A note on the Solow economic growth model with Richards population growth law

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Abstract. In their work on Solow model with Richards population growth law, Accinelli and Brida [1] give a global asymptotic stability result for the model's solution. In this paper, we present another proof of this fact.

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1 Introduction

One of the most important models elaborated to explain economic growth is the neoclassical growth model originated with the work of Solow [17], who proposed a model designed to show how growth in the capital stock, growth in the labor force, and advances in technology interact and how they affect a nation's total output. From simply being a tool for the analysis of the growth process, the Solow model has been generalized in several different directions (see, e.g., Accinelli and Brida [1]; Bucci and Guerrini [3]; Ferrara and Guerrini [4]-[7]; Germanà and Guerrini [8]; Guerrini [9]-[16]). In particular, Accinelli and Brida [1] have investigated the implications of studying the Solow model by modelling the population growth rate via a generalized logistic equation (Richards law [21]), and showed that with the Richards law, the intrinsic rate of population growth plays no role in determining long run equilibrium per worker level of capital. Moreover, they have presented a closed-form solution of the model and proved its stability. The purpose of this paper is to demonstrate this last statement in another way, using the Bendixson-DuLac and Poincarè-Bendixson theorems. For further research, it would be interesting to investigate the Ramsey model analogue following Udriste's ideas (see, e.g., [18]-[20]).

2 The model

There is a closed economy consisting of a single good, used either for consumption or investment, produced by physical capital K and population/labor L in a process described by a Cobb-Douglas production function $Y = K^{\alpha}L^{1-\alpha}$, $0 < \alpha < 1$. In this economy, output equals income, and the amount invested equals the amount saved.

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Let s be the fraction of output that is saved, i.e. the savings rate, so that $1-s$ is the fraction of output that is consumed. We assume that capital depreciates at the constant rate δ, i.e. at each point in time, a constant fraction of the capital stock wears out and, hence, can no longer be used for production. The net increase in the stock of physical capital at a point in time equals gross invest less depreciation, $\dot{K} = I - \delta K = sK^{\alpha}L^{1-\alpha} - \delta K$. Taking logarithms and differentiations on both sides of the equality, we have

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L},$$

so that we find that the capital per effective worker accumulates over time according to $\dot{k} = sk^{\alpha} - (\delta + \dot{L}/L)k$. If the labor force grows at a constant rate n, i.e. $\dot{L}/L = n$, then we get $\dot{k} = sk^{\alpha} - (\delta + n)k$, which is the fundamental differential equation of the neoclassical growth theory as put forward by Solow [17]. If, instead, we assume Richards law [18] for the evolution of population, i.e. $\dot{L}/L = r[1 - (L/L_{\infty})^{\beta}]$, where $r > 0$ is the intrinsic growth rate per capita, L_{∞} is the carrying capacity, and β is a positive real number, we obtain the modified Solow model introduced by Accinelli and Brida [1]. Within this setup, the corresponding model happens to be described by

$$\dot{k} = sk^{\alpha} - \left(\delta + \frac{\dot{L}}{L}\right)k, \qquad \frac{\dot{L}}{L} = r\left[1 - \left(\frac{L}{L_{\infty}}\right)^{\beta}\right]. \tag{2.1}$$

Accinelli and Brida [1] proved the dynamical system (2.1) to have a unique non-zero equilibrium (k_*, h_*) in $\mathbb{R}^2_+$, which is a sink, and showed the process described by the model to be globally asympotically stable, i.e. all solutions starting near the steady state remain near the steady state for all time, and furthermore they tend towards (k_*, h_*) as t grows to infinity.

3 Global asymptotic stability

In order to study closed orbits of system (2.1), we start by recalling the Bendixson-DuLac theorem (see, e.g., Boyce and DiPrima [2]), which states that if there exists a function $\varphi(k, L)$ such that $\partial(\varphi\dot{k})/\partial k + \partial(\varphi\dot{L})/\partial L$ has the same sign ($\neq 0$) almost everywhere in a simply connected region, then the plane autonomous system (2.1) has no periodic solutions. "Almost everywhere" means everywhere except possibly in a set of area 0, such as a point or line.

Lemma 3.1. *A limit cycle cannot occur in this model.*

Proof. Setting $\varphi(k, L) = k^{-1}L^{-1}$ gives

$$\frac{\partial(k^{-1}L^{-1}\dot{k})}{\partial k} + \frac{\partial(k^{-1}L^{-1}\dot{L})}{\partial L} = -(1-\alpha)sk^{\alpha-2}L^{-1} - r\beta L_{\infty}^{-\beta}k^{-1}L^{\beta-1} < 0.$$

Applying the Bendixson-DuLac theorem, we can conclude that there is no closed orbit. $\square$

Next, we recall the Poincarè-Bendixson Theorem (see, e.g., Boyce and DiPrima [2]), which, basically, asserts that any orbit which stays in a bounded region of a plane autonomous system either approaches a fixed point or a periodic orbit. Thus, chaotic behavior cannot arise.

Theorem 3.1. *Any solution of* (2.1) *converges to the steady state equilibrium* (k_*, L_*) *as* $t \to \infty$.

Proof. This will now be deduced from the Poincarè-Bendixson theorem. In fact, the Inada conditions on the production function imply that any solution to system (2.1) is bounded, i.e. there exists a compact set $\Omega \subset \mathbb{R}^2_+$ such that $(k, L) \subset \Omega$ for all t. As well, limit cycles are ruled out by Lemma 3.1. $\square$

References

[1] E. Accinelli and J. G. Brida, *Re-formulation of the Solow economic growth model with the Richards population growth law*, Growth, Math methods 0508006, EconWPA, 2005.

[2] W. E. Boyce and R. DiPrima, *Elementary differential equations and boundary value problems*, John Wiley and Sons, 1995.

[3] A. Bucci and L. Guerrini, *Transitional dynamics in the Solow-Swan growth model with AK technology and logistic population change*, B.E. Journal of Macroeconomics 9 (2009), 1-16.

[4] M. Ferrara and L. Guerrini, *On the dynamics of a three sector growth model*, International Review of Economics 55 (2008), 275-283.

[5] M. Ferrara and L. Guerrini, *The Ramsey model with logistic population growth and Benthamite felicity function revisited*, WSEAS Transactions on Mathematics 8 (2009), 41-50.

[6] M. Ferrara and L. Guerrini, *More on the Green Solow model with logistic population change*, WSEAS Transactions on Mathematics 8 (2009), 97-106.

[7] M. Ferrara and L. Guerrini, *A note on the Uzawa-Lucas model with unskilled labor*, Appl. Sci. 12 (2010), 90-95.

[8] C. Germanà and L. Guerrini, *On the closed-form solution of the improved labor augmented Solow-Swan model*, Appl. Sci. 7 (2005), 101-106.

[9] L. Guerrini, *The Solow-Swan model with a bounded population growth rate*, Journal of Mathematical Economics 42 (2006), 14-21.

[10] L. Guerrini, *Logistic population change and the Mankiw-Romer-Weil model*, Appl. Sci. 12 (2010), 96-101.

[11] L. Guerrini, *The Ramsey model with a bounded population growth rate*, Journal of Macroeconomics 32 (2010), 872-878.

[12] L. Guerrini, *A closed-form solution to the Ramsey model with logistic population growth*, Economic Modelling 27 (2010), 1178-1182.

[13] L. Guerrini, *The Ramsey model with AK technology and a bounded population growth rate*, Journal of Macroeconomics 32 (2010), 1178-1183.

[14] L. Guerrini, *Transitional dynamics in the Ramsey model with AK technology and logistic population change*, Economics Letters 109 (2010), 17-19.

[15] L. Guerrini, *The AK Ramsey model with von Bertalanffy population law and Benthamite function*, Far East Journal of Mathematical Sciences 45 (2010), 187-192.

[16] L. Guerrini, *A note on the Ramsey growth model with the von Bertalanffy population law*, Applied Mathematical Sciences 4 (2010), 3233-3238.

[17] R. M. Solow, *A contribution to the theory of economic growth*, Quarterly Journal of Economics 70 (1956), 65-94.

[18] C. Udriste and I. Tevy, *Nonholonomic approach of multitime maximum principle*, Balkan J. Geom. Appl. 14 (2) (2009), 101-116.

[19] C. Udriste, *Simplified multitime maximum principle*, Balkan J. Geom. Appl. 14 (1) (2009), 102-119.

[20] C. Udriste, O. Dogaru, I. Tevy and D. Bala, *Elementary work, Newton law and Euler-Lagrange equations*, Balkan J. Geom. Appl. 15 (2) (2010), 92-99.

[21] B. Zeide, *Analysis of growth equations*, Forest Science 39 (1993), 594-616.

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