

An algorithm for payoff space in C^1 parametric games

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Abstract. We present a new algorithm which determines the payoff-space of certain normal-form C^1 parametric games, and - more generally - of continuous families of normal-form C^1 games. This algorithm was implemented in MATLAB, and was applied to several real-life cases. It has the merit of providing the parametric expressions of the critical zone for any game in the considered family both in the bistrategy space and in the payoff space, and it allows to both graphically illustrate the disjoint union (with respect to the parameter set of the parametric game) of the family of all payoff spaces, and parametrically represent the union of all the critical zones. One of the main motivations of our paper is that, in the applications, many normal-form games naturally appear in a parametric form; moreover, some efficient models of *coopetition* are parametric games of the considered type. Specifically, our developed algorithm provides the parametric and graphical representation of the payoff space and of the critical zone of a parametric game in normal-form, supported by a finite family of compact intervals of the real line. It is a valuable tool in the study of simple normal-form C^1 -parametric games in two dimensions.

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Key words: normal form games; critical zone; payoff space.

1 Introduction and motivations

Our study determines the payoff space of normal-form C^1 -games in n dimensions, for n -players normal-form games, whose payoff functions are defined on compact intervals of the real line and are of class at least C^1 . In the particular case of two dimensions, the payoff space of normal-form C^1 -games is determined. We establish an appropriate procedure for the representation of the payoff space in normal-form C^1 parametric games.

The complete study of C^1 parametric games is strongly motivated not only by theoretical and pure mathematical reasons, but especially by their applications to Economics, Finance and Social Sciences. Indeed, many real interactions - between competitive or cooperative subjects - are modeled by games with time dependence

of strategy sets and of payoff functions. Moreover, recently, one operative and particularly efficient model of coopetition has been proposed and applied by D. Carfi et al. ([3]-[5]). The model is given by a particular type of parametric game, in which the parameter set is the cooperative strategy set of the game (see [7, 8]). This is the basic object which allows to pass from the standard normal-form definition of game (cf. [1, 2, 10, 11]) to their *coopetitive* extension. Our algorithm provides a general vision on the payoff space of a parametric game and of the Nash paths, which are of fundamental importance in applications.

2 Preliminaries on normal-form C^1 games

We shall consider n -person games in normal-form, as follows:

Definition 2.1 (of game in normal-form). Let $E = (E_i)_{i=1}^n$ be an ordered family of non-empty sets. We call *n -person game in normal-form with support E* , any function $f : {}^\times E \rightarrow \mathbb{R}^n$, where ${}^\times E$ denotes the Cartesian product $\times_{i=1}^n E_i$ of the family E . The set E_i is called the *strategy set of player i* , for every index i of the family E , and the product ${}^\times E$ is called the *strategy profile space*, or the *n -strategy space*, of the game.

We introduce the following terminology:

- the set $\{i\}_{i=1}^n$ of the first n positive integers is called *the set of the players* of the game;
- each element of the Cartesian product ${}^\times E$ is called a *strategy profile* of the game;
- the image of the function f , i.e., the set $f({}^\times E)$ of all real n -vectors of type $f(x)$, with x in the strategy profile space ${}^\times E$, is called the *n -payoff space*, or simply the *payoff space*, of the game f .

We further recall, for the sake of completeness, the definitions of usual order of the Euclidean n -space and of the Pareto boundary of a subset in \mathbb{R}^n , which are used throughout the paper.

Definition 2.2 (of natural order of \mathbb{R}^n). The natural order of \mathbb{R}^n is the binary relation \leq defined, for every $x, y \in \mathbb{R}^n$, by $x \leq y$, if the i -th component of x is less or equal to the i -th component of y , for all $i = 1, \dots, n$.

Definition 2.3 (of Pareto boundary). The *Pareto maximal boundary* of a game f is the subset of the n -strategy space of those n -strategies x , such that the corresponding payoff $f(x)$ is maximal in the n -payoff space, with respect to the usual order of the Euclidean n -space \mathbb{R}^n . If S denotes the strategy space ${}^\times E$, then we shall denote the maximal boundary of the n -payoff space by $\bar{\partial}f(S)$ and the maximal boundary of the game by $\bar{\partial}_f(S)$ or by $\bar{\partial}(f)$. In other terms, the maximal boundary $\bar{\partial}_f(S)$ of the game is the reciprocal image (via the function f) of the maximal boundary of the payoff space $f(S)$. We shall use similar terminology and notations for the *minimal Pareto boundary*.

The fundamental properties of Pareto boundaries are appropriately discussed in [5].

3 The method for C^1 games

The method for studying normal-form C^1 games is presented and applied in [3, 4, 6].

The context. We deal with a type of normal-form game f defined on the product of n compact non-degenerate intervals of the real line, such that f is the restriction to the n -strategy space of a C^1 function defined on an open set of \mathbb{R}^n which contains the n -strategy space S (in this case, a compact non-degenerate n -interval of the n -space \mathbb{R}^n). Before giving the main result of the method, we recall several basic notions.

3.1 Topological boundary

The *topological boundary* of a subset S of a topological space (X, τ) is the set defined by the following three equivalent properties:

- it is the closure of S minus the interior of S :

$$\partial S = \text{cl}(S) \setminus \text{int}(S);$$

- it is the intersection of the closure of S with the closure of its complement

$$\partial S = \text{cl}(S) \cap \text{cl}(X \setminus S);$$

- it is the set of those points x of X for which every neighborhood of x contains at least one point of S and at least one point from the complement of S .

The main result of our method states that

Theorem 3.1. *Let f be a C^1 function defined upon an open set O of the Euclidean space \mathbb{R}^n with values in \mathbb{R}^n . Then, for every part S of the open O , the topological boundary of the image of S via the function f is contained in the union*

$$f(\partial S) \cup f(C),$$

where C is the critical set of f in S - i.e., the set of the points x of S such that the Jacobian matrix $J_f(x)$ is not invertible. If, moreover, the function f is not continuous over a part H of O and is C^1 elsewhere in O , then the topological boundary of the image of S via the function f is contained in the union $f(\partial S) \cup f(C) \cup f(H)$, where C is the critical set of f in S .

4 Two players parametric games

In this article we shall use the following definitions for parametric games.

Definition 4.1. Let $E = (E_t)_{t \in T}$ and $F = (F_t)_{t \in T}$ be two families of non empty sets and let $f = (f_t)_{t \in T}$ be a family of functions

$$f_t : E_t \times F_t \rightarrow \mathbb{R}^2.$$

We define a parametric game over the strategy pair (E, F) and with family of payoff functions f , the pair $G = (f, >)$, where " $>$ " is the usual strict upper order of the

Euclidean plane \mathbb{R}^2 . We define the payoff space of the parametric game G as the union of all the payoff spaces of the game family $g = ((f_t, >))_{t \in T}$, i.e., the union of the payoff family

$$P = (f_t(E_t \times F_t))_{t \in T}.$$

We note that the family P can be identified with the multi-valued path

$$p : T \rightarrow \mathbb{R}^2 : t \mapsto f_t(E_t \times F_t),$$

and that the graph of this path p is a subset of the Cartesian product $T \times \mathbb{R}^2$. We note that, indeed, the mapping p is defined on an interval and is set-valued with values belonging to the power set $\mathcal{P}(\mathbb{R}^n)$; hence, we can also interpret this multi-function as an evolution curve of the section payoff spaces of our parametric game.

In particular, we focus on the following particular type of parametric games:

- parametric games in which the families E and F consist of only one set, respectively.

In the latter case we can identify a parametric game with a pair $(f, >)$, where f is a function from a Cartesian product $T \times E \times F$ into the plane \mathbb{R}^2 , where T , E and F are three non-empty sets.

Definition 4.2. When the triple (T, E, F) is a triple of subsets of normed spaces, we say that the parametric game $(f, >)$ is of class C^1 , if the function f is of class C^1 .

5 Numerical results

Consider a (loss) parametric game $(h, <)$, with strategy sets $E = F = [0, 1]$, parameter set $T = [0, 1]^2$ and biloss (disutility) function $h : \times(T, E, F) \rightarrow \mathbb{R}^2$, whose section

$$h_{(a,b)} : \times(E, F) \rightarrow \mathbb{R}^2$$

is defined by

$$h_{(a,b)}(x, y) = (x - (1 - a)xy, y - (1 - b)xy),$$

for all $(x, y) \in E \times F$ and $(a, b) \in T = [0, 1]^2$.

The above game is the von Neuman convexification of the finite game represented as

$$\begin{array}{cc} (a, b) & (1, 0) \\ (0, 1) & (0, 0) \end{array}.$$

Assume, now, that the parameter points (a, b) belong also to the 1-sphere \mathbf{S}_p^1 , with respect to the p -norm, in the Euclidean plane \mathbb{R}^2 , for some positive real p ; i.e., we assume that $a^p + b^p = 1$ for some positive real p . Consider, then, the restriction $g : S \times E \times F \rightarrow \mathbb{R}^2$ of the function h to the parameter set $S = \mathbf{S}_p^1 \cap T$.

By projection on the first factor of the product $S \times E \times F$, we can consider, instead of the parametric game g with parameter set $\mathbf{S}_p^1 \cap T$, the equivalent parametric game $(f, <)$, with parameter set $[0, 1]$ and a -payoff function f_a defined by

$$f_a(x, y) = \left(x - (1 - a)xy, y - (1 - (1 - a^p)^{1/p})xy \right),$$

for all $(x, y) \in E \times F$ and $a \in [0, 1]$.¹ In the following sections we shall consider the following sub-cases:

1. $(p = 1)$ $f_a(x, y) = (x - (1 - a)xy, y - axy), \forall x, y, a \in [0, 1]$.
2. $(p = 0.1)$ $f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^{0.1})^{10})xy), \forall x, y, a \in [0, 1]$.
3. $(p = 0.5)$ $f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^{0.5})^2)xy), \forall x, y, a \in [0, 1]$.
4. $(p = 2)$ $f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^2)^{0.5})xy), \forall x, y, a \in [0, 1]$.
5. $(p = 10)$ $f_a(x, y) = (x - (1 - xy), y - (1 - (1 - a^{10})^{0.1})xy), \forall x, y, a \in [0, 1]$.

Moreover, we shall present the following games:

6. $f_a(x, y) = (x + y + a, x - y + a^2),$ for all $x, y \in [0, 2]$ and $a \in [0, 1]$.
7. $f_a(x, y) = (x + y + a, x - y + |a|),$ for all $x, y \in [0, 2]$ and $a \in [-1, 1]$.

6 First game $p = 1$

Let $E = F = [0, 1]$ be the strategy sets and let a be a real fixed number from the interval $[0, 1]$. Consider the a -biloss (disutility) function of the parametric game $(f, <)$, defined by

$$f_a(x, y) = (x - (1 - a)xy, y - axy), \quad \forall (x, y) \in [0, 1].$$

The critical zone of the function f_a (represented in Figure 1 for every a in $[0, 1]$) is the set

$$\mathcal{C}(f_a) = \{(x, y) \in [0, 1]^2 : 1 - ax - (1 - a)y = 0\}.$$

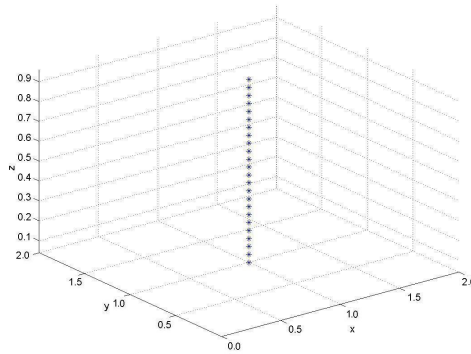


Figure 1: Disjoint union of the critical zones.

¹By equivalent parametric game, we mean the existence of the bijection $j : S \rightarrow [0, 1] : (a, b) \mapsto a$, whose inverse is the bijection $j^{-1} : [0, 1] \rightarrow S : a \mapsto (a, (1 - a^p)^{1/p})$.

The disjoint union of the family $(f_a(\partial(E \times F)))_{a \in T}$, i.e., the disjoint union of the transformations of the topological boundaries of the bistrategy space with respect to the parameter set, is shown in Figure 2.

Recalling that the action of a family of functions sharing a common domain on a subset of the common domain of the member-functions, is the family of the images (transformations) of the subset, we can consider the above disjoint union as a faithful representation of the action of the entire family $(f_a)_{a \in T}$, on the boundary $\partial(E \times F)$.

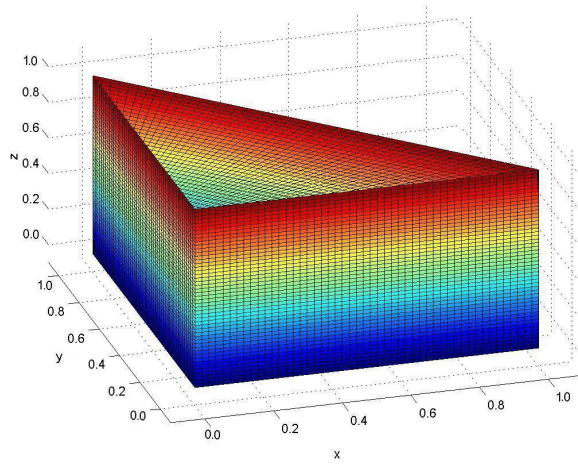


Figure 2: First Game: Disjoint union of transformations of the topological boundaries of the bistrategy space.

The disjoint union of transformations of the critical zones is shown in Figure 3.

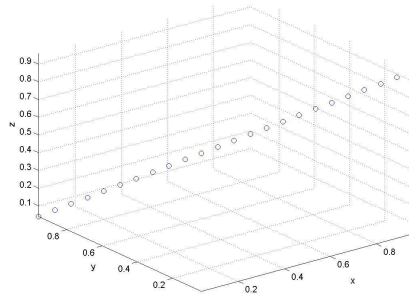


Figure 3: First Game: Disjoint union of transformations of the critical zones.

So, from the transformations of the topological boundaries and of the critical zones, we obtain the representation of the Payoff Space of the parametric game as disjoint union of the family $(f_a(E \times F))_{a \in T}$, i.e., the disjoint union of the transformations of the payoff spaces with respect to the parameter set, as shown in Figure 4. Moreover, we observe that this last disjoint union is the graph of the multivalued curve $c : T \rightarrow \mathbb{R}^2 : a \mapsto f_a(E \times F)$, after an irrelevant permutation

$$J : T \times \mathbb{R}^2 \rightarrow \mathbb{R}^2 \times T : (a, X, Y) \mapsto (X, Y, a).$$

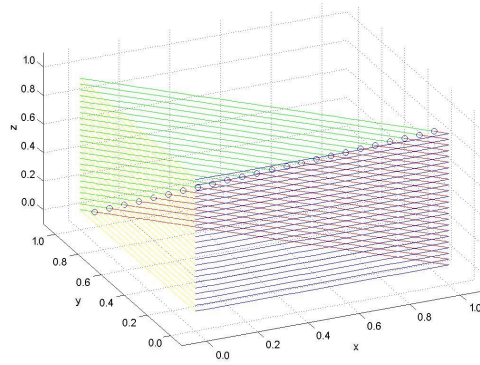


Figure 4: First Game: Disjoint union of payoff spaces.

7 Second game: $p = 0.1$

Let $E = F = [0, 1]$ be the strategy sets and let f_a be the a -biloss (disutility) function

$$f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^{0.1})^{10})xy),$$

for all x, y, a in $[0, 1]$. The critical zone of the a -biloss function is

$$\mathcal{C}(f_a) = \{(x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{0.1})^{10})x - (1 - a)y = 0\}.$$

The disjoint union of critical zones is shown in Figure 5. The disjoint union of transformations of the topological boundary of the bistrategy space is presented in Figure 6. The disjoint union of transformation of critical zones is illustrated in Figure 7. The Payoff Space (represented in Figure 8 for every a in T), is obtained from the union of transformations of the critical zone and of the topological boundary of the bistrategy space.

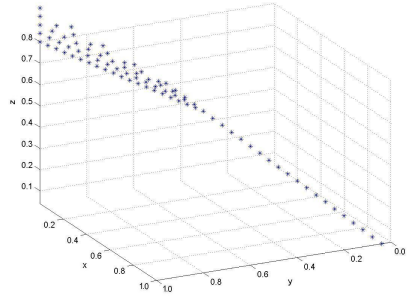


Figure 5: Second Game: Disjoint union of critical zones.

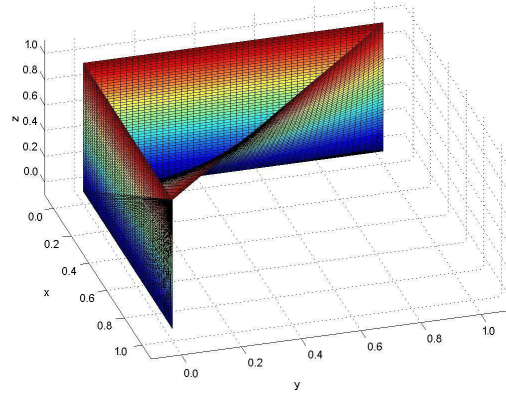


Figure 6: Second Game: Disjoint union of transformation of the topological boundary of the bistrategy space.

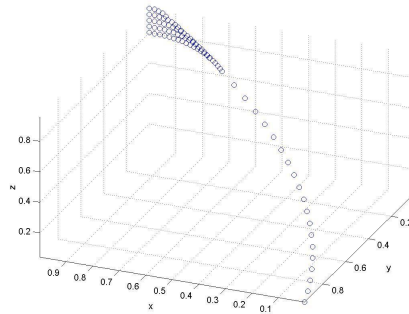


Figure 7: Second Game: Disjoint union of transformations of the critical zones.

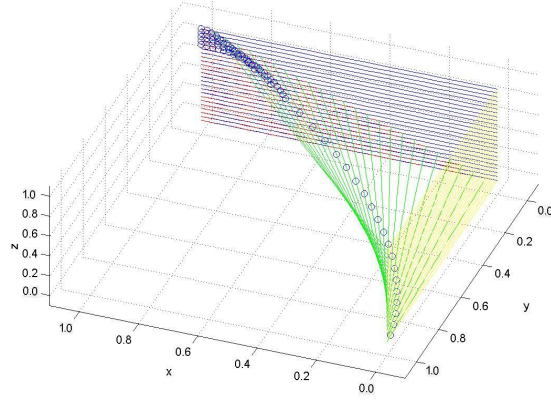


Figure 8: Second Game: Payoff space of the parametric game as disjoint union of partial payoff spaces.

8 Third game: $p = 0.5$

Let the strategy sets of the parametric game $G = (f, <)$ be $E = F = [0, 1]$ and let the a -biloss (disutility) function of G be defined by

$$f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^{0.5})^2)xy), \quad \forall x, y, a \in [0, 1].$$

The critical zones, in Figure 9, are the sets

$$\mathcal{C}(f_a) = \left\{ (x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{0.5})^2)x - (1 - a)y = 0 \right\},$$

with a varying in T . The transformations of the topological boundary of the bistrategy space are shown in Figure 10. The transformations of critical zones are presented in Figure 11. We obtain the payoff space as before, shown in Figure 12 as disjoint union.

9 Fourth game: $p = 2$

Let the strategy sets of the parametric game $G = (f, <)$ be $E = F = [0, 1]$ and let the a -biloss (disutility) function of G be defined by

$$f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^2)^{0.5})xy), \quad \forall x, y, a \in [0, 1].$$

The a -critical zone is

$$\mathcal{C}(f_a) = \left\{ (x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^2)^{0.5})x - (1 - a)y = 0 \right\}.$$

So the Payoff Space overlap the transformation of the topological boundary, as shown in Figure 13 for every a .

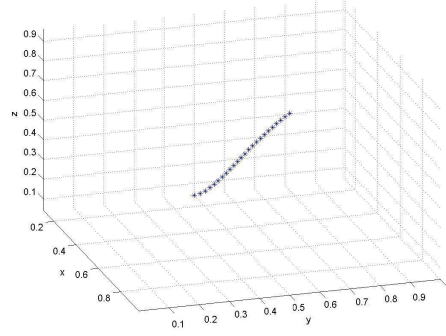


Figure 9: Second Game: Critical zones.

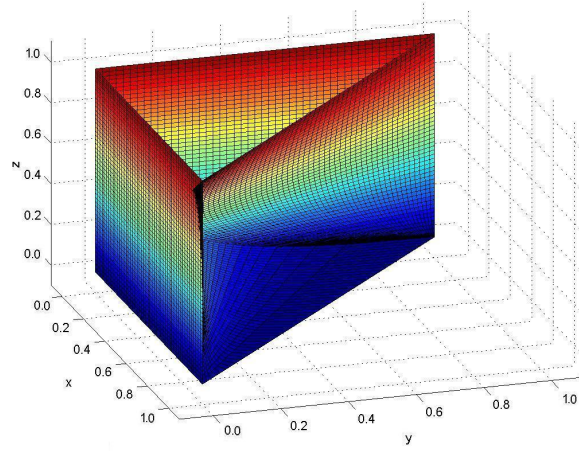


Figure 10: Third Game: Transformations of the topological boundary of the bistrategy space.

10 Fifth game: $p=10$

Let strategy sets be $E = F = [0, 1]$ and biloss (disutility) function be

$$f_a(x, y) = (x - (1 - a)xy, y - (1 - (1 - a^{10})^{0.1})xy), \quad \forall x, y, a \in [0, 1].$$

The critical zones, as depicted in Figure ??, are

$$\mathcal{C}(f_a) = \left\{ (x, y) \in [0, 1]^2 : 1 - (1 - (1 - a^{10})^{0.1})x - (1 - a)y = 0 \right\},$$

with a varying in T .

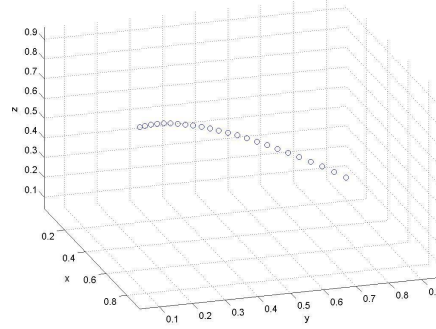


Figure 11: Third Game: Transformations of critical zones.

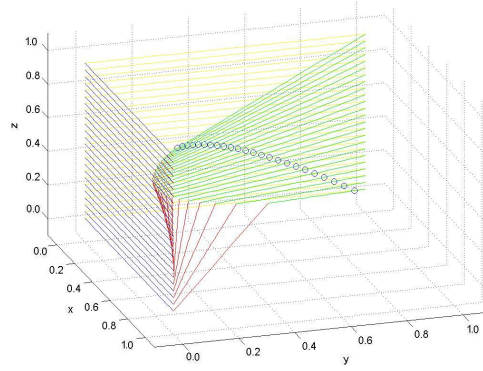


Figure 12: Third Game: Payoff space in form of disjoint union.

11 Sixth game

In this section we present a new game, where the strategy sets are $E = F = [0, 2]$, the parameter set is $T = [0, 1]$ and the a -biloss (disutility) function is defined by

$$f_a(x, y) = (x + y + a, x - y + a^2), \quad \forall x, y, a \in [0, 1].$$

The critical zone is void, so the payoff spaces overlap the transformations of the topological boundary, as shown in Figure 15.

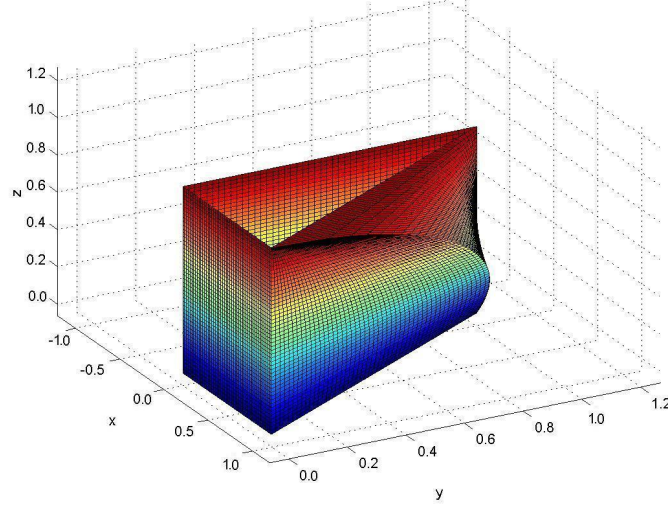


Figure 13: Fourth Game: Transformations of the topological boundary.

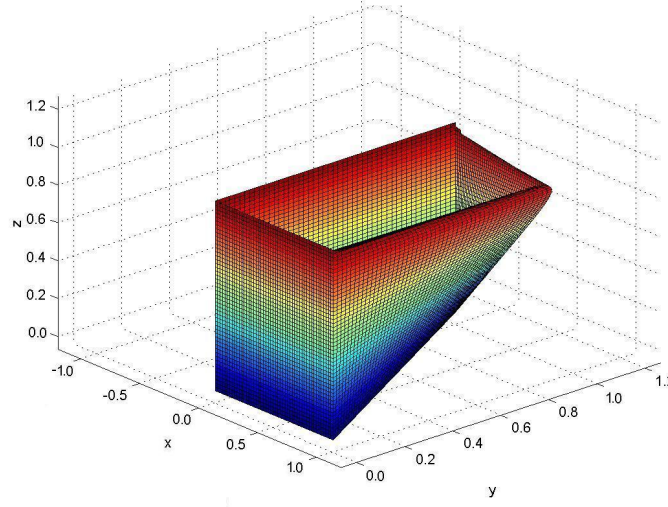


Figure 14: Fifth Game: Transformations of the topological boundary.

12 Seventh game

We further describe a new game, whose strategy sets are $E = F = [0, 2]$, the parameter set is $T = [-1, 1]$ and the a -biloss (disutility) function is

$$f_a(x, y) = (x + y + a, x - y + |a|), \quad \forall x, y, a \in [-1, 1].$$

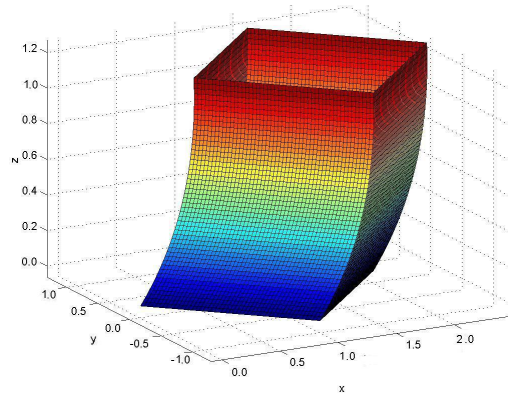


Figure 15: Sixth Game: Transformation of the topological boundary

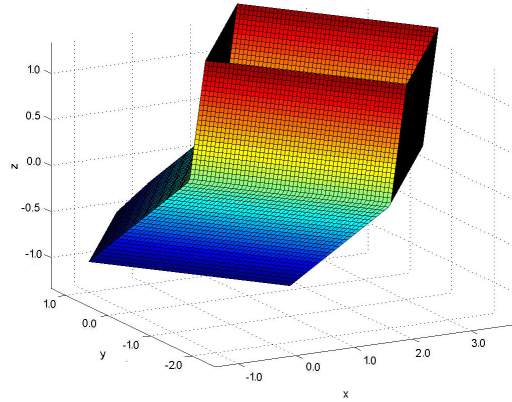


Figure 16: Seventh Game: Transformations of the topological boundary

The critical zone is empty, so the payoff spaces overlap the transformations of the topological boundary, as shown in Figure 15.

Conclusions

In this paper we present an algorithm which completely describes the payoff space of a C^1 parametric game in normal-form. Specifically, it allows to determine a parametric and graphical representation of its payoff space and of its critical zone. The reasons of this work arise not only from theoretical and pure mathematical motivations, but especially from multiple applications to Economics, Finance and Social Sciences.

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