Univalence condition and properties for two integral operators

N. Ularu and D. Breaz

Abstract. In this paper we consider the class $\mathcal{N}(\beta)$ and two integral operators $I(f,g)(z)$ and $I_1(f,g)(z)$, for analytic functions f,g in the open unit disk $\mathcal{U}$, where g is a functions that belongs to the family $\mathcal{B}(\mu,\beta)$. For this two operators we obtain some properties in this class and the univalence of $I_1(f,g)(z)$.

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1 Introduction and preliminaries

Let $\mathcal{U} = \{z : |z| < 1\}$ be the unit disk and $\mathcal{A}$ the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition $f(0) = f'(0) - 1 = 0$. We denote by $\mathcal{S}$ the class of univalent and regular functions. In this paper, we introduce two integral operators, defined by:

$$I(f,g)(z) = \int_0^z f'(t)e^{g(t)}dt \tag{1.2}$$

and

$$I_1(f,g)(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\alpha} dt, \tag{1.3}$$

where $f,g \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. The operator $I_1(f,g)$ was introduced and studied by Ularu and Breaz in [7].

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We further consider some classes of analytical functions. Many authors studied this classes, proving some interesting properties. We say that a function $f(z) \in \mathcal{S}^*_\beta$ is in the class of starlike functions of order β if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad (z \in \mathcal{U})$$

for some $\beta(0 \leq \beta < 1)$. The class of starlike functions was introduced by Robertson in [6]. As well, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_\beta$ if and only if

$$\mathrm{Re}\,(f'(z)) > \beta, \qquad (z \in \mathcal{U})$$

for some $\beta(0 \leq \beta < 1)$.

Let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ that contains all the functions $f(z)$, which satisfy the inequality

$$\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} < \beta \qquad (\beta > 1, z \in \mathcal{U}).$$

Uralegaddi, Ganigi and Sarangi in [8] and Owa and Srivastava in [5] introduced and studied the class $\mathcal{N}(\beta)$.

The family $\mathcal{B}(\mu, \beta)$, $\mu \geq 0$, $0 \leq \beta < 1$, consisting of the functions $f(z)$, which satisfy the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1 - \beta \qquad (z \in \mathcal{U}). \tag{1.4}$$

was studied by Frasin and Jahangiri in [3]. The family $\mathcal{B}(\mu, \beta)$ is a comprehensive class of analytic functions that includes various new classes of analytic univalent functions, for example, $\mathcal{B}(1, \beta) \subset \mathcal{S}^*_\beta$, and $\mathcal{B}(0, \beta) \equiv \mathcal{R}_\beta$. The subclass $\mathcal{B}(2, \beta) \equiv \mathcal{B}(\beta)$ has been introduced by Frasin and Darus [2].

To prove the univalence condition for the operator $I_1(f, g)(z)$ we will use the Becker univalence criterion given by the following result:

Theorem 1.1. **[1]** *If the function f is regular in the unit disk $\mathcal{U}$, $f(z) = z + a_2z^2 + \ldots$ and*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then the function f is univalent in $\mathcal{U}$.

Lemma 1.1. (General Schwarz-Lemma, [4]) *Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \; z \in \mathcal{U}_R \tag{1.5}$$

the equality (in the inequality (1.5) for $z \neq 0$) can hold only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

The purpose of this paper is to prove the univalence condition for the integral operator $I_1(f, g)(z)$ and to show that the operators $I(f, g)(z)$ and $I_1(f, g)(z)$ are in the class $\mathcal{N}(\delta)$ by using functions from the class $\mathcal{B}(\mu, \beta)$.

2 Main results

Theorem 2.1. *Let α be a complex number with $Re\alpha \geq 1$, M be a positive real number $(M \geq 1)$ and the functions $f, g \in \mathcal{A}$ of the form (1.1). If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathcal{U}, \quad |g'(z)| \leq M, \quad z \in \mathcal{U}$$

and

$$|\alpha| \leq \frac{3\sqrt{3}}{2(M+1)},$$

then the function $I_1(f,g)(z)$ defined by (1.3) is in the class $\mathcal{S}$.

Proof. From (1.3), we have that

$$I_1'(f,g)(z) = \left(f'(z)e^{g(z)}\right)^{\alpha} \tag{2.1}$$

and

$$I_1''(f,g)(z) = \alpha\left(f'(z)e^{g(z)}\right)^{\alpha-1} \cdot \left[e^{g(z)}(f''(z) + f'(z)g'(z))\right]. \tag{2.2}$$

From (2.1) and (2.2) we get

$$\frac{I_1''(f,g)(z)}{I_1'(f,g)(z)} = \alpha\left(\frac{f''(z)}{f'(z)} + g'(z)\right)$$

and

$$\begin{aligned}(1-|z|^2)\left|\frac{zI_1''(f,g)(z)}{I_1'(f,g)(z)}\right| &= (1-|z|^2)\left|z\alpha\left(\frac{f''(z)}{f'(z)} + g'(z)\right)\right| \\ &\leq (1-|z|^2)|z||\alpha|(1+M).\end{aligned} \tag{2.3}$$

Let us consider the function $T : [0,1] \to \mathbb{R}, T(x) = x(1-x^2), x = |z|$. Then we have

$$T(x) \leq \frac{2}{3\sqrt{3}} \tag{2.4}$$

for all $x \in [0,1]$. From (2.3), (2.4) we obtain

$$(1-|z|^2)\left|\frac{zI_1''(f,g)(z)}{I_1'(f,g)(z)}\right| \leq 1. \tag{2.5}$$

From Theorem 1.1 and (2.5) we yield that the function $I_1(f,g)(z)$ is in the class $\mathcal{S}$. □

Corollary 2.1. *Let α be a complex number with $Re\alpha \geq 1$ and let be the functions $f, g \in \mathcal{A}$ of the form (1.1). If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 1, \quad z \in \mathcal{U}, \quad |g'(z)| < 1, \quad z \in \mathcal{U}$$

and

$$|\alpha| \leq \frac{3\sqrt{3}}{4},$$

then the function $I_1(f,g)(z)$ defined by (1.3) is in the class $\mathcal{S}$.

Proof. We consider $M = 1$ in Theorem 2.1. $\square$

Example 2.1. *We consider the analytical functions $f(z) = z+\frac{1}{2}z^2$ and $g(z) = z\cdot e^z$. Because the functions $f(z)$ and $g(z)$ satisfy the conditions from Theorem 2.1, we obtain that*

$$I(z) = \int_0^z \left[(1+t)\cdot e^{t\cdot e^t}\right]^\alpha dt$$

is in the univalent functions class $\mathcal{S}$.

Theorem 2.2. *Let the functions $f, g \in \mathcal{A}$, with g in the class $\mathcal{B}(\mu,\beta), \mu \geq 1, 0 \leq \beta < 1$. If $|g(z)| < M$, for M a positive real number ($M \geq 1$), $z \in \mathcal{U}$ and $\left|\frac{f''(z)}{f'(z)}\right| < 1$, then the integral operator $I(f,g)(z)$ defined by (1.2) is in $\mathcal{N}(\rho)$, where*

$$\rho = 2 + (2-\beta)M^\mu.$$

Proof. From (1.2) we obtain that

$$\frac{zI''(f,g)(z)}{I'(f,g)(z)} = z\left(\frac{f''(z)}{f'(z)} + g'(z)\right).$$

Thus we have

$$\begin{aligned}
\operatorname{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}+1\right) &= \operatorname{Re}\left[z\left(\frac{f''(z)}{f'(z)}+g'(z)\right)+1\right] \\
&< |z|\left(\left|\frac{f''(z)}{f'(z)}\right| + |g'(z)|\right)+1 \\
&< |z|\left(1+\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu\right|\left|\frac{g(z)}{z}\right|^\mu\right)+1.
\end{aligned} \tag{2.6}$$

Because $g \in \mathcal{B}(\mu,\beta), |g(z)| < M$, applying the General Schwarz Lemma and from (2.6), we obtain

$$\begin{aligned}
\operatorname{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}+1\right) &< \left[1+\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu - 1\right|+1\right)M^\mu\right]+1 \\
&< 2+(2-\beta)M^\mu = \rho,
\end{aligned}$$

which implies that the integral operator $I(f,g)(z)$ is in the class $\mathcal{N}(\rho)$ $\square$

Corollary 2.2. *Consider the functions $f, g \in \mathcal{A}$, with g in the class $\mathcal{R}_\beta, 0 \leq \beta < 1$. If $|g(z)| < M$, for M a positive real number ($M \geq 1$), $z \in \mathcal{U}$ and $\left|\frac{f''(z)}{f'(z)}\right| < 1$, then the integral operator $I(f,g)(z)$ defined by (1.2) is in $\mathcal{N}(\rho)$, where*

$$\rho = 4 - \beta.$$

Proof. We put $\mu = 0$ in Theorem 2.2. $\square$

For $\mu = 1$ in Theorem 2.2 we obtain:

Corollary 2.3. *Let the functions $f, g \in \mathcal{A}$, with g in the class $\mathcal{S}^*_\beta, 0 \leq \beta < 1$. If $|g(z)| < M$, for M a positive real number ($M \geq 1$), $z \in \mathcal{U}$ and $\left|\frac{f''(z)}{f'(z)}\right| < 1$, then the integral operator $I(f, g)(z)$ defined by (1.2) is in $\mathcal{N}(\rho)$, where*

$$\rho = 2 + (2 - \beta)M.$$

Theorem 2.3. *Let the functions $f, g \in \mathcal{A}$, with g in the class $\mathcal{B}(\mu, \beta), \mu \geq 1, 0 \leq \beta < 1$ and α a complex number with $\mathrm{Re}\alpha \geq 1$. If $|g(z)| < M$, for M a positive real number ($M \geq 1$), $z \in \mathcal{U}$ and $\left|\frac{f''(z)}{f'(z)}\right| < 1$, then the integral operator $I_1(f, g)(z)$ defined by (1.3) is in $\mathcal{N}(\rho)$, where*

$$\rho = |\alpha|(1 + (2 - \beta)M^\mu) + 1.$$

Proof. From (1.3) we obtain

$$\frac{zI_1''(f, g)(z)}{I_1'(f, g)(z)} = z\alpha\left(\frac{f''(z)}{f'(z)} + g'(z)\right)$$

So,

$$\begin{aligned} \mathrm{Re}\left(\frac{zI_1''(f, g)(z)}{I_1'(f, g)(z)} + 1\right) &= \mathrm{Re}\left[\alpha z\left(\frac{f''(z)}{f'(z)} + g'(z)\right) + 1\right] \\ &< |\alpha||z|\left(\left|\frac{f''(z)}{f'(z)}\right| + |g'(z)|\right) + 1 \\ &< |\alpha||z|\left(1 + \left|g'(z)\left(\frac{z}{g(z)}\right)^\mu\right|\left|\frac{g(z)}{z}\right|^\mu\right) + 1. \end{aligned} \tag{2.7}$$

Since $g \in \mathcal{B}(\mu, \beta), |g(z)| < M$ from General Schwarz Lemma and from (2.7), we infer

$$\begin{aligned} \mathrm{Re}\left(\frac{zI_1''(f, g)(z)}{I_1'(f, g)(z)} + 1\right) &< |\alpha|\left(1 + \left(\left|g'(z)\left(\frac{z}{g(z)}\right)^\mu - 1\right| + 1\right)M^\mu\right) + 1 \\ &< |\alpha|(1 + (2 - \beta)M^\mu) + 1 = \rho. \end{aligned}$$

So the integral operator $I_1(f, g)(z) \in \mathcal{N}(\rho)$. $\square$

Corollary 2.4. *Let the functions $f, g \in \mathcal{A}$, with g in the class $\mathcal{B}(1, \beta) \subset \mathcal{S}^*_\beta, 0 \leq \beta < 1$ and α a complex number with $\mathrm{Re}\alpha \geq 1$. If $|g(z)| < M$, for M a positive real number ($M \geq 1$), $z \in \mathcal{U}$ and $\left|\frac{f''(z)}{f'(z)}\right| < 1$, then the integral operator $I_1(f, g)(z)$ defined by (1.3) is in $\mathcal{N}(\rho)$, where*

$$\rho = |\alpha|(1 + (2 - \beta)M) + 1.$$

Proof. In Theorem 2.3 we put $\mu = 1$. $\square$

For $\mu = 0$ in Theorem 2.3 we obtain:

Corollary 2.5. *Let the functions $f, g \in \mathcal{A}$, with g in the class $\mathcal{R}_\beta, 0 \leq \beta < 1$ and α a complex number with $\mathrm{Re}\alpha \geq 1$. If $|g(z)| < M$, for M a positive real number ($M \geq 1$), $z \in \mathcal{U}$ and $\left|\frac{f''(z)}{f'(z)}\right| < 1$, then the integral operator $I_1(f, g)(z)$ defined by (1.3) is in $\mathcal{N}(\rho)$, where*

$$\rho = |\alpha|(3 - \beta) + 1.$$

3 Conclusions

The future research will study similar properties using other classes of analytical functions for the integral operators which were defined in this paper, further generalizing the integral operators and getting results related to convexity, starlikeness, univalence conditions, etc.

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