 log(\sqrt{\pi n^2})$$

and:

$$\Gamma_R(1/4 + j\rho_0/2, \pi n^2) = \Re\left\{\Gamma(1/4 + j\rho_0/2, \pi n^2)\right\} = \Re\left\{\int_{\pi n^2}^{+\infty} \nu^{-3/4 + j\rho_0/2} e^{-\nu}\, d\nu\right\}$$

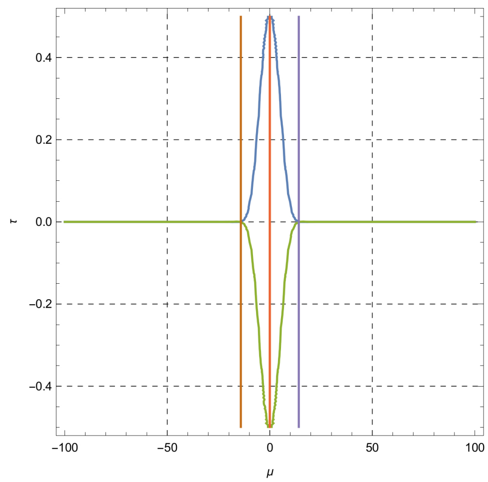

Figure 1: (Level curves) $\Psi_1\left(\frac{1}{2}+jt\right)=0$ and $\Psi_1\left(\frac{1}{2}-jt\right)=0$, it is observed the straight line intercepts at the first positive and negative zero of the zeta function: $\mu=\pm 14.1347$ and $\tau=0$

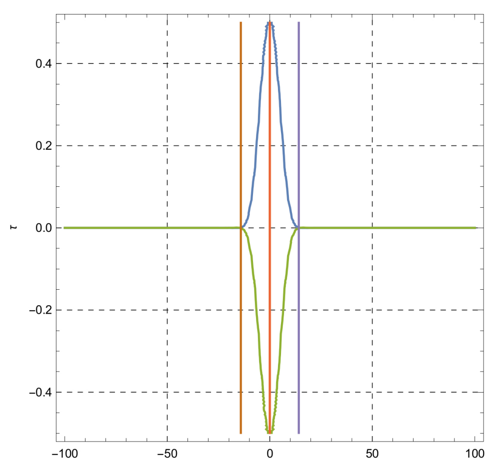

Figure 2: (Level curves) $\Psi_1\left(\frac{1}{2}+jt^*\right)=0$ and $\Psi_1\left(\frac{1}{2}-jt^*\right)=0$, it is observed the straight line intercepts at the first positive and negative zero of the zeta function: $\mu=\pm 14.1347$ and $\tau=0$

$$\Gamma_I(1/4 + j\rho_0/2, \pi n^2) = \Im\left\{\Gamma(1/4 + j\rho_0/2, \pi n^2)\right\} = \Im\left\{\int_{\pi n^2}^{+\infty} \nu^{-3/4+j\rho_0/2} e^{-\nu}\, d\nu\right\}.$$

Proof. From Lemma 2, the analytic expression of a generic zero, s_0, of the auxiliary function, $\tilde{\xi}(s)$, is obtained by searching among the united elements of the functional transformation S, where S is a contraction mapping (see Appendix C), i.e. zeros of the functional equation:

$$\Psi_1(s_0) = T\left[\frac{1}{s_0}\right] = 0, \tag{3.28}$$

where:

$$T\left[\frac{1}{s_0}\right] = S\left[\frac{1}{s_0}\right] - \frac{1}{s_0}. \tag{3.29}$$

For simmetry we obtain the zeros of the $\Psi_1(1 - s_0) = 0$. If s_0 is a zero, also (1-s_0), s_0^* and (1-s_0^*) are zeros of $\tilde{\xi}(s)$, then:

i) $S\left[\frac{1}{s_0^*}\right] = \frac{1}{s_0}$, is true iff $\rho_0 = 0$

ii) $S\left[\frac{1}{1-s_0^*}\right] = \frac{1}{s_0}$, is true iff $\sigma_0 = \frac{1}{2}$

iii) $S\left[\frac{1}{1-s_0}\right] = \frac{1}{s_0}$, is true iff $(\sigma_0 = \frac{1}{2}) \cap (\rho_0 = 0)$

Since $\forall \sigma \in (0,1) \Rightarrow s = \sigma \notin \Sigma \Rightarrow \rho_0 \neq 0$, by extending the Caccioppoli's theorem on the unit elements of the functional transformations [4], S admits a unique solution s_0, therefore result:

$$S\left[\frac{1}{s_0}\right] = S\left[\frac{1}{1 - s_0^*}\right] = \frac{1}{s_0} \tag{3.30}$$

as seen in Figures 3, 4, 5 and 6, and (3.25) can be written as:

$$\sum_{n=1}^{\infty} \left\{ (1/\pi n^2)^{s_1} \int_{\pi n^2}^{+\infty} \nu^{s_1 - 1} e^{-\nu}\, d\nu + (1/\pi n^2)^{s_1^*} \int_{\pi n^2}^{+\infty} \nu^{s_1^* - 1} e^{-\nu}\, d\nu \right\} = \frac{1}{s_0} + \frac{1}{1 - s_0}, \tag{3.31}$$

where:

$$s_1 = \frac{1 - s_0^*}{2} = \frac{1}{2}(1 - \sigma_0) + \frac{1}{2} j\rho_0, \tag{3.32}$$

$$s_1^* = \frac{1 - s_0}{2} = \frac{1}{2}(1 - \sigma_0) - \frac{1}{2} j\rho_0. \tag{3.33}$$

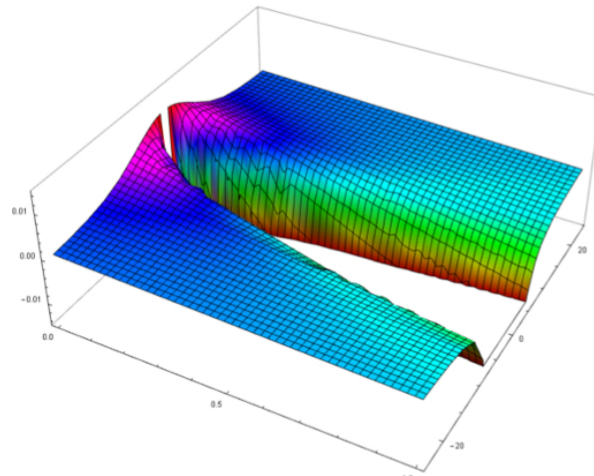

Figure 3: $\Re\{S[1/s_0]\} = \Re[1/s_0]$

It is noteworthy that in the (3.31) the integrals are upper incomplete Gamma functions ([10], [19]). For:

$$\Gamma(s_1, \pi n^2) = \int_{\pi n^2}^{+\infty} \nu^{s_1 - 1} e^{-\nu}\, d\nu = \Gamma_R(s_1, \pi n^2) + j\Gamma_I(s_1, \pi n^2) \tag{3.34}$$

and

$$u(n, s_1) = \left(\frac{1}{\pi n^2}\right)^{s_1}, \tag{3.35}$$

the relation (3.31) can be written as follows:

$$\frac{1}{2}\sum_{n=1}^{\infty} [\Gamma(s_1, \pi n^2)u(n, s_1) + \Gamma^*(s_1, \pi n^2)u^*(n, s_1)] = \Phi_R(\sigma_0, \rho_0) + j\Phi_I(\sigma_0, \rho_0), \tag{3.36}$$

with: $\Gamma_R(s_1, \pi n^2)$, $\Gamma_I(s_1, \pi n^2) \in \Re$; by observing that:

$$\Gamma({s_1}^*, \pi n^2) = \Gamma^*(s_1, \pi n^2),$$

the relation (3.36) can be written as:

$$\sum_{n=1}^{\infty} \left\{\Re[\Gamma(s_1, \pi n^2)u(n, s_1)] - \Im[\Gamma(s_1, \pi n^2)u(n, s_1)]\right\} = \Phi_R(\sigma_0, \rho_0) + j\Phi_I(\sigma_0, \rho_0). \tag{3.37}$$

By substituting (3.34) and (3.35) into (3.36), we infer:

$$\sum_{n=1}^{\infty} \frac{\Gamma_R(s_1, \pi n^2) cos[\alpha_n(\rho_0)] + \Gamma_I(s_1, \pi n^2) sin[\alpha_n(\rho_0)]}{(\pi n^2)^{\Re(s_1)}} = \Phi_R(\sigma_0, \rho_0) + j\Phi_I(\sigma_0, \rho_0), \tag{3.38}$$

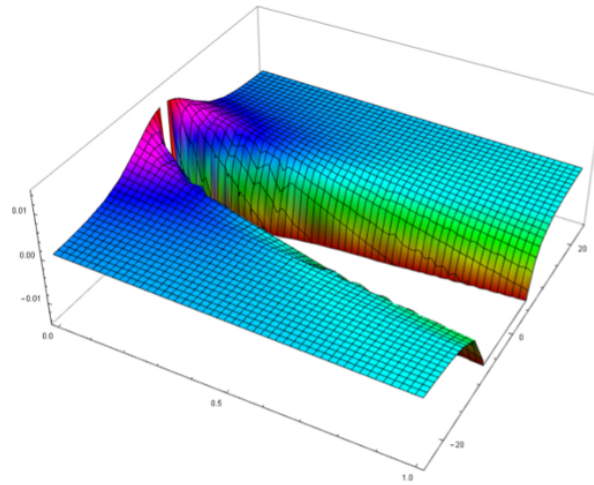

Figure 4: $\Re\{S[1/(1-s_0^*)]\} = \Re[1/s_0]$

where:

$$\alpha_n(\rho_0) = \alpha(\pi n^2; \rho_0) = \rho_0 \log(\sqrt{\pi n^2}). \tag{3.39}$$

Then (3.38) represents the equation of the zeros of the $\tilde{\xi}(s)$ function whose first member is a real function. Moreover, we have:

$$\Gamma_R({s_1}^*, \pi n^2) = \Gamma_R(s_1, \pi n^2),$$

$$\Gamma_I({s_1}^*, \pi n^2) = -\Gamma_I(s_1, \pi n^2),$$

$$\cos[\alpha_n(-\rho_0)] = \cos[\alpha_n(\rho_0)],$$

$$\sin[\alpha_n(-\rho_0)] = -\sin[\alpha_n(\rho_0)],$$

$$\Phi_R(\sigma_0, -\rho_0) = \Phi_R(\sigma_0, \rho_0),$$

$$\Phi_I(\sigma_0, -\rho_0) = -\Phi_I(\sigma_0, \rho_0).$$

Therefore, from (3.38), the zeros of the function $\tilde{\xi}(s)$ in the strip Σ can be obtained by resolving the system of the following two real equations in σ_0 and ρ_0:

$$\sum_{n=1}^{\infty} \frac{\Gamma_R(s_1, \pi n^2) cos[\alpha_n(\rho_0)] + \Gamma_I(s_1, \pi n^2) sin[\alpha_n(\rho_0)]}{(\pi n^2)^{\Re(s_1)}} = \Phi_R(\sigma_0, \rho_0) \tag{3.40}$$

$$0 = \Phi_I(\sigma_0, \rho_0), \tag{3.41}$$

For (3.4), we notice that (3.41) becomes:

$$0 = \frac{(2\sigma_0 - 1)\rho_0}{2\left\{[\sigma_0(1-\sigma_0) + \rho_0^2]^2 + (2\sigma_0 - 1)^2 \rho_0^2\right\}} \tag{3.42}$$

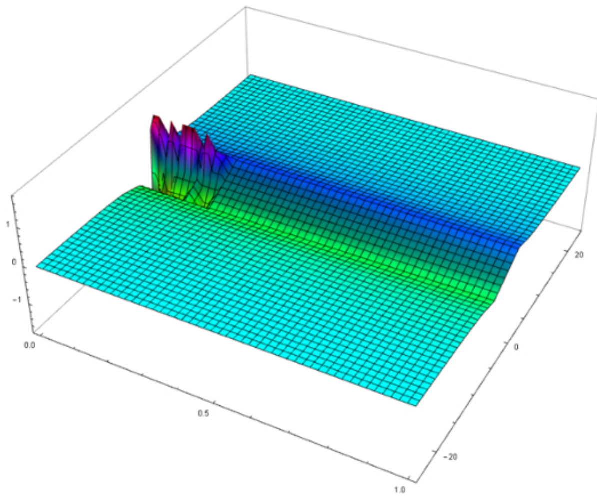

Figure 5: $\Im\{S[1/s_0]\} = \Im[1/s_0]$

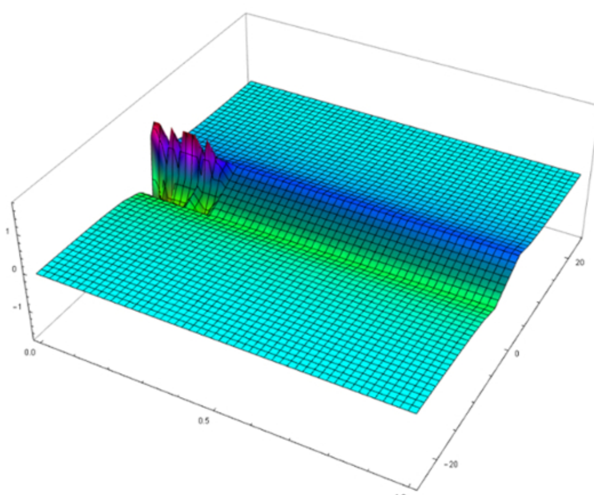

Figure 6: $\Im\{S[1/(1-s_0^*)]\} = \Im[1/s_0]$

and (3.42) is satisfied $\forall \rho_0 \in \mathbb{R} - \{0\} \Leftrightarrow \sigma_0 = 1/2$. Therefore, the system admits real solutions if and only if: $\sigma_0 = 1/2$ and for all ρ_0 values that satisfy:

(3.43)
$$\sum_{n=1}^{\infty} \frac{\Gamma_R(1/4 + j\rho_0/2, \pi n^2) cos[\alpha_n(\rho_0)] + \Gamma_I(1/4 + j\rho_0/2, \pi n^2) sin[\alpha_n(\rho_0)]}{(\pi n^2)^{1/4}} = \frac{2}{1 + 4\rho_0^2}.$$

Consequently, the Theorem is proved. □

Corollary 3.4. *All the zeros of the function $\zeta(s)$ in the strip Σ have real part equal to 1/2.*

Proof. Let s_0 be a generic zero of the Riemann zeta function:

$$\zeta(s_0) = 0.$$

For (2.3), we have:

$$\tilde{\xi}(s_0) = 0$$

For Theorem 1 we have that the real part s_0 is equal to 1/2: $\sigma_0 = 1/2$. Therefore, RH is proved. □

It can be noticed that the equation (3.24) is equivalent to:

$$|\Psi_1(\sigma_0 + j\rho_0)| = |\Psi_1(1 - \sigma_0 - j\rho_0)| \tag{3.44}$$

$$\angle\Psi_1(\sigma_0 + j\rho_0) - \angle\Psi_1(1 - \sigma_0 - j\rho_0) = (2k+1)\pi \tag{3.45}$$

with $k \in \mathbb{Z}$ and where $\angle\Psi_1$ is argument of Ψ_1, i.e., the difference of arguments is a multiple odd of π .

In Σ , $|\Psi_1(\sigma_0 + j\rho_0)|$ is a monodroma function, therefore, for reflection's principle $|\Psi_1(\sigma_0 + j\rho_0)| = |\Psi_1(\sigma_0 - j\rho_0)|$, the (3.44) is satisfied iff $1 - \sigma_0 = 1/2$:

$$|\Psi_1(\sigma_0 + j\rho_0)| = |\Psi_1(1 - \sigma_0 - j\rho_0)| \Leftrightarrow \sigma_0 = \frac{1}{2}, \tag{3.46}$$

i.e.,

$$|\Psi_1(1/2 + j\rho_0)| = |\Psi_1(1/2) - j\rho_0)| \tag{3.47}$$

is always satisfied, $\forall \rho_0 \in \mathbb{R} - \{0\}$. Therefore the equation of zeros is:

$$\angle\Psi_1(1/2 + j\rho_0) - \angle\Psi_1(1/2 - j\rho_0) = (2k+1)\pi, \tag{3.48}$$

i.e., it is equivalent at the new equation although it is formally different.

4 Calculating the first zeros of the Riemann zeta function through the equation of the zeros of the auxiliary function.

The equation of the zeros of the $\tilde{\xi}(s)$ function can be solved by truncating the series at the first member by means of a finite number of terms M, since it quickly converges to 0 for all the values of ρ greater than 0. This can be checked by setting M = 10 and varying ρ. The zeros are obtained at each phase inversion of π ([6], [7]), by means of the following formula:

$$\tilde{d} = log[d(\rho)] = log\,|d(\rho)| + log\,\{sgn[d(\rho)]\} = log\,|d(\rho)| + \frac{j\pi}{2}\,\{1 - sgn[d(\rho)]\}\,, \tag{4.1}$$

where d(ρ) is:

(4.2)

$$\sum_{n=1}^{M} \frac{\Gamma_R(1/4 + j\rho/2, \pi n^2)cos[\alpha_n(\rho)] + \Gamma_I(1/4 + j\rho/2, \pi n^2)sin[\alpha_n(\rho)]}{(\pi n^2)^{1/4}} - \frac{2}{1 + 4\rho^2}.$$

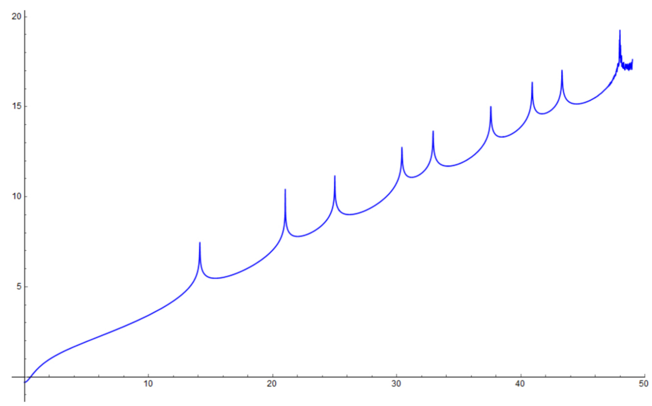

Figure 7: $-Log_{10}[\mathrm{d}(\rho)]$

As Figure 7 shows, the first nine zeros of the auxiliary function are the same of the first nine zeros of the Riemann zeta function. The trend of the graph was obtained by means of the software Mathematica and the values with approximation to the fourth decimal place were compared with the tables reported in ([15], [20]). The first zeros of the Riemann zeta function were calculated only with the attempt to verify the correctness of the equation. The trend of the curve in Figure 7 is similar to the trend of the function given by Sierra and Rodriguez-Laguna ([17]) based on the physical theory of Berry ([1]) and Galindo ([9]). In Figure 8, ρ ranges between 40 and 45. It is noteworthy the occurrence of two cuts of the real axis s corresponding to the zeros $\rho_7 = 40.9187$ e $\rho_8 = 43.3270$, respectively. For $\rho >$ 49, since the zeros of the auxiliary function are much more and denser than the zeros of the Riemann zeta function (because d(ρ) quickly converges to zero), it is not possible to distinguish them (Figs. 9, 10, 11). In fact, according to the equation, as can be seen from the Figure 12 and 13, for ρ_0 greater than 49, the function $\tilde{\xi}(s)$ is next to zero on all points of the straight line of equation $\sigma = 1/2$. If ρ_0 is 49 then:

$$\tilde{\xi}(0.5 + j49) = -1.36914 \times 10^{-14} - j4.97901 \times 10^{-29}$$

and

$$\zeta(0.5 + j49) = 0.666418 - j \times 0.203663.$$

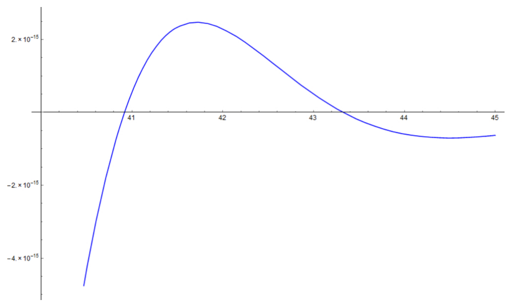

Figure 8: Two zeros of the Riemann zeta function

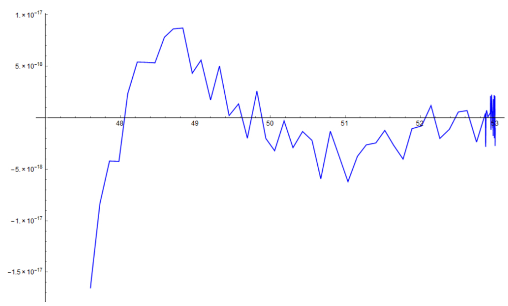

Figure 9: Trend for ρ ranging between 47 and 53 and M=10

The zeros of the Riemann zeta function can be calculated with high precision by using different techniques such as the Odlyzko - A. M. Schonhage algorithm ([13]) or R.B. Paris [14], for very high values of ρ.

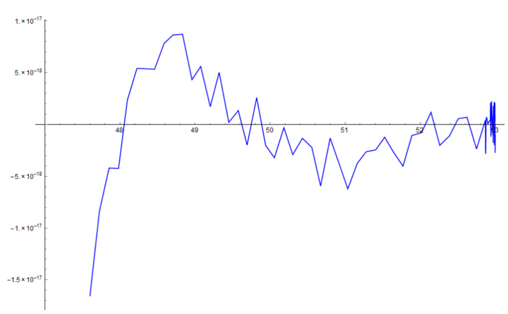

Figure 10: Trend for ρ ranging between 47 and 53 and M=50

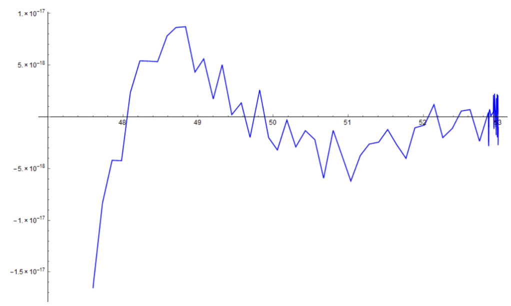

Figure 11: Trend for ρ ranging between 47 and 53 and M=100

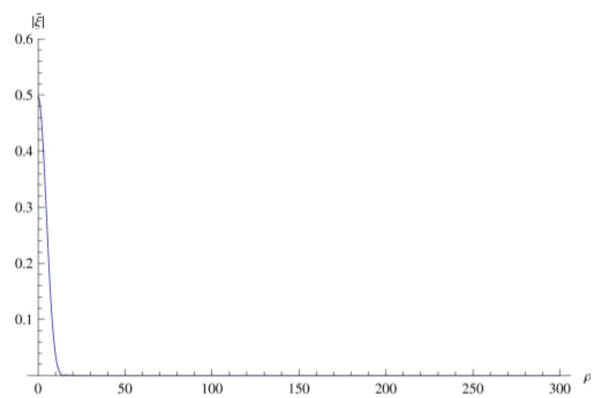

Figure 12: Trend of $|\tilde{\xi}(0.5 + j\rho)|$ for ρ ranging between 0 and 300

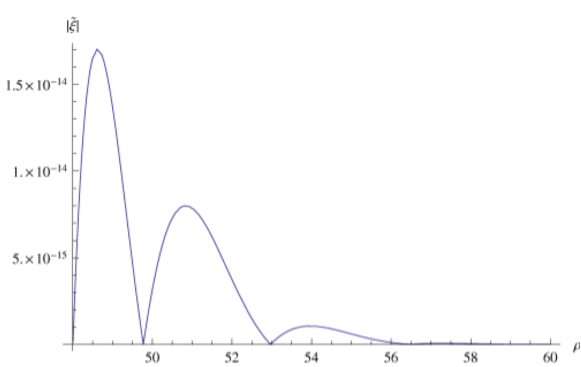

Figure 13: Trend of $|\tilde{\xi}(0.5 + j\rho)|$ for ρ ranging between 48 and 60

5 Link between $\tilde{\xi}(s)$ and $\xi(t)$

The function $\tilde{\xi}(s)$ in (2.3) was defined by using the following integral equation ([16]):

$$\Pi(s/2-1)(\pi)^{-s/2}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{+\infty} \psi(x)[x^{(s/2-1)} + x^{-(s+1)/2}]\,dx, \tag{5.1}$$

where $\Pi(s/2-1)$ is the gamma function in Euler's notation corresponding to Legendre's notation:

$$\Gamma(s/2) = \Pi(s/2-1). \tag{5.2}$$

In his paper, Riemann ([16]) introduces the function $\xi(t)$ of the complex variable t. By means of the gamma function expressed in Legendre's notation, we have:

$$\xi(t) = \Gamma(s/2+1)(s-1)(\pi)^{-s/2}\zeta(s). \tag{5.3}$$

By comparing the (2.3) and (5.3), we can state that the function $\tilde{\xi}(s)$ is linked to the Riemann function $\xi(t)$ by means of the following equation:

$$\xi(t) = \left[\frac{2\Gamma(s/2+1)}{s\;\Gamma(s/2)}\right]\;\tilde{\xi}(s). \tag{5.4}$$

By using the functional equation of the gamma function,

$$\Gamma(s/2+1) = \left(\frac{s}{2}\right)\Gamma(s/2), \tag{5.5}$$

we obtain:

$$\xi(t) = \tilde{\xi}(s) \tag{5.6}$$

with:

$$s = 1/2 + jt, \tag{5.7}$$

as stated by Bombieri ([3]). From (5.7), we have:

$$t = \rho + j(1/2 - \sigma) \tag{5.8}$$

since

$$s = \sigma + j\rho.$$

6 Comparison with Riemann-Siegel Z formula

By using the method described in section 3 to prove the RH, the auxiliary function $\tilde{\xi}(s)$ along the critical strip can be expressed through a series of upper incomplete gamma functions of the variable ρ:

$$\tilde{\xi}\left(\frac{1}{2} + j\rho\right) = \frac{1}{2} - \frac{1}{4}(1+4\rho^2)\Omega(\rho), \tag{6.1}$$

where:

$$\Omega(\rho) = \sum_{n=1}^{\infty} \frac{\Gamma_R(1/4 + j\rho/2, \pi n^2) cos[\alpha_n(\rho)] + \Gamma_I(1/4 + j\rho/2, \pi n^2) sin[\alpha_n(\rho)]}{(\pi n^2)^{1/4}}. \tag{6.2}$$

By replacing equation (6.1) in (2.3), we obtain the equivalent representation of the Riemann-Siegel Z function:

$$\zeta\left(\frac{1}{2} + j\rho\right) = \tilde{Z}(\rho) e^{-j\tilde{\Theta}(\rho)}, \tag{6.3}$$

where:

$$\tilde{Z}(\rho) = 2\pi^{1/4} \left[\frac{\Omega(\rho) - \frac{2}{(1+4\rho^2)}}{\mid \Gamma(1/4 + j\rho/2) \mid} \right] \tag{6.4}$$

and:

$$\tilde{\Theta}(\rho) = arg[\Gamma(1/4 + j\rho/2)] - \frac{\rho}{2} log(\pi). \tag{6.5}$$

The Figure 14, obtained by means of the software Mathematica, shows that for $0 \leq \rho \leq 48$ the graph of the representation $\tilde{Z}(\rho)$ is the same of the graph of the Riemann-Siegel $Z(\rho)$ formula. In Figure 15, the trend of $\tilde{Z}(\rho)$ differ from the trend of $Z(\rho)$

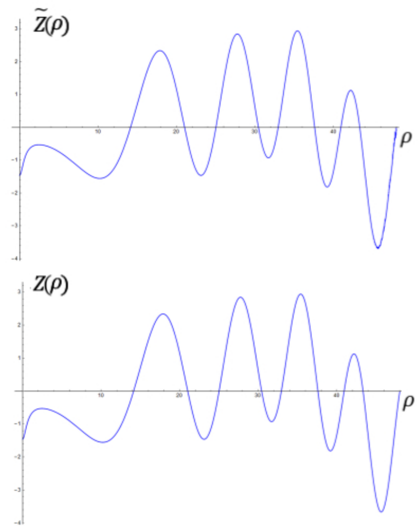

Figure 14: Comparison with Riemann-Siegel formula $Z(\rho)$ and $0 \leq \rho \leq 48$

for $\rho > 49$ because such representation tends to a 0/0 indeterminate form.

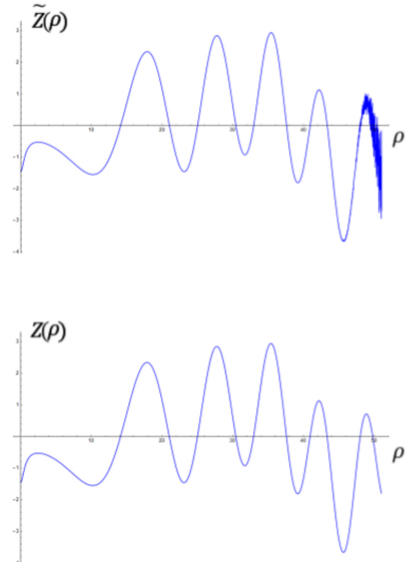

Figure 15: Comparison with Riemann-Siegel formula $Z(\rho)$ and $0 \leq \ \rho \leq \ 51$

7 Conclusions

In this work, we demonstrated that the real part of the zeros of the auxiliary function $\tilde{\xi}(s)$, for $\rho_0 \leq 49$, is equal to 1/2: $s_0 = 1/2 + \mathrm{j}\rho_0$ as well as the zeros of the Riemann zeta function. For ρ_0 greater than 49, the function $\tilde{\xi}(s)$ is close to zero at all points of the straight line of equation: $\sigma = 1/2$. Therefore RH was proved as Corollary 1 of Theorem 1. Moreover, it was demonstrated that the zeros of the Riemann function $\xi(t)$ are real and they are the same of the imaginary part of the zeros of the Riemann zeta function. In fact, if $1/2 + \mathrm{j}\rho_0$ is a generic zero of $\zeta(s)$, for (5.8), then $t_0 = \rho_0$.

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Appendices A-C

A

By (2.4) we have:

$$\int_{1}^{+\infty} \psi(x)[x^{(s_0/2-1)} + x^{-(s_0+1)/2}]\,dx = \frac{1}{s_0(1-s_0)}. \tag{A.1}$$

Then:

$$\frac{1}{s_0(1-s_0)} = \frac{1}{[(1-\sigma_0)\sigma_0 + \rho_0^2 + j\rho(1-2\sigma_0)]}, \tag{A.2}$$

i.e.,
(A.3)
$$\frac{1}{s_0(1-s_0)} = \frac{(1-\sigma_0)\sigma_0+\rho_0^2}{[(1-\sigma_0)\sigma_0+\rho_0^2]^2+\rho_0^2(1-2\sigma_0)^2]} + j\frac{\rho_0(2\sigma_0-1)}{[(1-\sigma_0)\sigma_0+\rho_0^2]^2+\rho_0^2(1-2\sigma_0)^2]}$$

Let:

$$x^{(s_0/2-1)} + x^{-(s_0+1)/2} = x^{-b}(x^a + x^{-a}); \tag{A.4}$$

by resolving respect to a and b, we obtain:

$$a = \frac{s_0}{2} - \frac{1}{4}; b = \frac{3}{4}. \tag{A.5}$$

Then

$$\int_1^{+\infty} \psi(x)[x^{(s_0/2-1)} + x^{-(s_0+1)/2}]\,dx = \int_1^{+\infty} \psi(x)x^{-3/4}[x^{(s_0/2-1/4)} + x^{-(s_0/2-1/4)}]\,dx \tag{A.6}$$

By observing that $x^a = e^{alog(x)}$, we obtain:

$$\begin{aligned}&\int_1^{+\infty} \psi(x)[x^{(s_0/2-1)} + x^{-(s_0+1)/2}]\,dx = \\ &= 2\int_1^{+\infty} \psi(x)x^{-3/4}cos[j/2(s_0-1/2)log(x)]\,dx\end{aligned} \tag{A.7}$$

where:

$$\begin{aligned}&cos[j/2(s_0-1/2)log(x)] = cosh\,[(\sigma_0-1/2)log(\sqrt{x})]\,cos\,[\rho_0 log(\sqrt{x})] + \\ &+ jsinh\,[(\sigma_0-1/2)log(\sqrt{x})]\,sin\,[\rho_0 log(\sqrt{x})]\end{aligned} \tag{A.8}$$

For (A.3), (A.7) and (A.8), we obtain (3.1) and (3.2).

B

By (A.8), we infer:

$$\phi(x;s_0) = j/2(\sigma_0 + j\rho_0 - 1/2)log(x) = -\rho_0/2log(x) + j/2(\sigma_0-1/2)log(x) \tag{B.1}$$

$$e^{j\phi(x;s_0)} = e^{j\rho_0/2log(x)-1/2(\sigma_0-1/2)log(x)} = x^{-s_0/2}x^{1/4} \tag{B.2}$$

$$e^{-j\phi(x;s_0)} = e^{-j\rho_0/2log(x)+1/2(\sigma_0-1/2)log(x)} = x^{s_0/2}x^{-1/4}. \tag{B.3}$$

From the first member of the equation (3.11), considering the integral term:

$$2\int_1^{+\infty} e^{-\pi n^2 x}x^{-3/4}cos[\phi(x;s_0)]\,dx = \int_1^{+\infty} e^{-\pi n^2 x}x^{-3/4}[e^{j\phi(x;s_0)} + e^{-j\phi(x;s_0)}]\,dx \tag{B.4}$$

we obtain:

$$\int_{1}^{+\infty} e^{-\pi n^2 x} x^{-3/4} e^{j\phi(x;s_0)}\, dx = \int_{\pi n^2}^{+\infty} e^{-\pi n^2 x} x^{-1/2} x^{-s_0/2}\, dx \tag{B.5}$$

and

$$\int_{1}^{+\infty} e^{-\pi n^2 x} x^{-3/4} e^{-j\phi(x;s_0)}\, dx = \int_{\pi n^2}^{+\infty} e^{-\pi n^2 x} x^{-1} x^{s_0/2}\, dx. \tag{B.6}$$

By using (3.12), we obtain:

$$\int_{1}^{+\infty} e^{-\pi n^2 x} x^{-3/4} e^{j\phi(x;s_0)}\, dx = \left(\frac{1}{\pi n^2}\right)^{(1-s_0)/2} \int_{\pi n^2}^{+\infty} e^{-\nu} \nu^{(1-s_0)/2}\, dx \tag{B.7}$$

and

$$\int_{1}^{+\infty} e^{-\pi n^2 x} x^{-3/4} e^{-j\phi(x;s_0)}\, dx = \left(\frac{1}{\pi n^2}\right)^{s_0/2} \int_{\pi n^2}^{+\infty} e^{-\nu} \nu^{s_0/2-1}\, dx. \tag{B.8}$$

C

Caccioppoli's theorem:

Let Σ a functional metric space, S a transformation of the $\Sigma \to \Sigma$ on Σ:

$$\forall\ \varphi \in \Sigma,\ \exists\ S[\varphi] \in \Sigma$$

and T a transformation of the Σ in $\Sigma\,'$:

$$\forall\ (\varphi, S[\varphi]),\ \exists\ T[\varphi] = \varphi - S[\varphi].$$

If the following conditions are met:

1) S is completely continuous
2) T is locally invertible
3) T trasforms the infinity in the infinity

then the functional equation:

$$\varphi = S[\varphi]$$

admits a unique solution.

Let Σ a complex functional metric space. Replacing φ with $1/s$, we define the distance in Σ:

$$d\left(\frac{1}{s_1}, \frac{1}{s_2}\right) = \left|\frac{1}{s_1} - \frac{1}{s_2}\right| \tag{C.1}$$

and

$$d\left(S\left[\frac{1}{s_1}\right], S\left[\frac{1}{s_2}\right]\right) = \left|S\left[\frac{1}{s_1}\right] - S\left[\frac{1}{s_2}\right]\right|. \tag{C.2}$$

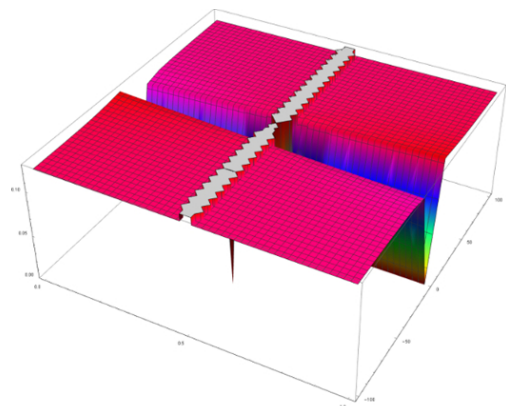

Figure 16: Lipschitz constant - c = 0.126

By applying the Caccioppoli's theorem extended to the complex field, S is a contraction mapping if, $\forall (s_1, s_2) \in \Sigma$, with $s_1 \neq s_2$, we have:

$$\left| S\left[\frac{1}{s_1}\right] - S\left[\frac{1}{s_2}\right] \right| \leq c \left| \frac{1}{s_1} - \frac{1}{s_2} \right| \tag{C.3}$$

with $0 \ < c \ < 1$. c is said a *Lipschitz constant*:

$$c = \sup_{\forall (s_1, s_2) \in \Sigma} \left[\frac{d\left(S\left[\frac{1}{s_1}\right], S\left[\frac{1}{s_2}\right]\right)}{d\left(\frac{1}{s_1}, \frac{1}{s_2}\right)} \right]$$

Let $\sigma_2 = 1 - \sigma_1, \rho_2 = \rho_1 + 1 - 2\sigma_1$, then c = 0.126, as see in Figure 16. Let:

$$a_n(s) = \pi^{-s/2}\Gamma(s/2, \pi n^2) \tag{C.4}$$

and

$$a_n(1-s) = \pi^{-(1-s)/2}\Gamma[(1-s)/2, \pi n^2]. \tag{C.5}$$

Then

$$S\left[\frac{1}{s}\right] = \int_1^{+\infty} \psi(x) x^{(s/2-1)}\, dx = \sum_{n=1}^{\infty} \frac{a_n(s)}{n^s} \tag{C.6}$$

and

$$S\left[\frac{1}{1-s}\right] = \int_1^{+\infty} \psi(x) x^{(-s/2-1/2)}\, dx = \sum_{n=1}^{\infty} \frac{a_n(1-s)}{n^{1-s}}. \tag{C.7}$$

We obtain:

$$\sum_{n=1}^{\infty} \frac{a_n(s)}{n^s} + \sum_{n=1}^{\infty} \frac{a_n(1-s)}{n^{1-s}} = \frac{1}{s} + \frac{1}{1-s}. \tag{C.8}$$

This is the equation of the auxiliary zeta function in the form generalized and extended Dirichlet series:

$$\tilde{L}[s, a(s)] = \sum_{n=1}^{\infty} \frac{a_n(s)}{n^s}. \tag{C.9}$$

Therefore (C.8) rewrites:

$$\tilde{L}[s, a(s)] + \tilde{L}[1-s, a(1-s)] = \frac{1}{s} + \frac{1}{1-s}. \tag{C.10}$$

Author's address:

Bruno Felice Filippo Flora
Via Matteotti 39, Locri (RC), 89044, Italy.
E-mail bruno.flora@pecprofessionisti.com