# On Post-Gluskin-Hosszu Theorem

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**Abstract.** The Post-Gluskin-Hosszu Theorem (also called Gluskin-Hosszu or Hosszu-Gluskin Theorem) refers to an $n$-ary group $\langle A, [\ \ ]\rangle$ and a binary group $\langle A, \circ\rangle$, defined on the same set $A$. E. Post stated and proved this Theorem, while considering instead of the group $\langle A, \circ\rangle$, the isomorphic to it associated group $A_0$. This reveals Post's basic contribution, and justifies the inclusion of his name as leading co-author of the Theorem. Apparently, M. Hosszu was not aware of Post's result, while L.M. Gluskin did not directly address $n$-ary groups in his research, focusing mainly on a large class on algebraic systems (positional operatives), for which he obtained a series of notable results, out of these, one of the consequences being exactly the Post-Gluskin-Hosszu Theorem.

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## 1 Introduction

According to W. Dörnte [2], we call *$n$-ary group* ($n \geq 2$) an universal algebra $\langle A, [\ \ ]\rangle$ with a given $n$-ary operation, $[\ \ \ ] : A^n \to A$, which is associative, i.e., for all $i \in \overline{1, n-1}$ in $A$ there holds the associativity condition

$$[[a_1 \dots a_n]a_{n+1} \dots a_{2n-1}] = [a_1 \dots a_i[a_{i+1} \dots a_{i+n}]a_{i+n+1} \dots a_{2n-1}],$$

and for all $i = \overline{1, n}$ and all $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n$, $b \in A$, the following equations is solvable in $A$:

$$[a_1 \dots a_{i-1}x_i a_{i+1} \dots a_n] = b.$$

We note that Dörnte's definition leads, for $n = 2$, to the usual definition of a binary group.

One can identify $n$-ary groups within the class of all universal algebras in various ways. A rather natural procedure is to point out first, among all the universal algebras, an algebra with an associative $n$-ary operation, and further to employ the following result:

**Theorem 1.1.** *Given a universal algebra $\langle A, [\ \ ]\rangle$ endowed with an $n$-ary associative operation, the following statements are equivalent:*

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*1)* $\langle A, [\ ]\rangle$ *is an* $n$*-ary group;*

*2)* **(E. Post [8], 1940)** *for any* $a_1, \ldots, a_n,\ b \in A$ *the following equations are both solvable in* $A$*:*

$$[xa_2 \ldots a_n] = b, \qquad [a_1 \ldots a_{n-1}y] = b;$$

*3)* **(E. Post [8], 1940)** *for any* $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n,\ b \in A$ *and some* $i \in \overline{2, n-1}$*,* $(n \geq 3)$ *the following equation is solvable in* $A$

$$[a_1 \ldots a_{i-1}xa_{i+1} \ldots a_n] = b;$$

*4)* **(A.N. Skiba, V.I. Tyutin [11], 1985)** *for any* $a, b \in A$*, the following equations are both solvable in* $A$

$$[x\underbrace{a\ \ldots\ a}_{n-1}] = b, \qquad [\underbrace{a\ \ldots\ a}_{n-1}y] = b;$$

*5)* **(A.N. Skiba, V.I. Tyutin [11], 1985)** *for any* $a, b \in A$ *and some* $i \in \overline{2, n-1}$ $(n \geq 3)$ *the following equation is solvable in* $A$

$$[\underbrace{a\ \ldots\ a}_{i-1}x\underbrace{a\ \ldots\ a}_{n-i}] = b;$$

*6)* **(A.M. Gal'mak [3, 4], 1991)** *for any* $a, b \in A$ *the following equations with* $n-1$ *unknowns are both solvable in* $A$

$$[x_1 \ldots x_{n-1}a] = b, \qquad [ay_1 \ldots y_{n-1}] = b;$$

*7)* **(A.M. Gal'mak [3, 4], 1991)** *for any* $a, b \in A$ *the following equation with* $n-2$ *unknowns* $(n \geq 3)$ *is solvable in* $A$

$$[ax_1 \ldots x_{n-2}a] = b.$$

For the necessary preliminaries regarding the theory of $n$-ary groups, we address the reader to the monographs [4, 5, 9].

The following result is usually known as the Gluskin-Hosszu or the Hosszu-Gluskin Theorem:

**Theorem 1.2.** **(L. M. Gluskin [6], M. Hosszu [7])** *On any* $n$*-ary group* $\langle A, [\ ]\rangle$ *one can define the binary operation* $\circ$*, a mapping* $\beta$ *and an element* $d \in A$ *such that* $\langle A, \circ\rangle$ *is a group,* $\beta$ *is its automorphism and the following relations hold true:*

$$[x_1x_2 \ldots x_n] = x_1 \circ x_2^{\beta} \circ \ldots \circ x_n^{\beta^{n-1}} \circ d, \qquad \forall x_1, \ldots, x_n \in A; \tag{1.1}$$

$$d^{\beta} = d; \tag{1.2}$$

$$x^{\beta^{n-1}} = d \circ x \circ d^{-1}, \qquad \forall x \in A. \tag{1.3}$$

As well, the converse of the Gluskin-Hosszu Theorem holds true.

**Theorem 1.3.** **(L. M. Gluskin [6], M. Hosszu [7])** *If the element* $d$ *of the group* $\langle A, \circ\rangle$ *and its automorphism* $\beta$ *satisfy the conditions* (1.2) *and* (1.3)*, then* $\langle A, [\ ]\rangle$ *is an* $n$*-ary group with the* $n$*-ary operation* (1.1)*.*

A short and elegant proof of this Theorem was provided by E.I. Sokolov [10]. He proved the Theorem by considering

$$x \circ y = [x \underbrace{a \dots a}_{n-2} y],$$

$$\beta : \quad x \to x^{\beta} = [\bar{a} x \underbrace{a \dots a}_{n-2}],$$

$$d = [\underbrace{\bar{a} \dots \bar{a}}_{n}],$$

where $\bar{a}$ is the *skew element* of $a$, i.e., the solution of the the equation

$$[x \underbrace{a \dots a}_{n-1}] = a.$$

If one fixes in the $n$-ary group $\langle A, [\ \ ]\rangle$ an element $a \in A$, then the operation $\circ$, mapping $\beta$ and element $d$, can be chosen as

$$x \circ_a y = [x a_1 \dots a_{n-2} y], \tag{1.4}$$

the mapping

$$\beta = \beta_a : x \to [a x a_1 \dots a_{n-2}] \tag{1.5}$$

and the element

$$d = d_a = [\underbrace{a \dots a}_{n}], \tag{1.6}$$

where $a_1 \dots a_{n-2}$ is a inverse sequence for the element $a$ in the in the $n$-ary group $\langle A, [\ ]\rangle$.

Then Theorem 1.2 can be rephrased as follows (see, e.g., [4]):

**Theorem 1.4.** *In any $n$-ary group $\langle A, [\ ]\rangle$ and for any $a \in A$ the following properties hold true:*

$$[x_1 x_2 \dots x_n] = x_1 \circ_a x_2^{\beta_a} \circ_a \dots \circ_a x_n^{\beta_a^{n-1}} \circ_a d_a, \qquad x_1, \dots, x_n \in A; \tag{1.7}$$

$$d_a^{\beta_a} = d_a; \tag{1.8}$$

$$x^{\beta_a^{n-1}} = d_a \circ_a x \circ_a d_a^{-1}, \quad x \in A, \tag{1.9}$$

*where $\langle A, \circ_a \rangle$ is a group and $\beta_a$ is its automorphism.*

While defining the operation $\circ_a$ and the mapping $\beta_a$, one may consider as inverse sequence $a_1 \dots a_{n-2}$ any of the sequences $\underbrace{a \dots a}_{i-1} \bar{a} \underbrace{a \dots a}_{n-i-2}$, $i \in \overline{1, n-2}$, and in particular, one of the sequences

$$\underbrace{a \dots a}_{n-3} \bar{a}, \qquad \text{or} \qquad \bar{a} \underbrace{a \dots a}_{n-3}.$$

**Remark 1.1.** The unity of the group $\langle A, \circ_a \rangle$ is the element $a$. Since

$$d_a \circ_a \bar{a} = [[\underbrace{a\ \ldots\ a}_{n}]\bar{a}\underbrace{a\ \ldots\ a}_{n-3}\bar{a}] = a,$$

it follows that $d_a^{-1} = \bar{a}$ is the inverse element in the group $\langle A, \circ_a \rangle$ for the element $d_a$.

If in (1.4)-(1.6) we replace the element $a$ with $\bar{a}$, then we get E.I. Sokolov's construction.

According to E.Post [8], the group $G$ is called to be *covering* for the $n$-ary group $\langle A, [\ ]\rangle$, if the set $A$ generates the group $G$, and the $n$-ary operation $[\ ]$ is related to the binary operation from the group $G$ by the relation:

$$[x_1x_2\ldots x_n] = x_1x_2\ldots x_n, \qquad x_1, x_2, \ldots, x_n \in A.$$

This equality shows that the $n$-ary operation $[\ ]$ coincides on the set $A$ with the $n$-ary operation, derived from the operation of the group $G$. For brevity, we shall say that the $n$-ary operation $[\ ]$ is *the derived* operation from the group operation of $G$. The subset

$$A_0 = \{x_1x_2\ldots x_{n-1} \mid x_1, x_2, \ldots, x_{n-1} \in A\} \subseteq G$$

is a normal subgroup of the group $G$ and it is called *associated* group ([8], [9]) for the $n$-ary group $\langle A, [\ ]\rangle$. If we fix the element $a \in A$, then it is easy to check ([8]) that

$$A_0 = Aa^{-1} = \{xa^{-1} \mid x \in A\},$$

where the symbol $a^{-1}$ denotes the inverse of the element $a$ in the group $G$. The element $a^{-1}$ itself can be represented in the group $G$ as the product

$$a^{-1} = a_1a_2\ldots a_{n-2}, \qquad a_1, a_2, \ldots, a_{n-2} \in A,$$

where $a_1\ldots a_{n-2}$ is an arbitrary inverse sequence in the $n$-ary group $\langle A, [\ ]\rangle$ for the element $a$.

The mapping

$$\varphi_a : u \to ua, \qquad u \in A_0$$

provides an isomorphism of the group $A_0$ onto the group $\langle A, \circ_a \rangle$, and the mapping

$$\psi_a : \ \ x \to xa^{-1}, \ \ x \in A$$

gives an isomorphism of the group $\langle A, \circ_a \rangle$ onto the group $A_0$. Hence, in particular, it follows that for any $a, b \in A$, the groups $\langle A, \circ_a \rangle$ and $\langle A, \circ_b \rangle$ are isomorphic. E.g., the isomorphism between $\langle A, \circ_a \rangle$ and $\langle A, \circ_b \rangle$ can be defined by

$$\tau = \psi_a\varphi_b : x \to xa^{-1}b = [xa_1\ldots a_{n-2}b], \ \ x \in A.$$

It is clear, that $\varphi_a$ and $\psi_a$ are inverse to each other mappings.

**Remark 1.2.** Since the equalities

$$[\underbrace{a\ \ldots\ a}_{n-1}\bar{a}] = a, \qquad a^{n-1}\bar{a} = a$$

are equivalent, then $\bar{a} = (a^{-1})^{n-2}$, where $a^{-1}$ is the inverse element of the element $a$ in the group $G$.

We shall further present an original reformulation of E. Post's result.

**Theorem 1.5 (E. Post [8, p. 245])** *Given any abstract 2-group $G_0$ to serve as associated group, an abstract element $s_0$ subject to the condition $s_0^{m-1} = t_0$, $t_0$ in $G_0$, and any automorphism $T$ of $G_0$, which carries $t_0$ into itself,* and whose $(m-1)$-st power is the automorphism of $G_0$ under $t_0$, to serve as the automorphism of $G_0$ under $s_0$, then there is one and only one associated abstract $m$-group $G$; conversely, every $m$-group can be thus determined.

Having in view Theorem 1.5, V.A. Artamonov noticed ([1]) that Hosszu and Gluskin's construction reduces to the covering group of the $n$-ary group. In order to prove this, it suffices to put down rigorously Post's verbose formulations.

The proof of the direct implication of Theorem 1.5 [8, p. 246] (which E. Post calls *the first part* starts by defining for any elements

$$s_{i_j} = t_{i_j} s_0, \;\; t_{i_j} \in G_0, j \in \overline{1,m}$$

from the coset $G = G_0 s_0$ a new element

$$c(s_{i_1} s_{i_2} \ldots \; s_{i_m}) = t_{i_1} s_0 t_{i_2} s_0 \;\; \ldots \;\; t_{i_m} s_0.$$

Then it is proved that this satisfies the equality

$$c(s_{i_1} s_{i_2} \ldots \; s_{i_m}) = (t_{i_1} \cdot T^{-1} t_{i_2} \;\ldots\; T^{-(m-1)} t_{i_m} \cdot t_0) s_0, \tag{1.10}$$

where $T$ is the automorphism from the statement of Theorem 1.5, which acts as

$$T: \; t \rightarrow s_0^{-1} t s_0, \quad t \in G_0,$$

and where the symbol $T^{-(j-1)} t_{i_j}$ denotes the image of the element $t_{i_j}$ under the action of the mapping $T^{-(j-1)}$. Then it is stated that the coset $G = G_0 s_0$ is an $m$-ary group relative to the $m$-ary operation $c$.

If we replace the notation $T^{-1}t$ (which stands for the image of the element $t$ via the mapping $T^{-1}$) by $t^{T^{-1}}$ and if we denote $T^{-1} = \beta$, then E. Post's equality (1.10) gets the form

$$c(s_{i_1} s_{i_2} \ldots \; s_{i_m}) = (t_{i_1} t_{i_2}^{\beta} \;\ldots\; t_{i_m}^{\beta^{m-1}} \; t_0) s_0. \tag{1.11}$$

The equality (1.11) differs from (1.1) and (1.7) only by notations and by the factor $s_0$, which ensures the passing from the element $t_{i_1} t_{i_2}^{\beta} \;\ldots\; t_{i_m}^{\beta^{m-1}} \; t_0$ of the set $G_0$ to the element $c(s_{i_1} s_{i_2} \ldots \; s_{i_m})$ of the set $G = G_0 s_0$.

In the following sections we shall thoroughly examine the direct and the inverse implications of Theorem 1.5, and prove the equivalence of the equalities from Theorem 1.4 with the corresponding equalities from Theorem 1.5.

## 2 The reverse implication of Theorem 1.5

For proving the reverse implication from Theorem 1.5 (which was called by E. Post *the second part*), we need the following

**Lemma 2.1**. Let $\langle A, [\ \ ]\rangle$ be an $n$-ary group, let $G$ and $A_0$ be its covering and associated groups, respectively, let $a$ be a fixed element of $A$, and let

$$\gamma: \ u \to aua^{-1}, \ \ \forall u \in G,$$
$$d_i = a^i, \quad i \in \{1, 2, \ldots\}.$$

Then the following statements hold true:

1) the restriction of $\gamma$ to $A$ is a automorphism of the $n$-ary group $\langle A, [\ \ ]\rangle$;

2) the restriction of $\gamma$ to $A_0$ is a automorphism of the group $A_0$;

3) the mapping $\gamma$ leaves unchanged each element $d_i$ for all $i \in \{1, 2, \ldots\}$;

4) the $i$-th power of the automorphism $\gamma$ acts on $G$ as an inner automorphism, defined by the element $d_i$, i.e., $u^{\gamma^i} = d_i u d_i^{-1}$, $u \in G$.

*Proof.* 1) It is obvious that the restriction of $\gamma$ to $A$ is a bijection. But since

$$\begin{aligned}[x_1 x_2 \ldots x_n]^\gamma &= a(x_1 x_2 \ldots x_n)a^{-1} \\ &= (ax_1a^{-1})(ax_2a^{-1}) \ldots (ax_na^{-1}) = [x_1^\gamma x_2^\gamma \ \ldots \ x_n^\gamma]\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in A$, then $\gamma$ is an automorphism of the $n$-ary group $\langle A, [\ \ ]\rangle$.

2) Since $\gamma$ is an inner automorphism of the group $G$ and $A_0$ is a normal subgroup in $G$, then the restriction of $\gamma$ to $A_0$ is an automorphism of the group $A_0$.

3) Since $d_i^\gamma = a(a^i)a^{-1} = a^i = d_i$, then $d_i^\gamma = d_i$.

4) Since $u^{\gamma^i} = a^i u (a^{-1})^i = a^i u (a^i)^{-1} = d_i u d_i^{-1}$, then $u^{\gamma^i} = d_i u d_i^{-1}$. □

If in Theorem 1.5 one translates the verbose formulations into mathematical formulas, then the converse claim of this Theorem gets the following form.

**Theorem 2.1 (E. Post [8])** *Let $\langle A, [\ ]\rangle$ be an $n$-ary group, let $G$ and $A_0$ be respectively its covering and associate groups. Let $a \in A$ and denote*

$$b = a^{n-1} \in A_0. \tag{2.1}$$

*Let $\gamma$ be the restriction to $A_0$ of the automorphism $\gamma$ from Lemma 1.1, i.e.,*

$$u^\gamma = aua^{-1}, \quad u \in A_0. \tag{2.2}$$

*Then $\gamma$ is an automorphism of the group $A_0$, and for any $x_1, x_2, \ldots, x_n \in A$, and $u \in A_0$, there hold true the following equalities:*

$$[x_1 x_2 \ldots x_n] = u_1 u_2^\gamma \ \ldots \ u_2^{\gamma^{n-1}} ba, \ \text{where } u_i = x_i a^{-1}, \ \forall i \in \overline{1, n}; \tag{2.3}$$

$$b^\gamma = b; \tag{2.4}$$

$$u^{\gamma^{n-1}} = bub^{-1}. \tag{2.5}$$

*Proof.* The fact that $\gamma$ is an automorphism of the group $A_0$ was shown in item 1) of Lemma 2.1. Further, we have

$$u_1 u_2^{\gamma} \dots u_n^{\gamma^{n-1}} ba = (xa^{-1})a(x_2a^{-1})a^{-1}aa(x_3a^{-1})a^{-1}a^{-1}\dots$$

$$\dots a^{n-2}(x_{n-1}a^{-1})\underbrace{a^{-1}\dots a^{-1}}_{n-2}\underbrace{a\dots a}_{n-1}(x_na^{-1})\underbrace{a^{-1}\dots a^{-1}}_{n-1}a^{n-1}a =$$

$$= x_1x_2x_3\dots x_{n-1}x_n = [x_1x_2\dots x_n],$$

which leads to (2.3).

Similarly, by replacing $b = d_{n-1}$ in item 3) of Lemma 2.1, we get (2.4), and by replacing $i = n-1$ and $b = d_{n-1}$ in item 4) of Lemma 2.1, it follows (2.5). □

Consider $\langle A, [\ \ ]\rangle$, $G$, $A_0$, $a$, $b$ and $\gamma$ like in Theorem 2.1. We specified in the introduction that the inverse element $a^{-1}$ in the group $G$ coincides with the product of elements $a_1a_2\dots a_{n-2}$, where $a_1\dots a_{n-2}$ is a inverse sequence for the element $a$ of the $n$-ary group $\langle A, [\ \ ]\rangle$. Hence, taking into account that the $n$-ary operation $[\ \ ]$ derived from the operation of the group $G$, the equalities (1.4)-(1.6) can be re-written in the following form

$$x \circ_a y = xa^{-1}y, \tag{2.6}$$

$$\beta_a:\ \ x \to axa^{-1}, \tag{2.7}$$

$$d_a = a^n, \tag{2.8}$$

where in the right hand sides of the equalities there are present the derivatives of the elements of the group $G$.

We shall show, that the equalities (2.3), (2.4) and (2.5) of Theorem 2.1 correspondingly infer the equalities (1.7), (1.8) and (1.9) of Theorem 1.4.

Since the mapping $\gamma$ is the inner automorphism of the group $G$ which is defined by the element $a$, it leaves unchanged the elements $a$ and $a^{-1}$, i.e. $a^{\gamma} = a$, $(a^{-1})^{\gamma} = a^{-1}$. Moreover, as shown before, the element $a^{-1}$ coincides with the product of elements $a_1a_2\dots a_{n-2}$, where $a_1\dots a_{n-2}$ is the inverse sequence for the element $a$ in the $n$-ary group $\langle A, [\ \ ]\rangle$. We shall use as well the equalities (2.6) and (2.8), and also the fact that $\gamma$ is an automorphism of the group $G$. Then from (2.3) we subsequently obtain the chain of equalities:

$$[x_1x_2\dots x_n] = (x_1a^{-1})(x_2a^{-1})^{\gamma}\dots(x_na^{-1})^{\gamma^{n-1}}a^{n-1}a,$$

$$[x_1x_2\dots x_n] = (x_1a^{-1})x_2^{\gamma}(a^{-1})^{\gamma}\dots x_n^{\gamma^{n-1}}(a^{-1})^{\gamma^{n-1}}a^n,$$

$$[x_1x_2\dots x_n] = x_1a^{-1}x_2^{\gamma}a^{-1}\dots x_n^{\gamma^{n-1}}a^{-1}a^n,$$

$$[x_1x_2\dots x_n] = x_1 \circ_a x_2^{\gamma} \circ_a \dots \circ_a x_n^{\gamma^{n-1}} \circ_a d_a.$$

Due to (2.7), the restriction of the mapping $\gamma$ to $A$ coincides with the mapping $\beta_a$. Therefore, the change in the last equality from above of $\gamma$ into $\beta_a$ leads to the equality (1.7). In this way, from (2.3), it follows (1.7).

From (2.4) we subsequently obtain

$$b^{\gamma}a = ba, \quad b^{\gamma}a^{\gamma} = ba,$$

$$(ba)^\gamma = ba, \quad (a^{n-1}a)^\gamma = a^{n-1}a, \quad (a^n)^\gamma = a^n,$$

whence, from (2.8) and from the fact that the mappings $\gamma$ and $\beta_a$ coincide on the set $A$, it follows (1.8). In this manner, (2.4), implies (1.8).

Similarly, by replacing $u = xa^{-1}$ in (2.5), we subsequently obtain

$$\begin{aligned}
&(xa^{-1})^{\gamma^{n-1}} = a^{n-1}xa^{-1}(a^{-1})^{n-1},\\
&x^{\gamma^{n-1}}(a^{-1})^{\gamma^{n-1}} = a^n a^{-1}xa^{-1}(a^{-1})^{n-1},\\
&x^{\gamma^{n-1}}a^{-1} = a^n a^{-1}xa^{-1}(a^{-1})^{n-1},\\
&x^{\gamma^{n-1}} = a^n \circ_a x \circ_a (a^{-1})^{n-2}.
\end{aligned}$$

The remarks 1.1 and 1.2 yield $(a^{-1})^{n-2} = d_a^{-1}$, and hence, from the last equalities and from the fact that the mappings $\gamma$ and $\beta_a$ coincide on the set $A$, it follows (1.9). In this way, (2.5), implies (1.9).

Now we can affirm that Theorem 1.2 is a consequence of Theorem 2.1. As a matter of fact, the two Theorems are essentially equivalent, since by making all the reasonings in reverse order, we can see that Theorem 2.1 is a consequence of Theorem 1.2.

# 3 The direct implication of Theorem 1.5

If in Theorem 1.5 we transform the verbose statements into formulas, then the direct implication of this Theorem gets the following form.

**Theorem 3.1 (E. Post [8])** *Let the group $G$ have a subgroup $A_0$ and an element $a$ such that* (2.1) *is satisfied; let the subgroup $A_0$ have an automorphism $\gamma$, such that* (2.2)*,* (2.4) *and* (2.5) *hold true. Then the coset $A = A_0a$ is an $n$-ary group with the $n$-ary operation $[\ ]$, derived from the operation of the group $G$, and* (2.3) *holds true; moreover, the set $A_0$ may be represented in the form*

$$A_0 = \{x_1x_2\ldots x_{n-1} \mid x_1, x_2, \ldots, x_{n-1} \in A\}. \tag{3.1}$$

We shall further prove a more general version of Theorem 1.5.

**Theorem 3.2 (E. Post [8])**. *Let the group $G$ have a subgroup $A_0$ and an element $a$ such that* (2.1) *is satisfied; let the subgroup $A_0$ have an automorphism $\gamma$, such that* (2.4) *and* (2.5) *hold true. Then $\langle A = A_0a, [\ \ ]\rangle$ is an $n$-ary group with the $n$-ary operation* (2.3)*. If the action of the automorphism $\gamma$ is defined by* (2.2)*, then the $n$-ary operation $[\ \ ]$ is derived from the operation of the group $G$; moreover, the set $A_0$ can be represented in the form* (3.1)*.*

*Proof.* We readily notice that from the equality $A = A_0a$, it follows that $a \in A$, $A_0 = Aa^{-1}$. Then for any $x_1, x_2, \ldots, x_n \in A$, we have

$$u_1 = x_1a^{-1}, u_2 = x_2a^{-1}, \ldots, u_n = x_na^{-1} \in A_0.$$

Since $A_0$ is a group and $\gamma$ is its automorphism, then $u_1u_2^\gamma \ \ldots\ u_n^{\gamma^{n-1}}b \in A_0$, whence it follows that $u_1u_2^\gamma \ \ldots\ u_n^{\gamma^{n-1}}ba \in A$. Since the $n$-ary operation $[\ ]$ is defined by the equality (2.3), then $[x_1x_2\dots x_n]\in A$. As consequence, the set $A$ is closed relative to the $n$-ary operation $[\ ]$.

By using (2.3), (2.4) and (2.5), then for any $i\in\overline{0,n-1}$, we have

$$[x_i\dots x_i[x_{i+1}\dots x_{i+n}]x_{i+n+1}\dots x_{2n-1}]=$$
$$=u_1u_2^{\gamma}\ \dots\ u_i^{\gamma^{i-1}}([x_{i+1}\dots x_{i+n}]a^{-1})^{\gamma^i}u_{i+n+1}^{\gamma^{i+1}}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_i^{\gamma^{i-1}}(u_{i+1}u_{i+2}^{\gamma}\ \dots\ u_{i+n}^{\gamma^{n-1}}baa^{-1})^{\gamma^i}u_{i+n+1}^{\gamma^{i+1}}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_i^{\gamma^{i-1}}(u_{i+1}u_{i+2}^{\gamma}\ \dots\ u_{i+n}^{\gamma^{n-1}}b)^{\gamma^i}u_{i+n+1}^{\gamma^{i+1}}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_i^{\gamma^{i-1}}(u_{i+1}^{\gamma^i}u_{i+2}^{\gamma^{i+1}}\ \dots\ u_n^{\gamma^{n-1}}u_{n+1}^{\gamma^n}\ \dots\ u_{i+n}^{\gamma^{n-1+i}}b^{\gamma^i})u_{i+n+1}^{\gamma^{i+1}}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_n^{\gamma^{n-1}}(u_{n+1}\ \dots\ u_{i+n}^{\gamma^i})^{\gamma^{n-1}}bu_{i+n+1}^{\gamma^{i+1}}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_n^{\gamma^{n-1}}(bu_{n+1}^{\gamma}\ \dots\ u_{i+n}^{\gamma^i}b^{-1})bu_{i+n+1}^{\gamma^{i+1}}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_n^{\gamma^{n-1}}bu_{n+1}^{\gamma}\ \dots\ u_{i+n}^{\gamma^i}u_{i+n+1}^{\gamma^{i+1}}\ \dots\ a_{2n-1}^{\gamma^{n-1}}ba=$$
$$=u_1u_2^{\gamma}\ \dots\ u_n^{\gamma^{n-1}}bu_{n+1}^{\gamma}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba,$$

i.e.,

$$[x_1\dots x_i[x_{i+1}\dots x_{i+n}]x_{i+n+1}\dots x_{2n-1}]=u_1u_2^{\gamma}\ \dots\ u_n^{\gamma^{n-1}}bu_{n+1}^{\gamma}\ \dots\ u_{2n-1}^{\gamma^{n-1}}ba.$$

Hence, we get

$$[x_1\dots x_i[x_{i+1}\dots x_{i+n}]x_{i+n+1}\dots x_{2n-1}]=[x_1\dots x_j[x_{j+1}\dots x_{j+n}]x_{j+n+1}\dots x_{2n-1}]$$

for any $i,j\in\overline{0,n-1}$, which means exactly the associativity of the $n$-ary operation $[\ ]$.

For any $i\in\overline{1,n}$ and any $g_1,\dots,g_{i-1},g_{i+1},\dots,g_n$ and $h\in A$ we shall further study the solvability in $\langle A,\ [\ ]\rangle$ of the equation

$$[g_1\dots g_{i-1}tg_{i+1}\dots g_n]=h. \tag{3.2}$$

Since we have

$$g_1=u_1a_1,\dots,g_{i-1}=u_{i-1}a,\ \ g_{i+1}=u_{i+1}a,\dots,g_n=u_na,\ \ h=wa,$$

for some $u_1,\dots,u_{i-1},u_{i+1},\dots,u_n,\ w\in A_0$, and considering that $\gamma$ is an automorphism of the group $A_0$, it follows that

$$u_2^{\gamma},\ \dots,\ u_{i-1}^{\gamma^{i-2}},\ u_{i+1}^{\gamma^i},\ \dots,\ u_n^{\gamma^{n-1}}\ \in\ A_0.$$

The equation

$$u_1u_2^{\gamma}\ \dots\ u_{i-1}^{\gamma^{i-2}}s\,u_{i+1}^{\gamma^i}\ \dots\ u_n^{\gamma^{n-1}}b\ =\ w,$$

is solvable within the group $A_0$, i.e., there exists $v\in A_0$, such that

$$u_1u_2^{\gamma}\ \dots\ u_{i-1}^{\gamma^{i-2}}vu_{i+1}^{\gamma^i}\ \dots\ u_n^{\gamma^{n-1}}b\ =\ w.$$

Denoting by $\delta$ the inverse automorphism of the automorphism $\gamma$, and setting $u_i = v^{\delta^{i-1}}$, the last equality can be written as

$$u_1 u_2^{\gamma} \ \ldots \ u_{i-1}^{\gamma^{i-2}} u_i^{\gamma^{i-1}} u_{i+1}^{\gamma^{i}} \ \ldots \ u_n^{\gamma^{n-1}} b \ = \ w,$$

whence

$$u_1 u_2^{\gamma} \ \ldots \ u_{i-1}^{\gamma^{i-2}} u_i^{\gamma^{i-1}} u_{i+1}^{\gamma^{i}} \ \ldots \ u_n^{\gamma^{n-1}} ba \ = \ wa,$$

i.e.,

$$u_1 u_2^{\gamma} \ \ldots \ u_{i-1}^{\gamma^{i-2}} u_i^{\gamma^{i-1}} u_{i+1}^{\gamma^{i}} \ \ldots \ u_n^{\gamma^{n-1}} ba \ = \ h.$$

From the last equality, by setting $g_i = u_i a$, and considering (2.3), we get

$$[g_1 g_2 \ldots g_{i-1} g_i g_{i+1} \ldots g_n] = u_1 u_2^{\gamma} \ \ldots \ u_{i-1}^{\gamma^{i-2}} u_i^{\gamma^{i-1}} u_{i+1}^{\gamma^{i}} \ \ldots \ u_n^{\gamma^{n-1}} ba \ = \ h.$$

Hence $g_i$ is a solution of the equation (3.2). Consequently, according to the definition of W. Dörnte, $\langle A, [\ ]\rangle$ is an $n$-ary group.

By using (2.2) and (2.3), we get

$$\begin{aligned} [x_1 x_2 \ldots x_n] &= u_1 u_2^{\gamma} \ \ldots \ u_n^{\gamma^{n-1}} ba \\ &= x_1 a^{-1}(a x_2 a^{-1} a^{-1})(a^2 x_3 a^{-1} a^{-2}) \ldots \\ &\qquad \ldots (a^{n-2} x_{n-1} a^{-1} a^{-(n-2)})(a^{n-1} x_n a^{-1} a^{-(n-1)}) a^{n-1} a \\ &= x_1 x_2 x_3 \ldots x_{n-1} x_n, \end{aligned}$$

i.e.,

$$[x_1 x_2 \ldots x_n] = x_1 x_2 \ldots x_n.$$

It follows that the $n$-ary operation $[\ ]$ is derived from the operation of the group $G$. Using this fact, and the closedness of the set $A$ relative to the $n$-ary operation $[\ ]$, we get

$$x_1 \ldots x_{n-1} a = [x_1 \ldots x_{n-1} a] \in A$$

for any $x_1, \ldots, x_{n-1} \in A$, whence from the equality $A_0 = Aa^{-1}$ it follows

$$x_1 \ldots x_n = [x_1 \ldots x_n a] a^{-1} \in Aa^{-1} = A_0.$$

Then

$$\{x_1 \ldots x_{n-1} \mid x_1, \ldots, x_{n-1} \in A\} \subseteq A_0.$$

Since $\langle A, [\ ]\rangle$ is an $n$-ary group, then there exist $a_1, \ldots, a_{n-2} \in A$ such that

$$[a a a_1 \ldots a_{n-2}] = a,$$

whence using the fact that the $n$-ary operation $[\ ]$ is derived from the operation of the group $G$, it follows that

$$a a a_1 \ldots a_{n-2} = a.$$

The last equality infers $a^{-1} = a_1 \ldots a_{n-2}$. But since $A_0 = Aa^{-1}$, then

$$A_0 = A a_1 \ldots a_{n-2} \subseteq \{x_1 \ldots x_{n-1} \mid x_1, \ldots, x_{n-1} \in A\}.$$

From the proved inclusions, it results (3.1). $\square$

**Remark 3.1.** Since the $n$-ary operation $[\ \ ]$ of the $n$-ary group $\langle A = A_0a, [\ \ ]\rangle$ from the Theorems 3.1 and 3.2 is derived from the operation of the group $G$, then the subgroup of the group $G$, generated by the coset $A = A_0a$, is the covering group for the $n$-ary group $\langle A = A_0a, [\ \ ]\rangle$. If we remove from the premises of the Theorems 3.1 and 3.2 the assumption that the group $G$ is generated by the coset $A = A_0a$, then we get two more versions of the direct implication of Theorem 1.5, as described below.

**Theorem 3.3 (E. Post [8])** *Let the group $G$ have a subgroup $A_0$ and an element $a$ such that the group $G$ is generated by the coset $A = A_0a$ and* (2.1) *holds true; let $\gamma$ be an automorphism of the subgroup $A_0$, such that* (2.2), (2.4) *and* (2.5) *are satisfied. Then $\langle A = A_0a, [\ \ ]\rangle$ is an $n$-ary group with the $n$-ary operation which is defined by means of* (2.3)*; the covering group of this $n$-ary group is the group $G$, and its associated group is the subgroup $A_0$.*

**Theorem 3.4 (E. Post [8])** *Let the group $G$ have a subgroup $A_0$ and an element $a$ such that the group $G$ is generated by the coset $A = A_0a$ and* (2.1) *holds true; let $\gamma$ be an automorphism of the subgroup $A_0$, such that* (2.4) *and* (2.5) *are satisfied. Then $\langle A = A_0a, [\ \ ]\rangle$ is an $n$-ary group with the $n$-ary operation defined by means of* (2.3)*; if the action of the automorphism $\gamma$ is defined by the rule* (2.2)*, then the covering group of this $n$-ary group is the group $G$, and its associated group is the subgroup $A_0$.*

## 4 Conclusions

The analysis of the Gluskin-Hosszu Theorem and of the corresponding result of E. Post (Theorem 1.5) developed in the present work shows that the considered statements are not only equivalent, but that they practically coincide. Slight differences between these statements can be explained by the fact that E. Post used in his Theorem the group $A_0$, while in the Gluskin-Hosszu's Theorem, there appears a certain isomorphic copy of the group – e.g., the group $\langle A, \circ_a \rangle$, like in Theorem 1.4. The change of the group $A_0$ with its isomorphic copy leads to the fact that in Gluskin-Hosszu's Theorem - unlike in Post's Theorem - in the equality which states the relation between the $n$-ary and the binary operations, the multiplier which allows to define the isomorphism of the considered groups, is missing.

In this way, we promote and sustain the inclusion of Post's name in the former Gluskin-Hosszu name of the Theorem, and provide the necessary scientific grounds in this matter of historical fairness.

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