A special relativistic approach of non equilibrium thermodynamics with internal variables

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Abstract. The extension of the Kluitenberg's theory to the special relativistic case for thermo-mechanical model with internal variables is presented. In particular, as in the classical scheme, it is assumed that the entropy density in an inertial frame of reference will depend on the density of internal energy, on the total strain tensor and on a tensorial internal variables. After having introduced the relativistic equilibrium stress tensor, the relativistic viscous stress tensor and the relativistic memory stress tensor, the expression for the entropy production is obtained.

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1 Introduction

It is well known that in non-equilibrium thermodynamics models the classical variables are not sufficient to describe some internal phenomena which occur inside the medium [10, 11]. Therefore these phenomena can be studied by introducing some additional variables, so called "internal variables" [12]. Several authors [2, 17] include internal variables in order to describe the behavior of materials in thermomechanics, for example materials with microstructure. Two different types of internal variables are distinguished: one is the internal degrees of freedom and the other one is internal variables of state that, differently from the first one, has not inertia and does not produce external work [1]. In this paper we will follow the terminology introduced in the Kluitenberg's framework that considers the internal variables such that its substantial time derivatives does not occur in the first law of thermodynamics [12]. Sometimes internal variables or internal degrees of freedom are introduced without thermodynamic basis under different denomination.

In this context, by mathematical point of view, it is useful to consider the entropy function depending of these variables (in addition to the usual variables), even if from physical point of view it is very difficult to associate them to a particular phenomena. On the other hand, several questions arise as: what are the variables that, from a physical point of view, has to be considered? How is it possible to define them? In

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some models it is not specified the physical nature of these variables and they are not defined, but they are used in an appropriate way relatively to the obtained results. The aim of this paper is to consider only mechanical phenomena in order to extend to the special relativistic framework the thermo-mechanical Kluitenberg's theory [10]. In agreement with Kluitenberg's theory, we introduce some internal variables such a way that the constitutive functions and, in particular, the entropy function depends on them.
It is assumed that the entropy will depend on the internal energy, the total strain ϵ_{ik} and some "internal" tensorial variable $\mathbf{\Omega}$ [10, 11] . By introducing a reference state R (for the medium), [11], and the Gibbs potential it is possible to split the total strain tensor into two parts $\epsilon_{ik}^{(0)}$ and $\epsilon_{ik}^{(i)}$ with $\epsilon_{ik} = \epsilon_{ik}^{(0)} + \epsilon_{ik}^{(i)}$; the term $\epsilon_{ik}^{(0)}$ vanishes in the state R while $\epsilon_{ik}^{(i)}$ is assumed as the internal variable. In other words $\epsilon_{ik}^{(i)}$ takes place of Ω_{ik}. In such a way, it is assumed that the internal variable is an independent variable in the scheme of entropy. It is important to remark that the introduction of the state R allows to the definition of $\epsilon_{ik}^{(i)}$ as the finding of ϵ_{ik} in this state. After the introduction of the two parts $\epsilon_{ik}^{(0)}$ and $\epsilon_{ik}^{(i)}$ in which ϵ_{ik} can be split, it is possible to define some tensorial quantities related to them which do not appear in previous models.
Moreover, it is important to observe that many other models can be obtained from this as particular case. The extension to relativistic case involves the fundamental problems concerning the transformation of the strain tensor in a frame reference changing and the way in which it can be splitted. The first question has been solved in a previous paper [4, 5] the second one is very important in order to compare with the classical Kluitenberg's theory and it will be solved in the section 4. This allows to the introduction of quantities which appear in classical model by a special relativistic point of view [8, 21].
In the last years the Kluitenberg's non equilibrium thermodynamics has been further developed by us, in [3, 6], in order to find some correlations between fundamental entities of the theory and directly experimental measurable functions. This leads to the experimental evaluation of some predicted results and some physical phenomena described by phenomenological and state coefficients appearing in the theory. Let us observe that $\epsilon_{ik}^{(0)}$ and $\epsilon_{ik}^{(i)}$ are quantity experimentally measurable even if, in a particular simple case, it can be shown that it is possible to obtain experimental measurements in relativistic case. In what follows, it will consider only one macroscopic phenomenon which influences the mechanical properties of the medium; it is simple to generalize to several microscopic phenomena.

2 Remarks on Kluitenberg's theory

In this section we will briefly recall some fundamental aspects of mechanical phenomena of the classical non-equilibrium Kluitenberg's theory that are useful for the relativistic extension. Kluitenberg's theory [10, 11], is based on the idea that the usual variables of non equilibrium thermodynamics are not sufficient to describe some relaxation mechanical phenomena that occur in a medium whenever subjected to perturbations.
In this theory it is introduced the following [13]

Postulate 2.1. *Let's consider an elastic medium which is represented here by a regular subset $\mathcal{B}$ of the point space $\mathcal{E}_3$. A thermodynamical state of $\mathcal{B}$ is represented by the values taken by the set of regular functions*

$$S = \{u, \epsilon_{ik}, \Omega_{ik}\} \tag{2.1}$$

in $\mathcal{B}$ at a given instant t; where u, ϵ_{ik} and Ω_{ik} are the specific internal energy, the strain tensor and tensorial symmetric variables respectively.

Postulate 2.2. *The specific entropy s is a function defined in $\mathcal{B} \times [0, \infty[$, as follows:*

$$s = s(u, \epsilon_{ik}, \Omega_{ik}). \tag{2.2}$$

Definition 2.1. The temperature T, the equilibrium stress tensor $\tau_{ik}^{(eq)}$ and the conjugate variable G_{ik} are

$$\frac{1}{T} = \frac{\partial s(u, \epsilon_{ik})}{\partial u},$$

$$\tau_{ik}^{(eq)} = -\rho T \frac{\partial s(u, \epsilon_{ik}, \Omega_{ik})}{\partial \epsilon_{ik}}, \tag{2.3}$$

$$G_{ik} = T \frac{\partial s(u, \epsilon_{ik}, \Omega_{ik})}{\partial \mathbf{\Omega}_{ik}}. \tag{2.4}$$

By differentiating the relation (2.2) and by virtue of the definitions 2.1., the following Gibbs' relation [10, 11] yields:

$$T ds = du - \nu \tau_{ik}^{(eq)} d\epsilon_{ik} + G_{ik} d\Omega_{ik}, \tag{2.5}$$

in which $\rho = \frac{1}{\nu}$ is the specific density and ν is the specific volume.

Postulate 2.3. *It exists a state R at the constant temperature T_0 in which it results:*

$$\tau_{ik}^{(eq)}(T_0) = \tau_{ik(0)}^{(eq)} = 0. \tag{2.6}$$

Theorem 2.4. *In the state R the strain ϵ_{ik} is a regular function only of $\mathbf{\Omega}$.*

Proof. The differentiation of the generalized free energy potential

$$g = u - Ts - \nu \tau_{ik}^{(eq)} \epsilon_{ik} \tag{2.7}$$

reads

$$dg = du - sdT - T ds - \epsilon_{ik} d(\nu \tau_{ik}^{(eq)}) - \nu \tau_{ik}^{(eq)} d\epsilon_{ik}, \tag{2.8}$$

on the other hand, the Gibbs' relation (2.5) allows to rearrange the equality (2.8) as follows

$$dg = -sdT - \epsilon_{ik} d(\nu \tau_{ik}^{(eq)}) - \nu\ G_{ik} \cdot d\Omega_{ik}. \tag{2.9}$$

As a consequence

$$\epsilon_{ik} = -\frac{\partial g(T, \nu \tau_{ik}^{(eq)}, \Omega_{ik})}{\partial(\nu \tau_{ik}^{(eq)})}, \tag{2.10}$$

that is

$$\epsilon_{ik} = f(T, \nu\tau_{ik}^{(eq)}, \Omega_{ik}); \tag{2.11}$$

hence this function in the state R will depend only on Ω_{ik}

$$f(T_0, \nu\tau_{ik(0)}^{(eq)}, \Omega_{ik}) = \epsilon_{ik}^{(i)}(\Omega_{ik}). \tag{2.12}$$

□

Proposition 2.5. *Under the hypothesis of invertibility of the function* $\epsilon_{ik}^{(i)}(\Omega_{ik})$*, i.e.* $\Omega_{ik} = \Omega_{ik}(\epsilon_{ik}^{(i)})$*, the function* ϵ_{ik} *can be expressed as*

$$\epsilon_{ik} = \epsilon_{ik}^{(0)} + \epsilon_{ik}^{(i)}. \tag{2.13}$$

Proof. From physical considerations it is supposed that the inverse function theorem can be applied in the range of physical interest. This means that the following function can be obtained:

$$\Omega_{ik} = \Omega_{ik}(\boldsymbol{\epsilon}^{(i)}). \tag{2.14}$$

If we substitute this equation into (2.2) and (2.10) it follows:

$$\begin{aligned} s &= \bar{s}(u, \epsilon_{ik}, \epsilon_{ik}^{(i)}), \\ \epsilon_{ik} &= \bar{\epsilon}_{ik}(T, \nu\tau_{ik}^{(eq)}, \epsilon_{ik}^{(i)}). \end{aligned} \tag{2.15}$$

Let us emphasize that (2.12) will specify that in the state R occurs a "strain phenomena" (described by vector variable $\boldsymbol{\Omega}$) although there is no stress and his time evolution is postulated independently from other variables. Let us introduce a new vector field $\epsilon_{ik}^{(0)}$

$$\epsilon_{ik}^{(0)} = \bar{\epsilon_{ik}}(T, \nu\tau_{ik}^{(eq)}, \epsilon_{ik}^{(i)}) - \epsilon_{ik}^{(i)}, \tag{2.16}$$

obviously $\epsilon_{ik}^{(0)}$ vanishes in the state R for all values of $\epsilon_{ik}^{(i)}$. Hence, from (2.16) it follows:

$$\epsilon_{ik} = \epsilon_{ik}^{(0)} + \epsilon_{ik}^{(i)}. \tag{2.17}$$

This equation shows that the strain ϵ_{ik} is additively composed of two parts $\epsilon_{ik}^{(0)}$ and $\epsilon_{ik}^{(i)}$. □

Definition 2.2. The tensor

$$\tau_{ik}^{(i)} = \rho T \frac{\partial s(u, \epsilon_{ik}, \epsilon_{ik}^{(i)})}{\partial \epsilon_{ik}^{(i)}} \tag{2.18}$$

is said inelastic stress tensor.

Definition 2.3. The tensor

$$\tau_{ik}^{(vi)} = \tau_{ik} - \tau_{ik}^{(eq)} \tag{2.19}$$

is called viscous stress tensor.

Let us remark that changes of $\epsilon_{ik}^{(0)}$ and $\epsilon_{ik}^{(i)}$ contribute to entropy production then they express irreversible processes. Moreover, if the viscous stress (2.19) vanishes, the variation of $\epsilon_{ik}^{(0)}$ does not contribute to the entropy production, i.e. changes in $\epsilon_{ik}^{(0)}$ represent reversible processes.

3 Remark on energy momentum tensor and first law of thermodynamics

In the present section we extend the Kluitenberg's theory to the special relativistic case. Let us suppose the medium in motion with respect to an inertial frame of reference Σ and we indicate Σ_0 the rest frame of reference of an element of fluid. Let us introduce the Galilean coordinates defined in terms of spatial and temporal variables (x, y, z, t) as follows:

$$x_0 = ct, \qquad x_1 = x, \qquad x_2 = y, \qquad x_3 = z,$$

where c is the scalar velocity of light in the vacuum. It is well known that the study of the motion of such a medium implies to consider different forms of density of energy flow. We will limit our considerations only on three forms of density of energy flow, do not taking into account electrodynamics phenomena.

In such a context we consider the following quantities, [16]-[23]:

i) E_i is the vector representing density of energy flow of not mechanical nature (as the heat),

ii) $\rho c^2 v_i$ is the density of energy flow due only to the motion of the medium, where ρ is the mass density,

iii) $v^j \phi_{ji}$ is the density of energy flow due to the action of the forces of stress flowing in the positive x_i direction, where ϕ_{ji} is the relativistic (no symmetric) stress tensor.

hence, the total density of energy flow L_i, is [22]

$$L_i = E_i + \rho c^2 v_i + v^j \phi_{ji}, \tag{3.1}$$

according to Einstein relation between mass and energy, it is possible to associate to this quantity the following total momentum density

$$H_i = \frac{L_i}{c^2} = \rho v_i + \frac{E_i}{c^2} + \frac{v^j \phi_{ji}}{c^2}. \tag{3.2}$$

Definition 3.1. The tensor

$$\chi_{\beta\delta} = \begin{cases} \chi_{ik} = H_i v_k + \phi_{ik} \\ \chi_{i0} = \chi_{0i} = cH_i \\ \chi_{00} = \rho c^2 \end{cases}, \tag{3.3}$$

in which Latin index assumes the values $1, 2, 3$ and Greek index assumes the values $0, 1, 2, 3$, is said energy momentum tensor [14, 16, 19, 22].

If ρF_i is the unitary volume force, the introduction of a four vector W_δ defines as [10]-[18]

$$W_\delta \equiv \Big(\frac{\rho v_i F_i}{c}, \rho F_i\Big) \tag{3.4}$$

allows us to write the equation

$$\frac{\partial \chi_{\beta\delta}}{\partial x^\beta} = W_\delta, \tag{3.5}$$

in which the temporal component represents the balance equation for the energy density and the spatial components represent the balance equation for the momentum density. Moreover, by introducing the four vector V^δ defined as

$$V^\delta \equiv \Big(\alpha, \alpha\frac{v_i}{c}\Big) \tag{3.6}$$

and by combining the (3.5) and (3.6) the first law of thermodynamics easily is obtained:

$$\frac{\partial \chi_{\beta\delta}}{\partial x^\beta} V^\delta = W_\delta V^\delta. \tag{3.7}$$

By substituting (3.4) into (3.7) the following temporal component holds:

$$\frac{\partial \chi_{00}}{\partial x^0} = \frac{2\rho F_i v_i}{c} - \frac{\partial \chi_{i0}}{\partial x^0} V^i - \frac{\partial \chi_{i0}}{\partial x^i} - \frac{\partial \chi_{ik}}{\partial x^k} V^i. \tag{3.8}$$

Let us remark that all these equations are relative to an arbitrary inertial frame Σ.

4 Relativistic thermodynamic approach

It is well known, that the tensor of order two $\chi_{\beta\delta}$, given by Definition 3.1., satisfies the following transformation law [15]:

$$\chi_{\beta\delta} = \frac{\partial x^\mu}{\partial x^{\delta'}} \frac{\partial x^\nu}{\partial x^{\beta'}} \overset{*}{\chi}_{\mu\nu}, \tag{4.1}$$

where $\overset{*}{\chi}_{\mu\nu}$ is the energy momentum tensor in a rest frame of reference and the law of transformation is related to Lorentz transformation.
In the following, a function evaluated in the rest frame of reference will be denoted

with the symbol "*" as upper index. Taking into account the relations (3.3) and (4.1), it is easy to obtain the following relation [19, 23]:

$$\chi_{00} = \alpha^2 \rho_0 c^2 + 2\frac{\alpha^2 v^i}{c^2}\overset{*}{E}_i + \frac{\alpha^2}{c^2} v^i v^k \overset{*}{\phi}_{ik} = \rho c^2, \tag{4.2}$$

where ρ_0 and $\overset{*}{\phi}_{ik}$ are respectively the mass density and the symmetric Cauchy stress tensor in a rest frame of reference, and where $\alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Our relativistic approach is completely based on the principle of special relativity which expresses the invariability of the law of the physic in two arbitrary inertial reference frames. This allows us to preserve the assumption on the insufficient number of the variables for the description of some internal phenomena which occur in a medium. Indeed, this cannot be proved, but we think that it is a reasonable assumption also in a relativistic approach [7, 9].

Postulate 4.1. *The entropy density ϕ in an arbitrary inertial frame of reference Σ, depends on the density of energy χ_{00} [13], on the total relativistic strain tensor γ_{ik} and on the relativistic internal variables $\overset{(i)}{\gamma}_{ik}$,*

$$\phi = \phi(\chi_{00}\gamma_{ik}, \overset{(i)}{\gamma}_{ik}) \tag{4.3}$$

and in Σ_0:

$$\overset{*}{\phi} = \overset{*}{\phi}(\overset{*}{\chi}_{00}, \overset{*}{\gamma}_{ik}, \overset{*(i)}{\gamma}_{ik}). \tag{4.4}$$

Since the entropy σ is an invariant quantity, it follows from (4.3) and (4.4) that

$$\phi = \alpha\overset{*}{\phi}, \tag{4.5}$$

such that

$$d\overset{*}{\sigma} = \overset{*}{\phi} d\overset{*}{V} \tag{4.6}$$

represents the entropy of the volume $d\overset{*}{V}$ in Σ_0. Here χ_{00} and $\overset{*}{\chi}_{00}$ represent the temporal components of the energetic tensor in Σ and Σ_0 respectively. From (4.3)-(4.5) follows:

$$\frac{\partial\phi}{\partial\chi_{00}} = \alpha\frac{\partial\overset{*}{\phi}}{\partial\overset{*}{\chi}_{00}}\frac{\partial\overset{*}{\chi}_{00}}{\partial\chi_{00}}, \tag{4.7}$$

recalling the expression of the temperature T in Σ_0, [4, 5], as

$$\frac{1}{\overset{*}{T}} = \frac{\partial\overset{*}{\phi}}{\partial\overset{*}{\chi}_{00}}, \tag{4.8}$$

the relation (4.7) becomes

$$\frac{\partial\phi}{\partial\chi_{00}} = \frac{1}{\alpha}\frac{1}{\overset{*}{T}}, \tag{4.9}$$

in which the (4.2) has been considered.

Definition 4.1. The relativistic temperature T, in a generic inertial frame Σ, is a function satisfying the relation

$$\frac{1}{T} = \frac{\partial \phi}{\partial \chi_{00}} = \frac{1}{\alpha \overset{*}{T}}. \tag{4.10}$$

In other words the last definition is equivalent to

$$T = \alpha \overset{*}{T}, \tag{4.11}$$

that is in agreement with Ott's transformation formula [20]. Now, from (4.3)-(4.5), it follows:

$$\frac{\partial \phi}{\partial \gamma_{rs}} = \alpha \frac{\partial \overset{*}{\phi}}{\partial \overset{*}{\gamma}_{ik}} \frac{\partial \overset{*}{\gamma}_{ik}}{\gamma_{rs}}. \tag{4.12}$$

By virtue of the well known classical relation in Σ_0, ([4, 5, 10, 11])

$$\frac{\partial \overset{*}{\phi}}{\partial \overset{*}{\gamma}_{ik}} = \frac{1}{\overset{*}{T}} \overset{*}{\phi}{}^{(eq)}_{ik} \tag{4.13}$$

and taking into account the (4.11), the expression (4.12) becomes:

$$T \frac{\partial \phi}{\partial \gamma_{rs}} = \alpha^2 \overset{*}{\phi}{}^{(eq)}_{ik} \frac{\partial \overset{*}{\gamma}_{ik}}{\partial \gamma_{rs}}. \tag{4.14}$$

Definition 4.2. The tensor in Σ

$$\phi^{(eq)}_{rs} = T \frac{\partial \phi}{\partial \gamma_{rs}} \tag{4.15}$$

is said equilibrium relativistic stress tensor.

Hence

$$\phi^{(eq)}_{rs} = \alpha^2 \overset{*}{\phi}{}^{(eq)}_{ik} \frac{\partial \overset{*}{\gamma}_{ik}}{\partial \gamma_{rs}}, \tag{4.16}$$

where $\overset{*}{\phi}{}^{(eq)}_{rs}$ is the equilibrium stress tensor in Σ_0.

Definition 4.3. The tensor

$$\phi^{(i)}_{rs} = T \frac{\partial \phi}{\partial \gamma^{(i)}_{rs}} = \alpha^2 \overset{*}{\phi}{}^{(i)}_{ik} \frac{\partial \overset{*}{\gamma}_{ik}}{\partial \gamma^{(i)}_{rs}} \tag{4.17}$$

is called relativistic affinity stress tensor.

(4.16) and (4.17) are unusable if it does not know the law of transformation of $\overset{*}{\gamma}_{ik}$. Let us remark that the quantities $\phi_{ik}^{(eq)}$, $\phi_{ik}^{(i)}$ have the same transformation law because of the intrinsic meaning of the stress as ratio between the force and the surface [22]. By indicating with S_{ik} a generic stress tensor (which can be $\phi_{ik}^{(i)}$, ϕ_{ik}, $\phi_{ik}^{(eq)}$) the transformation laws of the stress tensor are well known [19, 23], i.e. the following transformation laws yields, [22, 23]:

$$
\begin{aligned}
&S_{11} = \overset{*}{S}_{11} \quad S_{12} = \alpha\overset{*}{S}_{12} \quad S_{13} = \alpha\overset{*}{S}_{13},\\
&S_{21} = \frac{1}{\alpha}\overset{*}{S}_{21} \quad S_{22} = \overset{*}{S}_{22} \quad S_{23} = \overset{*}{S}_{23},\\
&S_{31} = \frac{1}{\alpha}\overset{*}{S}_{31} \quad S_{32} = \overset{*}{S}_{32} \quad S_{33} = \overset{*}{S}_{33}.
\end{aligned} \tag{4.18}
$$

From these relations follow two properties:

- relativistic stress tensor is not symmetric,
- every component in Σ will depend only on component with same index in Σ_0.

By using (4.18), (4.16) allows also to get the transformation law for the strain tensor, for this reason we prove the following theorem.

Theorem 4.2. *The transformation laws of the relativistic strain tensor is given by:*

$$
\begin{aligned}
&\gamma_{11} = \alpha^2\overset{*}{\gamma}_{11} \quad \gamma_{12} = \alpha\overset{*}{\gamma}_{12} \quad \gamma_{13} = \alpha\overset{*}{\gamma}_{13},\\
&\gamma_{21} = \alpha^3\overset{*}{\gamma}_{21} \quad \gamma_{22} = \alpha^2\overset{*}{\gamma}_{22} \quad \gamma_{23} = \alpha^2\overset{*}{\gamma}_{23},\\
&\gamma_{31} = \alpha^3\overset{*}{\gamma}_{31} \quad \gamma_{32} = \alpha^2\overset{*}{\gamma}_{32} \quad \gamma_{33} = \alpha^2\overset{*}{\gamma}_{33}.
\end{aligned}
$$

Proof. By taking into account the non-symmetry of stress tensor, (4.16) or (4.17) lead to

$$
\overset{*}{S}_{ik}\frac{\partial\overset{*}{\gamma}_{ik}}{\partial\gamma_{rs}} \neq \overset{*}{S}_{ik}\frac{\partial\overset{*}{\gamma}_{ik}}{\partial\gamma_{sr}}, \tag{4.19}
$$

from which follows the not symmetry of the strain tensor,

$$
\gamma_{rs} \neq \gamma_{sr}.
$$

Let us rewrite (4.16), where is denoted with S_{ik} the stress tensor:

$$
S_{rs} = \alpha^2\overset{*}{S}_{ik}\frac{\partial\overset{*}{\gamma}_{ik}}{\partial\gamma_{rs}} \tag{4.20}
$$

and let us compute the above relation for example for the component $r = 1,\ s = 3$, i.e. (see relations (4.18)):

$$
\begin{aligned}
&S_{13} = \alpha\overset{*}{S}_{13} = \alpha^2\Big(\overset{*}{S}_{11}\frac{\partial\overset{*}{\gamma}_{11}}{\partial\gamma_{13}} + \overset{*}{S}_{12}\frac{\partial\overset{*}{\gamma}_{12}}{\partial\gamma_{13}} + \overset{*}{S}_{13}\frac{\partial\overset{*}{\gamma}_{13}}{\partial\gamma_{13}}\Big) + ...\\
&+\alpha^2\Big(\overset{*}{S}_{31}\frac{\partial\overset{*}{\gamma}_{31}}{\partial\gamma_{13}} + ... + \overset{*}{S}_{33}\frac{\partial\overset{*}{\gamma}_{33}}{\partial\gamma_{13}}\Big).
\end{aligned} \tag{4.21}
$$

Since the term S_{13} is a function only of $\overset{*}{S}_{13}$ and by assuming that S_{ik} is different from zero, the above equation becomes:

$$\frac{\partial \overset{*}{\gamma}_{13}}{\partial \gamma_{13}} = \frac{1}{\alpha}. \tag{4.22}$$

(4.21) reduces to (4.22) if and only if

$$\frac{\partial \overset{*}{\gamma}_{rs}}{\partial \gamma_{ik}} = 0 \qquad i \neq r \qquad k \neq s. \tag{4.23}$$

This means that any components of the strain tensors in Σ depends on the component with the same index in Σ_0. By integrating the relation (4.22), the transformation law for the strain tensor of the component $\overset{*}{\gamma}_{13}$ is obtained:

$$\gamma_{13} = \alpha \overset{*}{\gamma}_{13} + c_1, \tag{4.24}$$

where c_1 is the integration constant that can be choosen equal to zero because of, in rest frame of reference Σ_0, has to satisfy:

$$\gamma_{13} = \overset{*}{\gamma}_{13}. \tag{4.25}$$

Analogously for all components, the transformation laws for relativistic strain are obtained. $\square$

As explained above, the law of transformation for the stress tensor is independent of the character of anelasticity, viscosity etc. of the tensors $\phi^{(i)}_{ik}$, $\phi^{(eq)}_{ik}$, i.e. all stress tensors have the same transformation law, therefore the transformation laws for any kind of strain tensor, $\overset{*}{\gamma}_{ik}$, $\gamma^{(i)}_{ik}$..., will assume the same form.

Definition 4.4. The tensor

$$\phi^{(vi)}_{ik} = \phi_{ik} - \phi^{(eq)}_{ik} \tag{4.26}$$

is said relativistic viscous stress tensor.

Definition 4.5. The tensor

$$\phi^{(i)}_{ik(m)} = \phi^{(i)}_{ik} - \phi^{(eq)}_{ik} \tag{4.27}$$

is called relativistic memory stress tensor.

Let us remark that if no viscous phenomena occur it follows

$$\phi_{ik} = \phi^{(eq)}_{ik}. \tag{4.28}$$

From (4.27), it follows

$$\phi_{ik} = \phi^{(vi)}_{ik} + \phi^{(i)}_{ik} - \phi^{(i)}_{ik(m)}. \tag{4.29}$$

5 Entropy production

In this section obtain the expression of relativistic entropy production.

Theorem 5.1. *The relativistic entropy density production is due to the following phenomena:*

- *the gradient of temperature;*
- *the flow of the unitary volume force;*
- *the equilibrium strain phenomena;*
- *the affinity stress phenomena*

and assumes the following expression

$$\Omega^{(\sigma)} = -\frac{\chi_{0i}}{T^2}\frac{\partial T}{\partial x_i} + \frac{\rho F_i v_i}{Tc} + \frac{\phi^{(eq)}_{ik}}{T}\frac{\partial \gamma_{ik}}{\partial x_0} + \frac{\phi^{(i)}_{ik}}{T}\frac{\partial \gamma^{(i)}_{ik}}{\partial x_0}. \tag{5.1}$$

Proof. From relation (4.3) one obtains:

$$\frac{\partial \phi}{\partial x^0} = \frac{\partial \phi}{\partial \chi_{00}}\frac{\partial \chi_{00}}{\partial x^0} + \frac{\partial \phi}{\partial \gamma_{ik}}\frac{\partial \gamma_{ik}}{\partial x^0} + \frac{\partial \phi}{\partial \gamma^{(i)}_{ik}}\frac{\partial \gamma^{(i)}_{ik}}{\partial x^0}; \tag{5.2}$$

by taking into account (4.11), (4.15) and (4.17), (5.2) becomes:

$$\frac{\partial \phi}{\partial x^0} = \frac{1}{T}\frac{\partial \chi_{00}}{\partial x^0} + \frac{1}{T}\phi^{(eq)}_{ik}\frac{\partial \gamma_{ik}}{\partial x^0} + \frac{1}{T}\phi^{(i)}_{ik}\frac{\partial \gamma^{(i)}_{ik}}{\partial x^0}. \tag{5.3}$$

By recalling the expressions (3.4), (3.7) and (3.8), it follows:

$$\Big(\frac{\partial \chi_{i0}}{\partial x_0} + \frac{\partial \chi_{ik}}{\partial x_k}\Big)V^i = \frac{\rho F_i v_i}{c}, \tag{5.4}$$

in which the term ρF_i is the unitary volume force, by substituting it into (5.3), it follows:

$$\frac{\partial \phi}{\partial x_0} = -\frac{1}{\alpha^2}\frac{\partial J^{(\sigma)}}{\partial x_i} + \Omega^{(\sigma)}, \tag{5.5}$$

where

$$J^{(\sigma)} = \frac{\chi_{0i}}{T}. \tag{5.6}$$

Then

$$\Omega^{(\sigma)} = -\frac{\chi_{0i}}{T^2}\frac{\partial T}{\partial x_i} + \frac{\rho F_i v_i}{Tc} + \frac{\phi^{(eq)}_{ik}}{T}\frac{\partial \gamma_{ik}}{\partial x_0} + \frac{\phi^{(i)}_{ik}}{T}\frac{\partial \gamma^{(i)}_{ik}}{\partial x_0} \tag{5.7}$$

and the assert is proved. □

Let us observe that the relation (5.1) restore the classical entropy production in the proper reference Σ_0.

6 Conclusions

The extension of a classical theory to the relativistic case generally presents some difficulties especially defining the relativistic form of the corresponding classical quantities. In particular, the main difficulty is to express the transformation law of the classical quantities. Our approach is based entirely on the principle of relativity and on the transformation law of the strain tensor obtained by us in a previous paper. In particular, by taking into account the classical non equilibrium thermodynamics approach, we introduce a relativistic entropy function, as in the classical case, depending on the corresponding relativistic variables. Moreover, by invoking the principle of relativity, even in the relativistic approach, it is possible to split the strain tensor as the sum of an "elastic" and "inelastic" parts as in the classical case. If this is not true we can evidence by experiments the motion of an inertial frame with respect to another contradicting the principle of special relativity. Obviously, it is possible to obtain the classical case if we refer to a rest reference frame. By considering the definition of temperature, introduced by us in a previous paper, [5], we have been able to define the corresponding relativistic stress tensor (equilibrium, inelastic) and therefore to place the foundations in order to treat every quantity of the classical Kluitenberg's theory in a relativistic framework. In some sense, the quantities introduced by us are the fundamental in order to develop an relativistic Kluitenberg's extension. Of course, even for those quantities we obtain the classical form if we refer to a rest reference frame. It is our opinion this is a very simple and natural approach to extend the Kluitenberg's ideas to relativistic case and it allows to develop the theory. This approach can be useful for the study of astrophysical problems in which it needs a more complete description, as in the black holes.

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References

[1] A. Berezovki, J. Engelbrecht and G. A. Maugin, *Thermoelasticity with dual internal variables*, Journal of Thermal Stresses, 34 (2011), 413-430.

[2] A. Berezovki, J. Engelbrecht J. and G.A. Maugin, *Internal variables and generalized continuum theories*, IUTAM Symposium on Progress in the Theory and Numerics of Configurational Mechanics, Springer Science + Business Media 2009.

[3] V.Ciancio, F. Farsaci, G. A. Bartolotta, *Phenomenological and state coefficients in viscoanelastic medium of order one (with memory)*, Computational Sciences and its Applications - ICCSA, 3980 (2006), 821-827.

[4] V. Ciancio F. Farsaci, *Relativistic elastic tensor*, Appl. Sci. 13 (2011), 21-29.

[5] V. Ciancio, F. Farsaci, P. Rogolino, *On the entropy production in relativistic thermodynamics*, Diff. Geom. - Dyn. Syst. (2010), 41-48.

[6] V. Ciancio, F. Farsaci, P. Rogolino *Mechanical model for relaxation phenomena in viscoanelastic media of order one*, Physica B, 405 (16) (2010), 3208-3212.

[7] S. Hayward, *Relativistic thermodynamics*, Class Quantum Grav. 15 (10) (1998), 3147-3162.

[8] H. Heino Von Borzeszkowski, T. Chrobok, *On special and generalized relativistic thermodynamics*, Atti dell'Accademia Peloritana dei Pericolanti Classe Science Fisiche, Matematiche e Naturali, LXXXVI (2008), 1-11.

[9] J. Jorne, P. S. Jorne, *Reltivistic thermodynamics of irreversible processes*, Chemical Engineering Communications, 21 (4-6) (1983), 361-367.

[10] G. A. Kluitenberg, *A thermodynamic derivation of stress-strain relations for Burgers media and related substances*, Physica, 38 (1968), 513-548.

[11] G. A. Kluitenberg, *Thermodynamical theory of elasticity and plasticity*, Physica, 28 (1962), 217-232.

[12] G. A. Kluitenberg, *Plasticity and non equilibrium thermodynamics*, in "'The constitutive Law in Thermoplasticity"', CISM Course and lecture, Springer-Verlag, Wien-New York 1984.

[13] L. D. Landau and E. M. Lifshitz, *Fluid mechanics*, Pergamon Press 6, 1959.

[14] L. D. Landau, E. M. Lifshitz, *Course of theoretical physics*, Editori Riuniti, Ed. Mir. 2, 1976.

[15] L. D. Landau and E. M. Lifshitz, *Field Theory* (in Italian), Editori Riuniti, Ed. Mir., 1981.

[16] T. Levi Civita, *Basics of Relativistic Mechanics* (in Italian), Zanichelli Bologna, 1986.

[17] G.A. Maugin and W. Muschik, *Thermodynamics with internal variables. Part I. General Concepts*, J. Non-Equilib. Thermodyn. 19 (1994), 217–249.

[18] G.A. Maugin and W. Muschik, *Thermodynamics with internal variables. Part II. Applications*, J. Non-Equilib. Thermodyn. 19 (1994), 250–289.

[19] C. Moller, *The theory of Relativity*, Clarendon Press, 1952.

[20] H. Ott, *Lorentz transformation of heat and temperature* (in German), Z. Phys, 175 (1) (1963), 70-104.

[21] M. Requardt, *Thermodynamics meets special relativity or what is real in physics?*, ArXiv 08012639 (2008), 2639:1-2639:27.

[22] W. Rindler, *Restricted Relativity* (in Italian), Ed. Cremonese Roma, 1971.

[23] R. C. Tolman, *Relativity, Thermodynamics and Cosmology*, Clarendon Press, 1958.

[24] P. Van, A. Berezovski and J. Engelbrecht, *Internal variables and dynamic degrees of freedom*, J. Non-Equilib. Thermodyn. 33 (3) (2008), 235-254.

[25] S. Weinberg, *Gravitation and Cosmology*, John Wiley and Sons, 1972.

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