Some global properties of pseudo-cyclic Ricci symmetric manifolds

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Abstract. The present paper deals with a study of some global properties of pseudo cyclic Ricci symmetric manifolds. It is shown that in a compact, orientable pseudo cyclic Ricci symmetric manifold without boundary, there exists no non-zero Killing (resp., projective Killing, conformal Killing) vector field, and also it is proved that in such a manifold a harmonic vector field reduces to a parallel vector field.

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1 Introduction

It is well known that a Riemannian manifold is locally symmetric due to Cartan if its curvature tensor R satisfies $\nabla R = 0$, where ∇ denotes the Riemannian connection. The notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by A. G. Walker (see [15]), weakly symmetric manifold by L. Tamássy and T. Q. Binh (see [14]), semisymmetric manifold by Z. I. Szabó (see [13]), pseudosymmetric manifold by R. Deszcz (see [5]) and pseudosymmetric manifold by M. C. Chaki (see [2]). However, the notion of pseudosymmetry by R. Deszcz is different to that by M. C. Chaki (see [2]). Again, a Riemannian manifold is Ricci symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. Every locally symmetric manifold is Ricci symmetric but not conversely. However, the converse is true in dimension three. During the last five decades, the notion of Ricci symmetry has been weakened by many authors such as Ricci-recurrent manifolds (see [10]), Ricci semisymmetric manifolds (see [13]), pseudo Ricci symmetric manifolds by M. C. Chaki (see [3]) and Ricci pseudosymmetric manifold by R. Deszcz (see [5]). It may be noted that the notion of Ricci pseudosymmetry by R. Deszcz (see [5]) is not equivalent to that by M. C. Chaki (see [3]). In this connection we can mention the work of I. E. Hirică (see [6]) and J. P. Jaiswal and R. H. Ojha (see [7]) who have studied the pseudosymmetric Riemannian manifolds and weakly pseudo-projectively symmetric

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manifolds respectively. A Riemannian manifold $(M^n, g), n > 2$, is said to be pseudo Ricci symmetric (see [3]) if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X), \tag{1.1}$$

where A is a nowhere vanishing 1-form and ∇ denotes the Riemannian connection. Such an n-dimensional manifold is denoted by $(PRS)_n$. Pseudo Ricci symmetric manifold has also been studied by K. Arslan et. al. (see [1]).

By extending the notion of pseudo Ricci symmetric manifold, recently the present authors (see [11]) introduced the notion of pseudo cyclic Ricci symmetric manifold. A Riemannian manifold $(M^n, g), n > 2$, is called *pseudo cyclic Ricci symmetric manifold* if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following:

$$\begin{aligned} &(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ &\quad = 2A(X)S(Y,Z) + A(Y)S(X,Z) + A(Z)S(Y,X) \\ &\quad = 2A(Y)S(Z,X) + A(Z)S(X,Y) + A(X)S(Y,Z) \\ &\quad = 2A(Z)S(X,Y) + A(X)S(Y,Z) + A(Y)S(Z,X), \end{aligned} \tag{1.2}$$

where A is a nowhere vanishing 1-form associated to the vector field ρ such that $A(X) = g(X, \rho)$ for all X. Such an n-dimensional manifold is denoted by $(PCRS)_n$. Every $(PRS)_n$ is $(PCRS)_n$ but not conversely (see [11]).

Section 2 is concerned with preliminaries. Section 3 deals with some global properties of $(PCRS)_n$ and it is proved that under certain condition such a manifold does not admit non-zero Killing (resp., projective Killing and conformal Killing) vector field. Also it is shown that any harmonic vector field on a compact orientable $(PCRS)_n$ reduces to a parallel vector field.

2 Preliminaries

Let Q be the symmetric endomorphism of the tangent bundle corresponding to the Ricci tensor S, i.e., $S(X,Y) = g(QX,Y)$ for all vector fields X, Y. Let $\{e_i : i = 1, 2, \cdots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Let r be the scalar curvature of $(PCRS)_n$. Then the Ricci tensor of a $(PCRS)_n$ is of the form (2.1) (see [11]):

Proposition 2.1. *In a $(PCRS)_n, n > 2$, the scalar curvature can not vanish and the Ricci tensor is of the form*

$$S(X,Y) = rT(X)T(Y), \tag{2.1}$$

where $T(X) = g(X, \lambda) = \frac{1}{\sqrt{A(\rho)}} A(X)$, λ being a unit vector field associated with the nowhere vanishing 1-form T.

From (2.1) it follows that a $(PCRS)_n$ is a special type of quasi-Einstein manifold (see [4], [8], [9], [11], [12]). Again from (2.1), we get

$$S(X,X) = r[g(X,\lambda)]^2 \quad \text{for all } X \tag{2.2}$$

and $S(\lambda, \lambda) = r$. Let θ be the angle between λ and an arbitrary vector X. Then

$$\cos\theta = \frac{g(X,\lambda)}{\sqrt{g(\lambda,\lambda)}\sqrt{g(X,X)}} = \frac{g(X,\lambda)}{\sqrt{g(X,X)}}.$$

Hence $[g(X,\lambda)]^2 \leq g(X,X) = |X|^2$. Consequently, from (2.2) it follows that

$$S(X,X) \leq r|X|^2. \tag{2.3}$$

Let $l^2 = \sum_{i=1}^{n} S(Qe_i, e_i)$ be the square of the length of Ricci tensor. Then (2.1) infers

$$\begin{aligned} l^2 &= \sum_{i=1}^{n} S(Qe_i, e_i) = r\sum_{i=1}^{n} T(Qe_i)T(e_i) \\ &= r\sum_{i=1}^{n} g(Qe_i,\lambda)g(e_i,\lambda) = rg(Q\lambda,\lambda) = rS(\lambda,\lambda) = r^2. \end{aligned}$$

This leads to the following:

Proposition 2.2. *The length of the Ricci tensor of a $(PCRS)_n, n > 2$, is r.*

3 Global properties of $(PCRS)_n$

This section is concerned with a compact, orientable $(PCRS)_n, n > 2$, without boundary with λ as the generator. Then we prove the following:

Theorem 3.1. *If in a compact, orientable $(PCRS)_n, n > 2$, without boundary, the scalar curvature r is negative, then there exists no non-zero Killing vector field in this manifold.*

Proof. It is known that (see [16]) for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \Big[S(X,X) - |\nabla X|^2 - (div\, X)^2\Big] dv \leq 0, \tag{3.1}$$

where dv denotes the volume element of M. If X is a Killing vector field, then $div\, X = 0$ (see [17]). Hence (3.1) takes the following form

$$\int_M \Big[S(X,X) - |\nabla X|^2\Big] dv = 0. \tag{3.2}$$

Let us consider that $r < 0$. Then by virtue of (2.3) we have

$$\int_{M=(PCRS)_n} \Big[r|X|^2 - |\nabla X|^2\Big] dv \geq \int_M \Big[S(X,X) - |\nabla X|^2\Big] dv,$$

which yields by virtue of (3.2) that

$$\int_M \Big[r|X|^2 - |\nabla X|^2\Big] dv \geq 0.$$

If $r < 0$, then the last relation reduces to $\int_M [r|X|^2 - |\nabla X|^2] dv = 0$. Hence $X = 0$. $\square$

Definition 3.1. (see [17]) A vector field X in a Riemannian manifold $(M^n, g), n > 2$, is said to be projective Killing vector field if it satisfies

$$(\pounds_X \nabla)(Y, Z) = \omega(Y)Z + \omega(Z)Y$$

for any vector fields Y and Z, ω being a certain 1-form and $\pounds$ is the operator of Lie differentiation.

Theorem 3.2. *If in a compact, orientable $(PCRS)_n, n > 2$, without boundary, the scalar curvature is such that $r \leq 0$, then a projective Killing vector field has vanishing covariant derivative, and if $r < 0$, then there exists no non-zero projective Killing vector field in this manifold.*

Proof. We know that (see [16]) for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \Big[S(X,X) - \frac{1}{4}|d\xi|^2 - \frac{n-1}{2(n+1)}(div\, X)^2\Big] dv = 0, \tag{3.3}$$

where ξ is an 1-form corresponding to the vector field X. We now assume that $r \leq 0$. Therefore (2.3) yields $S(X,X) \leq 0$ and hence from (3.3) we obtain $d\xi = 0$ and $div\, X = 0$. This implies that X is harmonic as well as a Killing vector field. Consequently its covariant derivative vanishes. □

Definition 3.2. (see [17]) A vector field X in a Riemannian manifold $(M^n, g), n > 2$, is said to be conformal Killing vector field if it satisfies $\pounds_X g = 2\rho g$ for any vector field X, where ρ is given by $\rho = -\frac{1}{n}(div\, X)$ and $\pounds$ is the operator of Lie differentiation.

Theorem 3.3. *If in a compact, orientable $(PCRS)_n, n > 2$, without boundary, the scalar curvature r is negative, then there exists no non-zero conformal Killing vector field in this manifold.*

Proof. It is known from (see [16]) that for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \Big[S(X,X) - |\nabla X|^2 - \frac{n-2}{n}(div\, X)^2\Big] dv = 0,$$

where dv denotes the volume element of M. Now we assume that the scalar curvature r is negative. Then proceeding similarly as before we obtain $\nabla X = 0$, $div\, X = 0$. □

Theorem 3.4. *In a compact, orientable $(PCRS)_n, n > 2$, without boundary, any harmonic vector field is a parallel vector field which is orthogonal to the generator λ.*

Proof. It is known from (see [16]) that for a vector field X in a Riemannian manifold M, the following relation holds

$$\int_M \Big[S(X,X) + |\nabla X|^2\Big] dv = 0, \tag{3.4}$$

where dv denotes the volume element of M. Then by virtue of (2.2), (3.4) reduces to

$$\int_M \Big[r|g(X,\lambda)|^2 + |\nabla X|^2\Big] dv = 0,$$

which implies that $g(X, \lambda) = 0$ and $\nabla X = 0$. □

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References

[1] K. Arslan, R. Ezentas, C. Murathan and C. Özgür, *On pseudo Ricci symmetric manifolds*, Balkan J. Geom. Appl., 6 (2001), 1–5.

[2] M. C. Chaki, *On pseudosymmetric manifolds*, An. Stiint. Ale Univ., "AL. I. Cuza" Iasi, Romania, 33 (1987), 53–58.

[3] M. C. Chaki, *On pseudo Ricci symmetric manifolds*, Bulg. J. Phys., 15 (1988), 526–531.

[4] P. Debnath and A. Konar, *On quasi Einstein manifold and quasi Einstein space-time*, Diff. Geom. Dyn. Syst., 12 (2010), 73–82.

[5] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belg. Math. Soc., Series A, 44 (1992), 1–34.

[6] I. E. Hirică, *On pseudosymmetric Riemann spaces*, Balkan J. Geom. Appl., 14 (2009), 42–49.

[7] J. P. Jaiswal and R. H. Ojha, *On weakly pseudo-projectively symmetric manifolds*, Diff. Geom. Dyn. Syst., 12 (2010), 83–94.

[8] C. Özgür, *On a class of generalized quasi Einstein manifolds*, Appl. Sci., 8 (2006), 138–141.

[9] C. Özgür and S. Sular, *On $N(k)$-quasi Einstein manifold satisfying certain condition*, Balkan J. Geom. Appl., 13 (2008), 74–79.

[10] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, J. Lond. Math. Soc., 27 (1952), 287–295.

[11] A. A. Shaikh and S. K. Hui, *On pseudo cyclic Ricci symmetric manifolds*, Asian-European J. Math., 2 (2009), 227–237.

[12] A. A. Shaikh and A. Patra, *On quasi-conformally flat quasi-Einstein spaces*, Diff. Geom. Dyn. Syst., 12 (2010), 201–212.

[13] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$. The local version*, J. Diff. Geom., 17 (1982), 531–582.

[14] L. Tamássy and T. Q. Binh, *On weakly symmetric and weakly projective symmetric Rimannian manifolds*, Coll. Math. Soc. J. Bolyai, 50 (1989), 663–670.

[15] A. G. Walker, *On Ruses spaces of recurrent curvature*, Proc. London Math. Soc., 52 (1950), 36–64.

[16] Y. Watanabe, *Integral inequalities in compact orientable manifolds, Riemannian or Kahlerian*, Kodai Math. Sem. Rep., 20 (1968), 261-271.

[17] K. Yano, *Integral Formulas in Riemannian Geometry*, Marcel Dekker, New York, 1970.

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