Application of the Lie groups of transformations for an approximate solution of MHD flow of a visco-elastic second grade fluid

N. Islam, B. Islam, H. P. Mazumdar and J. Lahiri

Abstract. An approximate solution to the problem of steady boundary layer flow of a viscous incompressible electrically conducting second grade fluid over a stretching sheet is presented. Scaling group transformation is applied to reduce the partial differential equation into an ordinary differential equation and then least square method is used to minimize the residual of the equation. The effects of the magnetic field on the flow characteristics are studied through numerical computations with different values of the Hartman Number.

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Key words: Lie group, boundary layer, visco-elastic fluid.

1 Introduction

The partial differential equations governing the motion of applied flow problems are non-linear in nature and hence cannot be solved easily. Whenever possible these differential equations are reduced to ordinary differential equations by employing transformations to obtain similar solutions. Ames[1], Bluman-Cole[3],Hansen[7], Ibragimov[8], Olver[10], Seshadri-Na[14] and Stephani[16] have discussed application of groups and symmetries to partial differential equations arising from natural phenomena and technological problems. Symmetry groups are invariant transformations which do not alter the structural form of the equation under investigation. The advantage of the symmetry method is that it can be applied successfully to non-linear partial differential equations governing the motion of fluid. Sophus Lie developed a transformation, currently known as Lie group of transformation, which map a given differential equation to itself. The differential equations remain invariant under some continuous group of transformations usually known as symmetries of a differential equation. In this paper, we apply Lie's scaling group of transformation to the problem of visco-elastic second grade fluid over a stretching sheet in the presence of transverse magnetic field. Many natural phenomena and technological problems are susceptible to MHD analysis.

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The object of the present work is to investigate the effects of the transverse magnetic field and the amplification factor on the flow part of a stretching sheet.

2 Formulation of the problem

We consider the laminar flow of a viscous incompressible, electrically conducting second grade fluid past a stretching sheet. Many authors, as Sakiadis[13], Beard-Walters[2], Rajagopal *et al*[12], Siddappa-Abel[15] and others have investigated the boundary layer flow of visco-elastic fluid past a stretching sheet. The results of this type of investigation are considered important to gain insight into polymer processing industry. Beard-Walters[2], derived the steady two dimensional boundary layer equations for the visco-elastic second grade fluid past a stretching sheet on which Mazumdar *et al*[9] applied the scaling group of transformations. In the presence of transverse magnetic field, the equation developed by Beard-Walters for second grade visco-elastic fluid flow past a stretching sheet may reduce simply to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - k[\frac{\partial}{\partial x}(u\frac{\partial^2 u}{\partial y^2}) + \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}] - \frac{\sigma B^2 U}{\rho}, \tag{2.2}$$

where u and v are the components of velocity in the x and y directions respectively, $\nu = \mu/\rho$ is the kinematic coefficient of viscosity, μ is the viscosity, ρ is the density of the fluid, k is a positive parameter that depicts the visco-elastic property of the fluid, σ is electrical conductivity of the fluid and B is the strength of applied transverse magnetic field.
The boundary conditions for $x \geq 0$, are given by

$$\begin{cases} u = Ax \; and \; v = 0 \; at \; y = 0, \\ and \; u \to 0 \; as \; y \to \infty \end{cases} \tag{2.3}$$

The sheet is moving in its own plane with a speed proportional to the distance from the origin, A being the constant of proportionality.

3 Reduction of equations to non-dimensional form

Let us first convert the equations into dimensionless form by introducing characteristic length L, half length of the sheet, the characteristic velocity U defined as $U = AL$ let $R_e = UL/\nu$ be Reynold's number. We take the dimensionless variables as

$$\begin{cases} \bar{x} = x/L, \quad \bar{y} = (y/L)\sqrt{R_e}, \\ \bar{U} = u/U, \quad \bar{v} = (v/U)\sqrt{R_e}. \end{cases} \tag{3.1}$$

Using the variables defined in (3.1) into equations (2.1),(2.2) and (2.3), we infer

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{3.2}$$

and

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \bar{k}[\frac{\partial}{\partial \bar{x}}(\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}) + \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2} + \bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}] - M\bar{u}, \tag{3.3}$$

where $\bar{k} = \frac{kR_e}{L^2} = \frac{kA}{\nu}$, $M = \frac{\sigma B^2 L}{\rho U} = \frac{\sigma B^2}{\rho A}$ and M is known as *the Hartman number*. The boundary conditions are

$$\begin{cases} \bar{u} = \bar{x} \ and \ \bar{v} = 0 \ at \ \bar{y} = 0, \\ \bar{u} \to 0 \ as \ \bar{y} \to \infty. \end{cases} \tag{3.4}$$

Let us define Stream functions ψ as follows:

$$\bar{u} = \frac{\partial \psi}{\partial \bar{y}} \ and \ \bar{v} = \frac{\partial \psi}{\partial \bar{x}}. \tag{3.5}$$

In terms of stream function ψ, equations (3.3) and (3.4) take the form

$$\begin{aligned} \frac{\partial \psi}{\partial \bar{y}}\frac{\partial^2 \psi}{\partial \bar{x}\partial \bar{y}} - \frac{\partial \psi}{\partial \bar{x}}\frac{\partial^2 \psi}{\partial \bar{y}^2} &= \frac{\partial^3 \psi}{\bar{y}^3} - \bar{k}\left[\frac{\partial^2 \psi}{\partial \bar{x}\partial \bar{y}}\frac{\partial^3 \psi}{\partial \bar{y}^3} + \right. \\ &\left. \frac{\partial \psi}{\partial \bar{y}}\frac{\partial^4 \psi}{\partial \bar{x}\partial \bar{y}^3} - \frac{\partial^2 \psi}{\partial \bar{y}^2}\frac{\partial^3 \psi}{\partial \bar{x}\partial \bar{y}^2} - \frac{\partial \psi}{\partial \bar{x}}\frac{\partial^4 \psi}{\partial \bar{y}^4}\right] - M\tfrac{\partial \psi}{\partial \bar{y}}. \end{aligned} \tag{3.6}$$

and the boundary conditions,

$$\begin{cases} \dfrac{\partial \psi}{\partial \bar{y}} = \bar{x} \ and \ \dfrac{\partial \psi}{\partial \bar{x}} = 0 \ at \ \bar{y} = 0, \\ \dfrac{\partial \psi}{\partial \bar{y}} \to 0 \ as \ \bar{y} \to \infty. \end{cases} \tag{3.7}$$

4 Application of scaling group of transformations

Let us consider the scaling group of transformations by Patel- Timol [11] as

$$\begin{cases} x^* = \bar{x}e^l \\ y^* = \bar{y}e^m \\ \psi^* = \psi e^n, \end{cases} \tag{4.1}$$

where l, m, n are very small transformation parameters. The transformation (4.1) is known as point-transformation which co-ordinates $(\bar{x}, \bar{y}, \psi)$ to the (x^*, y^*, ψ^*). Substitution of (4.1) into (3.6) yields

$$\begin{aligned} e^{(l+2m-2n)}\left[\frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^*\partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}}\right] &= e^{(3m-n)}\frac{\partial^3 \psi^*}{\partial y^{*3}} \\ - \bar{k}e^{(l+4m-2n)}\left[\frac{\partial^2 \psi^*}{\partial x^*\partial y^*}\frac{\partial^3 \psi^*}{\partial y^{*3}} + \frac{\partial \psi^*}{\partial y^*}\frac{\partial^4 \psi^*}{\partial x^*\partial y^{*3}} - \right. \\ \left. \frac{\partial^2 \psi^*}{\partial y^{*2}}\frac{\partial^3 \psi^*}{\partial x^*\partial y^{*2}} - \frac{\partial \psi}{\partial x^*}\frac{\partial^4 \psi^*}{\partial y^{*4}}\right] - Me^{(m-n)}\tfrac{\partial \psi^*}{\partial y^*}. \end{aligned} \tag{4.2}$$

If the system remains invariant under the group of transformations (4.1), we must have the following relations among the transformation parameters $l + 2m - 2n = 3m - n = l + 4m - 2n = m - n$, which yields

$$m = 0 \ \ and \ \ l = n. \tag{4.3}$$

In view of (4.3), equation (4.2) reduces to

$$\begin{aligned} \frac{\partial \psi^*}{\partial y^*}\frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*}\frac{\partial^2 \psi^*}{\partial y^{*2}} = & \frac{\partial^3 \psi^*}{\partial y^{*3}} - \bar{k}[\frac{\partial^2 \psi^*}{\partial x^* \partial y^*}\frac{\partial^3 \psi^*}{\partial y^{*3}} \\ + \frac{\partial \psi^*}{\partial y^*}\frac{\partial^4 \psi^*}{\partial x^* \partial y^{*3}} - \frac{\partial^2 \psi^*}{\partial y^{*2}}\frac{\partial^3 \psi^*}{\partial x^* \partial y^{*2}} - & \frac{\partial \psi}{\partial x^*}\frac{\partial^4 \psi^*}{\partial y^{*4}}] - M\frac{\partial \psi^*}{\partial y^*,} \end{aligned} \tag{4.4}$$

and the boundary conditions transform to

$$\begin{cases} \dfrac{\partial \psi^*}{\partial y^*} = x^* \ and \ \dfrac{\partial \psi^*}{\partial x^*} = 0 \ at \ y^* = 0, \\ \dfrac{\partial \psi^*}{\partial y^*} \to 0 \ as \ y \to \infty. \end{cases} \tag{4.5}$$

Transformation (4.1) is reduced to one parameter group of transformations e.g. ,

$$\begin{cases} x^* = \bar{x}e^l \\ y^* = \bar{y} \\ \psi^* = \psi e^l. \end{cases} \tag{4.6}$$

5 Absolute invariants

The generator corresponding to the one parameter infinitesimal Lie group of point transformation (4.6) is

$$X = \bar{x}\frac{\partial}{\partial \bar{x}} + \psi\frac{\partial}{\partial \psi}. \tag{5.1}$$

The invariant $G(\bar{x}, \bar{y}, \psi)$ corresponding to X is obtained by solving the differential equation

$$\bar{x}\frac{\partial G}{\partial \bar{x}} + \psi\frac{\partial G}{\partial \psi} = 0.$$

The auxiliary equation is $\frac{d\bar{x}}{\bar{x}} = \frac{d\psi}{\psi}$, which gives $\frac{\psi}{\bar{x}}$= constant.
We consider the new variable

$$G(\eta) = \frac{\psi^*}{\bar{x}}, \ with \ \eta = \bar{y}^*. \tag{5.2}$$

Substitution of (5.2) into (4.4) together with (4.5) yield

$$(G')^2 - GG'' = G''' - \bar{k}[\ 2G'G''' - (G'')^2 - GG^{iv}] - MG', \tag{5.3}$$

and the boundary conditions:

$$\begin{cases} G'(\eta) = 1, \ G(\eta) = 0 \ when \ \eta = 0, \\ G'(\eta) \to 0 \ as \ \eta \to \infty, \end{cases} \tag{5.4}$$

where $G' = \frac{dG}{d\eta}, \ G'' = \frac{d^2G}{d\eta^2}$ etc.

6 Solution of the problem

Let us introduce two variables ζ and f as

$$\zeta = \alpha\eta \ , \ f = \alpha G. \tag{6.1}$$

where α (> 0) is an amplification factor. In view of (6.1), equation (5.3) is transformed to

$$\alpha^2[f''' - \bar{k}\ 2f'f''' - (f'')^2 - ff^{iv}] - Mf' - (f')^2 + ff'' = 0 \tag{6.2}$$

and boundary conditions become

$$\begin{cases} f = 0, \ \dfrac{df}{d\zeta} = 0 \ when \ \zeta = 0, \\ \dfrac{df}{d\zeta} \to 0 \ as \ \zeta \to \infty. \end{cases} \tag{6.3}$$

Satisfying all the boundary conditions, we may choose $f(\zeta)$ in the following form:

$$f(\zeta) = \frac{1}{2}\left[\frac{3}{2} - e^{\zeta} - \frac{1}{2}e^{-2\zeta}\right].$$

Therefore,

$$\frac{df}{d\zeta} = \frac{1}{2}(e^{-\zeta} + e^{-2\zeta}) = \frac{dG}{d\eta}. \tag{6.4}$$

Substituting these in equation (6.2), we have

(6.5)

$$R(\zeta,\alpha) = \frac{1}{8}\{(4-3\bar{k})\alpha^2-3-4M\}e^{-\zeta}+\frac{1}{4}\{(8-12\bar{k})\alpha^2-3-2M\}e^{-2\zeta}+\frac{1}{8}(5\bar{k}\alpha^2+1)e^{-3\zeta};$$

$R(\zeta,\alpha)$ is called also the *defect function*. Our aim is to minimize $R(\zeta,\alpha)$ using least square method and obtain the value of α from the formula

$$\frac{\partial}{\partial\alpha}\int_0^\infty R^2(\zeta,\alpha)\ d\zeta = 0. \tag{6.6}$$

Substituting (6.5) in (6.6) and effecting integration and then differentiation, we obtain

$$\alpha^2[\bar{k}(871\bar{k} - 1548) + 3440] + 190M(9\bar{k} - 8) + 1681\bar{k} - 1494 = 0. \tag{6.7}$$

α being real, we get

$$\{\bar{k}(871\bar{k} - 1548) + 3440\}\{190M(9\bar{k} - 8) + 1681\bar{k} - 1494\} < 0. \tag{6.8}$$

But, since $\bar{k}\ (871\bar{k} - 1548) + 3440 = 871\bar{k}^2 - 1548\bar{k} + 3440$ is always positive ($1548^2 - 4\times871\times3440 < 0$), we infer that (6.8) yields $190M\ (9\bar{k}-8) + 1681\bar{k} - 1494 < 0$, which on simplification gives

$$\bar{k} < \frac{1494 + 1520M}{1681 + 1710M}. \tag{6.9}$$

α can be calculated as well from (6.7) for different choices of M and $\bar{k}$ satisfying (6.9)

7 Numerical results and discussions

For different values of Hartman number M and the parameter $\bar{k}$, which is associated with the visco-elastic fluid, the numerical calculations are carried out and are exhibited in the graphs. The velocity profiles $G'(\zeta)$ are determined from (6.1) and (6.4) for $M \in \{0.0, 0.5, 1.0, 1.5, 2.0\}$, and then $G'(\eta)$ vs η is plotted for $\bar{k}$= 0.25, 0.5, 0.75 in Fig.1, Fig.2, Fig.3, respectively. Its is seen that $G'(\eta)$ decreases as M increases. Thus, the effect of the magnetic field is to reduce the velocity component parallel to the stretching surface. It is also seen that the velocity decreases rapidly near the stretching sheet and this rate is higher as $\bar{k}$ increases.

The dimensionless stream function $G(\zeta)$ is given by

$$G(\zeta) \;=\; \frac{1}{2\alpha}\left(\frac{3}{2} - e^{-\zeta} - \frac{1}{2}e^{-2\zeta}\right).$$

In each of the Fig.4, Fig.5, and, Fig.6, the stream function $G(\zeta)$ is plotted against $\zeta(=\alpha\eta)$ for different values of M and $\bar{k}$. Figures show that stream function decreases with the increase of the values in the Hartman number M as well as $\bar{k}$. Far downstream the streamlines approach asymptotically to the x-axis as $x \;\to \infty$.

In order to examine the accuracy of the appropriate method, we plot the defect function $R(\zeta)$ vs ζ in each of the figures from Fig.7 to Fig.13 for different values of M and $\bar{k}$. It is seen from the figures that the error bounds may be considered satisfactory with low values of the magnetic parameter M and also the low values of the parameter $\bar{k}$. Again, $\bar{k}$ is plotted against M, in Fig.14 from the relation $\bar{k} = \frac{1494+1520M}{1681+1710M}$. It is seen that $\bar{k}$ increases with the increasing values of M and approaches asymptotically to the line $\bar{k} = 0.9$.

8 Appendix. Computer plots

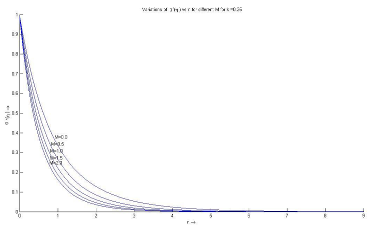

Fig. 1

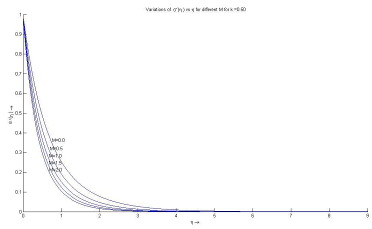

Fig. 2.

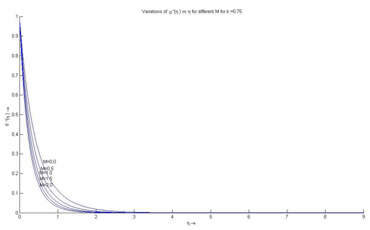

Fig. 3

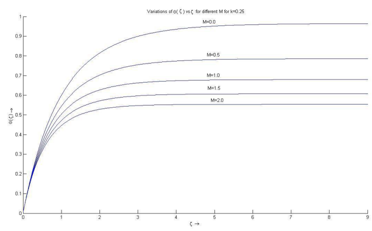

Fig. 4.

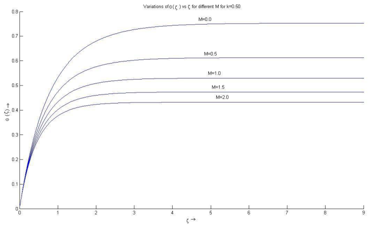

Fig. 5

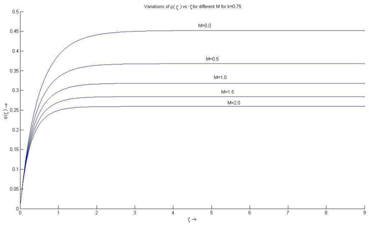

Fig. 6.

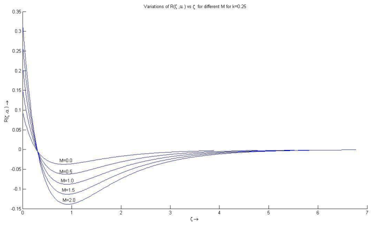

Fig. 7

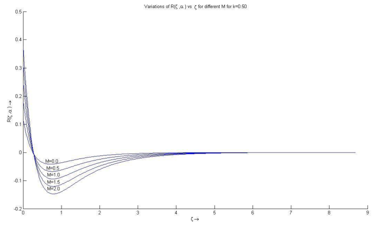

Fig. 8.

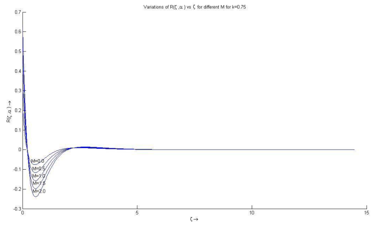

Fig. 9

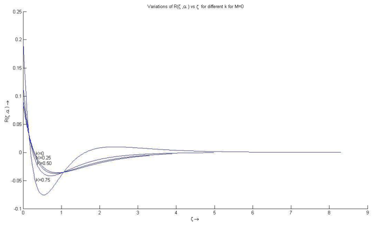

Fig. 10.

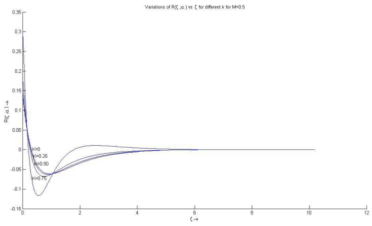

Fig. 11

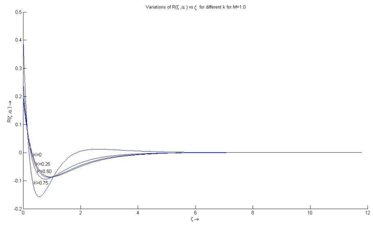

Fig. 12.

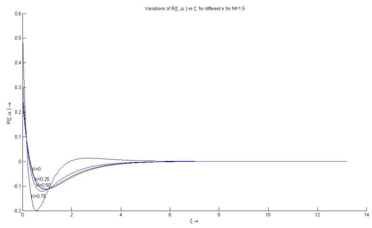

Fig. 13

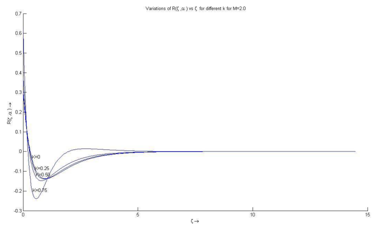

Fig. 14.

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Authors' addresses:

Nurul Islam
Department of Mathematics, Ramakrishna Mission Residential College (Autonomous), Narendrapur, 700103 Kolkata, India.
E-mail: dr.i.nurul@gmail.com

Bipul Islam
Department of Computer Science and Engineering,
Netaji Shubash Engineering College,
Garia (Panchpota), 700150 Kolkata, India.
E-mail: mib.speaking@gmail.com

Himadri Pai Mazumdar
Physics and Applied Mathematics Unit, Indian Statistical Institute,
203, B.T. Road, 700108 Kolkata, India.
E-mail: hpm@isical.ac.in

Jhuma Lahiri
Department of Mathematics, M.R.B. Vidyamandir,
Madhyamgram, 24 Pgs(N), India.