Quasi-nilpotent equivalence of S-decomposable and S-spectral systems

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Abstract. The notions of quasi-nilpotent equivalence and commutator for operators were introduced by I. Colojoară and C. Foiaş in 1965 and 1967 ([8], [10]) and the generalizations of these notions to operator systems appeared in the papers [17], [18] of Şt. Frunză.
This work is trying to extend significant results for spectral equivalence of decomposable and spectral operators (systems) and to characterize the spectral equivalence of those by equality of spectral capacities (spectral measures) to S-decomposable and S-spectral systems. We shall see that not all results for decomposable (spectral) systems can be extended and generalized to S-decomposable (S-spectral) systems.

1 Introduction

In this paper we recall several notations and definitions from the specialized literature, which will be further needed.

Let X be a Banach space and let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X. Let $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ be a system of commuting operators, let $Y \subset X$ be an invariant subspace to a, let $b = a|Y = (a_1|Y, a_2|Y, ..., a_n|Y)$ be the restriction of a to Y and let $\dot{a} = (\dot{a}_1, \dot{a}_2, ..., \dot{a}_n)$ be the quotient system induced by a on the quotient space $\dot{X} = X|Y$.

The system $a = (a_1, a_2, ..., a_n) \subset \mathbf{B}(X)$ is said to be *nonsingular* on X if the Koszul complex $E(a, X)$ is exact, where

$E(X, a) : 0 \to X = \Lambda^n[\sigma, X] \xrightarrow{\delta_n} \Lambda^{n-1}[\sigma, X] \xrightarrow{\delta_{n-1}} \ldots \xrightarrow{\delta_3} \Lambda^2[\sigma, X] \xrightarrow{\delta_2} \Lambda^1[\sigma, X] \xrightarrow{\delta_1} \Lambda^0[\sigma, X] = X \to 0$

or, equivalently, the complex $F(a, X)$ is exact, where

$F(X, a) : 0 \to X = \Lambda^0[\sigma, X] \xrightarrow{\delta^0} \Lambda^1[\sigma, X] \xrightarrow{\delta^1} \Lambda^2[\sigma, X] \xrightarrow{\delta^2} \ldots \xrightarrow{\delta^{n-2}} \Lambda^{n-1}[\sigma, X] \xrightarrow{\delta^{n-1}} \Lambda^n[\sigma, X] = X \to 0$ (see [24]).

The complement in $\mathbb{C}^n$ of the set of those elements $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ for which the system $z - a = (z_1 - a_1, z_2 - a_2, \ldots, z_n - a_n)$ is nonsingular on X is said to be the *spectrum of a on X* and is denoted by $\sigma(a, X)$. The complement of the reunion of all open sets V in $\mathbb{C}^n$ having the property that there is a form $\varphi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, C^{\infty}(V, X)]$ satisfying the equality $sx = (\alpha \oplus \bar{\partial})\varphi$ is said to be the *spectrum of $x \in X$ with respect to a* and is denoted by $sp(a, x)$ (see [18]). The complement in $\mathbb{C}^n$ of the set of all $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ such that there are an open neighborhood V of z and n X-valued analytic functions $f_1, f_2, \ldots, f_n$ on V, satisfying the identity $x \equiv (\zeta_1 - a_1)f_1(\zeta) + \cdots + (\zeta_n - a_n)f_n(\zeta)$, $\zeta \in V$ is called *the local analytic spectrum of x with respect to a* and is denoted by $\sigma(a, x)$ (see [18]). In [16], J. Eschmeier proved that $sp(a, x) = \sigma(a, x)$.

We shall say that the system $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ verifies the cohomology property (L) if $H^{n-1}(C^{\infty}(G, X), \alpha \oplus \bar{\partial}) = 0$, for any open set $G \subset \mathbb{C}^n$ ([24]).

We denote by S_a the complement in $\mathbb{C}^n$ of the set of those points $\omega \in \mathbb{C}^n$ for which there is an open polydisc $D_\omega \ni \omega$ with the property that $H^p(A(D_\omega, X), \alpha_a) = 0$, for $0 \le p \le n - 1$ (where $\alpha_a(z) = z - a$, $A(\Omega, X)$ is the space of all X-valued analytic functions on Ω, $z \in \mathbb{C}^n$, $\Omega \subset \mathbb{C}$ open set). The set S_a will be called the *analytic spectral residuum* of the system a. If $S_a = \varnothing$, then we say that the system a has the *single-valued extension property* (or a verifies the cohomology property (L)) ([18], [27]).

Definition 1.1. Let X be a Banach space, let $\mathbf{B}(X)$ be the algebra of all linear bounded operators on X, let $\mathcal{P}_X$ be the set of the projectors on X and let $\mathfrak{B}_S^n$ be the family of all Borelian sets of $\mathbb{C}^n$ that have the property $B \cap S = \varnothing$ or $S \subset B$, $B \in \mathfrak{B}_S^n$, where $S \subset \mathbb{C}^n$ is a compact fixed set.

A map $E_S : \mathfrak{B}_S^n \to \mathcal{P}_X$ is called a $(\mathbb{C}^n, X)$ *type S-spectral measure* if

(1) $E_S(\varnothing) = 0$, $E_S(\mathbb{C}^n) = I$;

(2) $E_S(B_1 \cap B_2) = E_S(B_1)E_S(B_2)$, $B_1, B_2 \in \mathfrak{B}_S^n$;

(3) $E_S\left(\bigcup_{m=1}^{\infty} B_m\right)x = \sum_{m=1}^{\infty} E_S(B_m)x$, $B_m \in \mathfrak{B}_S^n$, $B_p \cap B_m = \varnothing$ if $p \ne m$, $x \in X$.

A commuting system $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ is called *S-spectral system* if there is a $(\mathbb{C}^n, X)$ type S-spectral measure E_S such that

(4) $a_j E_S(B) = E_S(B)a_j$, $B \in \mathfrak{B}_S^n$, $1 \le j \le n$;

(5) $\sigma(a, E_S(B)X) \subset \overline{B}$, $B \in \mathfrak{B}_S^n$.

For $S = \varnothing$, we have $\mathfrak{B}_\varnothing^n = \mathfrak{B}^n$, $\varnothing$-spectral measure is spectral measure and $\varnothing$-spectral system is a spectral system.

Remark 1.1. *A commuting system $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ is S-spectral if and only if it is written as a direct sum $a = b \oplus c$, where b is a spectral system and $\sigma(c, X) \subset S$.*

Indeed, if a is S-spectral, then one easily verifies that the map $E : \mathfrak{B}(\mathbb{C}^n) \to \mathcal{P}_X$ (where $\mathfrak{B}(\mathbb{C}^n) = \mathfrak{B}_{\varnothing}^n$) defined by $E(B) = E_S(B \cap \complement S)$ is a spectral measure for $b = a|E_S(\complement S)X$, $B \in \mathfrak{B}(\mathbb{C}^n)$, while $c = a|E_S(S)X$, $\sigma(c, X) = \sigma(a, E_S(S)X) \subset S$. Conversely, if $b = (b_1, b_2, \ldots, b_n) \subset \mathbf{B}(X_1)$ is spectral and $c = (c_1, c_2, \ldots, c_n) \subset \mathbf{B}(X_2)$ is non spectral, with $\sigma(c, X_2) \not\supset \sigma(b, X_1)$, by putting $S = \sigma(c, X_2)$, $X = X_1 \oplus X_2$, $a = b \oplus c$, the map $E_S : \mathfrak{B}_S^n \to \mathcal{P}_X$ defined by the equalities $E_S(B) = E(B) \oplus 0$, if $B \cap S = \varnothing$ and $E_S(B) = E(B) \oplus I_2$, for $B \supset S$, $B \in \mathfrak{B}_S^n$, is a S-spectral measure of a, where E is the spectral measure of b and I_2 is the identity operator in X_2.

Definition 1.2. Let X be a Banach space, let $\mathcal{S}(X)$ be the family of all closed linear subspaces of X, let $S \subset \mathbb{C}^n$ be a compact set and let $\mathfrak{F}_S(\mathbb{C}^n)$ be the family of all closed sets $F \subset \mathbb{C}^n$ which have the property: either $F \cap S = \varnothing$ or $F \supset S$.

We shall call *S-spectral capacity* an application $\mathcal{E}_S : \mathfrak{F}_S(\mathbb{C}^n) \to \mathcal{S}(X)$ that meets the properties:

1. $\mathcal{E}_S(\varnothing) = \{0\}$, $\mathcal{E}_S(\mathbb{C}^n) = X$;

2. $\mathcal{E}_S \left(\bigcap_{i=1}^{\infty} F_i \right) = \bigcap_{i=1}^{\infty} \mathcal{E}_S(F_i)$, for any sequence $\{F_i\}_{i \in \mathbb{N}} \subset \mathfrak{F}_S(\mathbb{C}^n)$;

3. for any open finite S-covering $\{G_S\} \bigcup \{G_j\}_{j=1}^m$ of $\mathbb{C}^n$ we have

$$X = \mathcal{E}_S(\overline{G}_S) + \sum_{j=1}^{m} \mathcal{E}_S(\overline{G}_j).$$

A commuting system of operators $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ is said to be S-*decomposable* if there is a S-spectral capacity $\mathcal{E}_S$ such that:

4. $a_j \mathcal{E}_S(F) \subset \mathcal{E}_S(F)$, for any $F \in \mathfrak{F}_S(\mathbb{C}^n)$, $1 \leq j \leq n$;

5. $\sigma(a, \mathcal{E}_S(F)) \subset F$, for any $F \in \mathfrak{F}_S(\mathbb{C}^n)$.

In case that $S = \varnothing$, $\mathfrak{F}_{\varnothing}(\mathbb{C}^n) = \mathfrak{F}(\mathbb{C}^n)$ is the family of all closed sets $F \subset \mathbb{C}^n$, the $\varnothing$-spectral capacity is said to be spectral capacity and the system is decomposable.

Theorem 1.2. *Let $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ be a commuting system of S-decomposable operators, let $\mathcal{E}_S$ be a $(\mathbb{C}^n, X)$ type S-spectral capacity and let $S_a = \varnothing$ (a has the single-valued extension property, i.e. a verifies the cohomology property (L) ([18, 1.5.2]). Then $\mathcal{E}_S(F) = X_a(F) = X_{[a]}(F)$, for any $F \in \mathfrak{F}_S(\mathbb{C}^n)$.*

Proof. In [16], J. Eschmeier proved that the two local spectra of $x \in X$ with respect to a, $sp(a, x)$ and $\sigma(a, x)$, are equal ([18], 1.5.1, 1.5.2), in short $sp(a, x) = \sigma(a, x)$, hence $X_a(F) = X_{[a]}(F)$, $F \subset \mathbb{C}^n$ closed. In [7] (3.5.9), it was shown that $\mathcal{E}_S(F)$ is a spectral maximal space of a, where $F \in \mathfrak{F}_S(\mathbb{C}^n)$. Using the definition of resolvent set of $y \in Y$ with respect to a and the resolvent set of the restriction $a|Y$ on Y, where Y is a closed linear subspace of X invariant to $a = (a_1, a_2, \ldots, a_n)$, we have $\rho(a, y) = r(a, y) \supset r(a, Y)$; hence it results that $\sigma(a, y) = sp(a, y) \subset sp(a, Y)$. Because $sp(a, \mathcal{E}_S(F)) \subset F$, we have $\mathcal{E}_S(F) \subset X_{[a]}(F)$.

To prove that $X_{[a]}(F) \subset \mathcal{E}_S(F)$ it is enough to verify that for any open set G, with $F \subset G$, F, $\overline{G} \in \mathfrak{F}_S(\mathbb{C}^n)$, we have $X_{[a]}(F) \subseteq \mathcal{E}_S(\overline{G})$, therefore $X_a(F) \subseteq \bigcap\{\mathcal{E}_S(\overline{G}), F \subset G\} = \mathcal{E}_S(\{\bigcap \overline{G}, F \subset G\}) = \mathcal{E}_S(F)$.

Let $G \subset \mathbb{C}^n$ be an open set with $F \subset G$, let F_1, $F_2 \subset \mathbb{C}^n$ be closed such that $F \subset F_1 \subset G$, $F_2 \cap F = \varnothing$ and let $X = \mathcal{E}_S(F_1) + \mathcal{E}_S(F_2)$. For $x \in X_{[a]}(F)$ we have $x = x_1 + x_2$, $x_1 \in \mathcal{E}_S(F_1)$, $x_2 \in \mathcal{E}_S(F_2)$. According to [18, Theorem 1.5.7.]), it results that there are two forms ψ on $r(a, x)$, respectively ψ_2 on $r(a, x_2)$ such that $sx = (\alpha \oplus \overline{\partial})\psi$ and $sx_2 = (\alpha \oplus \overline{\partial})\psi_2$, hence $sx_1 = (\alpha \oplus \overline{\partial})(\psi - \psi_2)$ on $r(a, x) \cap r(a, x_2)$. Further, the proof is the same as that of [18] (Theorem 2.2.1.), with only one condition, namely $F \in \mathfrak{F}_S(\mathbb{C}^n)$ ($F \supset S$ or $F \cap S = \varnothing$). $\qquad \square$

Theorem 1.3. *A S-spectral system $a = (a_1, a_2, \ldots, a_n) \subset \mathbf{B}(X)$ having the S-spectral measure E_S is S-decomposable with its S-spectral capacity $\mathcal{E}_S$ given by*

$$\mathcal{E}_S(F) = E_S(F)X, \ \ F \in \mathfrak{F}_S(\mathbb{C}^n).$$

Proof. Obviously we have

$$\mathcal{E}_S(\varnothing) = E_S(\varnothing)X = OX = \{0\}, \ \ \mathcal{E}_S(\mathbb{C}^n) = E_S(\mathbb{C}^n)X = IX = X,$$

hence the relation (1) from Definition 1.2 is verified.

To prove the relation (3) of Definition 1.2 we use the subadditivity of a measure

$$Y = \mathcal{E}_S(\overline{G}_S) + \sum_{i=1}^{n} \mathcal{E}_S(\overline{G}_i) = E_S(\overline{G}_S)X + \sum_{i=1}^{n} E_S(\overline{G}_i)X \supseteq$$

$$\supseteq E_S\left(\overline{G}_S \cup \left(\bigcup_{i=1}^{n} \overline{G}_i\right)\right)X = E_S(\mathbb{C})X = IX = X,$$

hence we have $Y \supseteq X$; but $Y \subseteq X$ is a subspace, hence $Y = X$, where $\overline{G}_S$, $\overline{G}_i$ $(i = 1, 2, \ldots, n)$ are not disjoint.

From the equality $a_i E_S(F) = E_S(F)a_i$, $i = 1, 2, \ldots, n$, it results that

$$a_i \mathcal{E}_S(F) = a_i E_S(F)X = E_S(F)a_i X \subset E_S(F)X = \mathcal{E}_S(F).$$

We also have

$$sp(a, \mathcal{E}_S(F)) = sp(a, E_S(F)X) \subset F.$$

It remains to be verified the relation (2) of Definition 1.2.

From $E_S(B) = E_S(B \cap B) = E^2(B)$, for any $B \in \mathfrak{B}_S(\mathbb{C}^n)$, we observe that the values of the S-spectral measure are linear continuous projectors on X, thus the subspace $\mathcal{E}_S(F) = E_S(F)X$ is closed, for any $F \in \mathfrak{F}_S(\mathbb{C}^n)$.

The equality (2), $\mathcal{E}_S\left(\bigcap_{i \in I} F_i\right) = \bigcap_{i \in I} \mathcal{E}_S(F_i)$, is proved in the same way as in [18, Proposition 3.1.3], with the only indication that $F_i \in \mathfrak{F}_S(\mathbb{C}^n)$, $i = 1, 2, \ldots, n$. $\qquad \square$

Remark 1.4. *A direct and simple proof of the previous theorem can be given using the following observation (and also Proposition 3.1.3., [18]): A S-spectral system $a = (a_1, a_2, \ldots, a_n)$ is a direct sum between a spectral system $b = (b_1, b_2, \ldots, b_n) = a|E_S(\complement S)X$ and a system $c = a|E_S(S)X$, with $\sigma(c, E_S(S)X) \subset S$. Because the system b is decomposable, then a is S-decomposable.*

Definition 1.3. Let X be a Banach space and $\mathbf{B}(X)$ the algebra of all linear bounded operators on X. For $a \in \mathbf{B}(X)$ we define the operators $l(a)$ and $r(a)$ given by

$$l(a)b = ab \ \text{ and } \ r(a)b = ba, b \in \mathbf{B}(X).$$

The linear operators $l(a)$ and $r(a)$ commute, i.e. $l(a)r(b) = r(b)l(a)$, for all a, $b \in \mathbf{B}(X)$.

The operator $C(a, b) = l(a) - r(b)$ is called *the commutator* of a and b, where a, $b \in \mathbf{B}(X)$.

For $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \subset \mathbf{B}(X)$ two commuting operator systems, we define the system

$$C(a, b) = (C(a_1, b_1), C(a_2, b_2), \ldots, C(a_n, b_n))$$

called *the commutator* of a and b. In this case, $C(a, b)$ is also a commuting operator system.

Taken $k = (k_1, k_2, \ldots, k_n)$ a system of positive integers we use the notations

$$C^k(a, b) = C^{k_1}(a_1, b_1) \cdot \ldots \cdot C^{k_n}(a_n, b_n) \ \text{ and } \ (a - b)^{[k]} = C^k(a, b)id,$$

where $C^{k_j}(a_j, b_j) = (C(a_j, b_j))^{k_j}$, $1 \le j \le n$.

For any $u \in \mathbf{B}(X)$ and $1 \le j \le n$, we have

$$C^{(k+1)_j}(a, b)u = a_j C^k(a, b)u - C^k(a, b)ub_j$$

and

$$(a - b)^{[(k+1)_j]} = a_j(a - b)^{[k]} - (a - b)^{[k]}b_j$$

(see 4.1.1., 4.1.2., 4.1.3, [18]).

Theorem 1.5. *Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \subset \mathbf{B}(X)$ be two systems of S-decomposable operators with $S_a = S_b = \varnothing$ and let $\mathcal{E}_{Sa}$, $\mathcal{E}_{Sb}$ be its S-spectral capacities. If a is spectral equivalent to b, then its S-spectral capacities are equal, $\mathcal{E}_{Sa} = \mathcal{E}_{Sb}$.*

Proof. According to the result obtained by J. Eschmeier in [16], the two local spectra $sp(a, x)$ and $\sigma(a, x)$ are equal, so also its spectral spaces $X_a(F)$ and $X_{[a]}(F)$ are equal, because from $\sigma(a, x) = sp(a, x) \subset F$, it results that $x \in X_a(F)$, so $x \in X_{[a]}(F)$, that is to say $X_a(F) \subset X_{[a]}(F)$; conversely, in a similar way, we prove that $X_{[a]}(F) \subset X_a(F)$. But according to Theorem 1.2, the S-spectral capacities $\mathcal{E}_{Sa}$, $\mathcal{E}_{Sb}$ are given by

$$\mathcal{E}_{Sa}(F) = X_a(F)$$

$$\mathcal{E}_{Sb}(F) = X_b(F)$$

and from Corollary 4.1.19., [18], it results that

$$\sigma(a, x) = sp(a, x) = \sigma(b, x) = sp(b, x),$$

hence

$$\mathcal{E}_{Sa}(F) = \mathcal{E}_{Sb}(F), \ \ \text{for any} \ \ F \in \mathfrak{F}_S(\mathbb{C}^n), \ \text{i.e.} \ \mathcal{E}_{Sa} = \mathcal{E}_{Sb}.$$

$$\square$$

With the same arguments as above, we obtain the following result:

Proposition 1.6. *If a, b are two S-decomposable systems with its S-spectral capacities $\mathcal{E}_{Sa}$, $\mathcal{E}_{Sb}$, then*

$$\lim_{k \to \infty} \|(a - b)^{[k]}\|^{\frac{1}{k}} = 0 \ \ \text{implies that} \ \ \mathcal{E}_{Sb}(F) \subset \mathcal{E}_{Sa}(F), \ \ F \in \mathfrak{F}_S(\mathbb{C}^n).$$

Theorem 1.7. *If $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \subset \mathbf{B}(X)$ are two operator systems spectral equivalent and a is S-decomposable, then b is S-decomposable.*

Proof. It is enough to show that the S-spectral capacity $\mathcal{E}_{Sa}$ of a is also the S-spectral capacity of b. We prove that the relations (4) and (5) form Definition 1.2 are verified by $\mathcal{E}_{Sa}$ for b, i.e. $b_j \mathcal{E}_{Sa}(F) \subset \mathcal{E}_{Sa}(F)$, $1 \le j \le n$, $sp(b, \mathcal{E}_{Sa}(F)) \subseteq F$, $F \in \mathfrak{F}_S(\mathbb{C}^n)$.
   Because $\sigma(b, x) = \sigma(a, x)$ ([18], 4.1.19.) and according to Theorem 1.2 we have

$$\mathcal{E}_{Sa}(F) = X_a(F) = \{x \in X; \sigma(a, x) = \sigma(b, x) \subset F\} = X_b(F).$$

For a closed linear subspace Y of X invariant to both a and b, we obviously have that the restrictions $a|Y$ and $b|Y$ are spectral equivalent, hence $a|\mathcal{E}_{Sa}(F)$ and $b|\mathcal{E}_{Sa}(F)$ are spectral equivalent and according to theorem of equality of spectra we have

$$sp(b, \mathcal{E}_{Sa}(F)) = sp(a, \mathcal{E}_{Sa}(F)) \subset F, \ \ F \in \mathfrak{F}_S(\mathbb{C}^n).$$

Definition 1.5.1., [18] shows that $\sigma(b, b_j x) \subset \sigma(b, x)$, $1 \le j \le n$, hence $b_j \mathcal{E}_{Sa}(F) \subset \mathcal{E}_{Sa}(F)$, $1 \le j \le n$. $\square$

Proposition 1.8. *Let $a = (a_1, a_2, \ldots, a_n)$, $b = (b_1, b_2, \ldots, b_n) \subset \mathbf{B}(X)$ be two operator systems spectral equivalent such that a is S-spectral. Then b is also S-spectral.*

Proof. A simple and direct proof can be given using the similar result form the case of spectral operators and also using the observation: An S-spectral operator is a direct sum between a spectral operator and an operator whose spectrum is in S.
   Another laborious way is to use the proof for spectral operators by [9], [17], [18] in the case of S-spectral operators with non-essential changes. Noting by $E_S = E_{Sa}$ the S-spectral measure of a, it can be shown that E_S verifies the relations (4) and (5) from Definition 1.1 for b, hence E_S is a S-spectral measure of b, therefore b is S-spectral. $\square$

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