Semigroup of generalized inverses of matrices

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Abstract. The paper is divided into two principal parts. In the first one, we give the set of generalized inverses of a matrix A a structure of a semigroup and study some algebraic properties like factorization and commutativity. We also define an equivalence relation in order to establish an isomorphism between the quotient semigroup and the semigroup of projectors on $R(A)$. In the second part, we study a relation between semigroups associated to equivalent matrices and establish a correspondence between the set of matrices and the set of associated semigroups. We also study some algebraic properties in this set like intersection and partial order.

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1 Introduction

In the following, K represents the real or the complex field. Let A be an $m \times n$ matrix over K. The generalized inverse, or the $\{1\}-$inverse of A is an $n \times m$ matrix X over K, which satisfies the matrix equation $AXA = A$. If in addition X satisfies the equation $XAX = X$, then X is said to be a reflexive generalized inverse or $\{1, 2\}-$inverse of A. It is known that there are infinitely sets of $\{1\}-$inverses and $\{1, 2\}-$inverses of a matrix A.

Many studies on generalized inverses and their applications have been done (see [1], [2], [5]). Some of them are algebraic; like the sum (see [4]) and the reverse order law of a product (see [7]). In some areas of binary matrices, the usage of the generalized inverse of a matrix might provide results as well, like the $C\mathbb{P}$ property for binary matrices (see [6]).

In this paper we will study two principal parts. In the first one, we will give the set of $\{1\}-$ inverses of a matrix A a structure of a semigroup and study some algebraic properties like factorization and commutativity. To study the property of factorization in these semigroups, we will use the Moore-Penrose inverse. We will also define an equivalence relation in order to establish an isomorphism between the quotient semigroup and the semigroup of projectors on $R(A)$. The second part concerns the relation between semigroups associated to equivalent matrices and the

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correspondence between the set $M_{m\times n}(K)$ and the set of the associated semigroups. Some algebraic properties in this set will be treated; like the intersection of semigroups and isomorphisms. Also, ordering of sets in the real inner product space, is a recent subject. In [3], this subject addressed as main result, and proved using the Lorentz-Minkowski distance for ordering certain sets. We shall also follow this pattern by using generalized inverses for ordering sets of the introduced semigroups by minus partial order.

2 Preliminary notes

Definition 2.1. Let A be an $m\times n$ matrix over K. The Moore-Penrose inverse of A denoted by A^+ is the $n\times m$ matrix X over K which satisfies the four equations

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA. \tag{2.1}$$

For every matrix there exists only one Moore-Penrose inverse. If X is at least a $\{1\}-$ inverse of A, then AX and XA are projectors on $R(A)$ and $R(X)$ the range spaces of A and X respectively and $rank(AX) = rankA = rank(XA)$. For more properties and remarks, see ([1]), ([5]).

Denote by $A^{\{1\}}$ and by $A^{\{1,2\}}$ the sets of all $\{1\}-$ inverses and $\{1,2\}-$ inverses of A respectively. We will denote by small letters the sub-matrices of a matrix X and by I and 0 the identical and zero matrices or identical and zero sub-matrices. Lemmas without proofs are either easy to proof or well known in the precedent references.

Lemma 2.1. *Let A be an $m\times n$ matrix over K of rank r. Then*
1) There exist non singular matrices P and Q such that $A = Q^{-1}\begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix}P$.
2) $A^+ = P^{-1}\begin{pmatrix} a_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}Q$.
3) The elements of $A^{\{1\}}$ are of the form $P^{-1}\begin{pmatrix} a_r^{-1} & e \\ f & g \end{pmatrix}Q$ and the elements of $A^{\{1,2\}}$ are of the form $P^{-1}\begin{pmatrix} a_r^{-1} & e \\ f & fa_re \end{pmatrix}Q$.

The first assertion is an elementary lemma in linear algebra. The second one and the third need only a direct verification in (2.1). For this purpose the proof was omitted.

3 Main results

3.1 Semigroup on $A^{\{1\}}$

3.1.1 Factorization and commutativity

The main important point in the theorem below is factorization of $A^{\{1,2\}}$. For this purpose we introduce two classes of $A^{\{1\}}$:

$$P_{A^+A} = \left\{ X \in A^{\{1\}} / X = P^{-1} \begin{pmatrix} a_r^{-1} & x \\ 0 & 0 \end{pmatrix} Q \right\}$$
$$P_{AA^+} = \left\{ X \in A^{\{1\}} / X = P^{-1} \begin{pmatrix} a_r^{-1} & 0 \\ x & 0 \end{pmatrix} Q \right\}.$$

The notations P_{A^+A} and P_{AA^+} are justified by the fact that these two classes are the sets of fixed points under left and right multiplications by orthogonal projectors A^+A and AA^+ . In fact, for any $X \in A^{\{1\}}$,
$A^+AX = X \Rightarrow X = P^{-1} \begin{pmatrix} a_r^{-1} & x \\ 0 & 0 \end{pmatrix} Q$ and $XAA^+ = X \Rightarrow X = P^{-1} \begin{pmatrix} a_r^{-1} & 0 \\ x & 0 \end{pmatrix} Q$.

Theorem 3.1. *Let $*$ be a law on $A^{\{1\}}$ defined as follows: for any X, $Y \in A^{\{1\}}$; $X * Y = XAY$. Then*

1) $A^{\{1\}}$ is a semigroup.
2) For any X, $Y \in A^{\{1\}}$ we have $X * Y \in A^{\{1,2\}}$
3) $A^{\{1,2\}}$ is an ideal of $A^{\{1\}}$.
4) $P_{AA^+} * P_{A^+A} = A^{\{1,2\}}$ and $P_{AA^+} \cap P_{A^+A} = \{A^+\}$.
5) For any X and Y in $A^{\{1\}}$ we have
$X * Y = Y * X \Leftrightarrow AX = AY$ and $XA = YA$.

Proof. 1) $A(XAY)A = (AXA)YA = AYA = A$. Then $X * Y = XAY \in A^{\{1\}}$. The associativity of $*$ is obtained from the associativity of a product of matrices. So $A^{\{1\}}$ is a semigroup. 2) Let X and Y be in $A^{\{1\}}$. Since $A(XAY)A = A$ and $(XAY)A(XAY) = X(AYA)(XAY) = X(AXA)Y = XAY$, we obtain $X * Y = XAY \in A^{\{1,2\}}$. 3) As $A^{\{1,2\}} \subset A^{\{1\}}$, then 3 is just a simple consequence of 2.
4) First, remark that P_{AA^+} and P_{A^+A} are subsemigroups of $A^{\{1,2\}}$. Let $A = Q^{-1} \begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix} P$. For any $Z = P^{-1} \begin{pmatrix} a_r^{-1} & x \\ y & ya_rx \end{pmatrix} Q \in A^{\{1,2\}}$ there exist $Y = P^{-1} \begin{pmatrix} a_r^{-1} & 0 \\ y & 0 \end{pmatrix} Q \in P_{AA^+}$ and $X = P^{-1} \begin{pmatrix} a_r^{-1} & x \\ 0 & 0 \end{pmatrix} Q \in P_{A^+A}$ such that $Y * X = P^{-1} \begin{pmatrix} a_r^{-1} & 0 \\ y & 0 \end{pmatrix} \begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_r^{-1} & x \\ 0 & 0 \end{pmatrix} Q^{-1} = P^{-1} \begin{pmatrix} a_r^{-1} & x \\ y & ya_rx \end{pmatrix} Q = Z$. Consequently, we have $P_{AA^+} * P_{A^+A} = A^{\{1,2\}}$. However, by direct computation, we fined that $P_{A^+A} * P_{AA^+} = \{A^+\}$. For $Z = p \begin{pmatrix} a_r^{-1} & x \\ y & ya_rx \end{pmatrix} Q^{-1} \in P_{AA^+} \cap P_{A^+A}$ we have $x = 0$ and $y = 0$. Then $Z = P \begin{pmatrix} a_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} = A^+$. Therefore $P_{AA^+} \cap P_{A^+A} = \{A^+\}$. This assertion can be replaced by uniqueness of factorization of elements of $A^{\{1,2\}}$. 5) For any X and Y in $A^{\{1\}}$ we have $XAY = YAX \Rightarrow AXAY =$

$AYAX \Rightarrow AY = AX$ and $XAY = YAX \Rightarrow XAYA = YAXA \Rightarrow XA = YA$. Conversely, $AY = AX \Rightarrow XAY = XAX$ and $YA = XA \Rightarrow YAX = XAX$. Hence we have $X * Y = Y * X \Leftrightarrow AX = AY$ and $XA = YA$. □

Note that we have not really commutativity on $A^{\{1\}}$, but we have a kind of commutativity modulo projectors.

It is easy to prove that the set of projectors of a vector space on a subspace is a semigroup under ordinary composition. As many studies on projectors and their properties have been done, we will exploit this to make properties of $A^{\{1\}}$ more easy. This claim will be reached by defining an equivalence relation in $A^{\{1\}}$, thus we will have an isomorphism between the quotient semigroup of $A^{\{1\}}$ and the semigroup of projectors on $R(A)$. It will cause no confusion if we use the same letter to designate a projector and its associated matrix.

Theorem 3.2. *Let $A \in M_{m\times n}(K)$, Π be the semigroup of projectors of K^m on $R(A)$. Then we have*
1) For any $P \in \Pi$, there exists $X \in A^{\{1\}}$ such that $AX = P$.
2) There is an equivalence relation $\sim$ on $A^{\{1\}}$ such that under the quotient law of $*$, $\frac{A^{\{1\}}}{\sim}$ is a semigroup isomorphic to Π.

Proof. 1) Let $P \in \Pi$ and $A^- \in A^{\{1\}}$. As AA^- is a projector on $R(A)$, we have $AA^-P = P$ and $PA = A$. Thus if we take $X = A^-P$, we have $AXA = AA^-PA = AA^-A = A$ and $AX = P$ which means that $\Pi = \left\{AX/X \in A^{\{1\}}\right\}$.
2) Let $\sim$ be a relation in $A^{\{1\}}$ defined by $X \sim Y \Leftrightarrow AX = AY$. Then it is easy to check that $\sim$ is an equivalence relation in $A^{\{1\}}$. Let χ be the canonical map from $A^{\{1\}}$ on $\frac{A^{\{1\}}}{\sim}$. Then for every $\chi(X), \chi(Y) \in \frac{A^{\{1\}}}{\sim}$, the quotient law of $*$ is defined by $\chi(X)\chi(Y) = \chi(X * Y)$. It easy to show that $\frac{A^{\{1\}}}{\sim}$ is a semigroup and χ is an homomorphism. It immediately follows that there exists a map ψ from $\frac{A^{\{1\}}}{\sim}$ to Π defined by $\psi(\chi(X)) = AX$. Now we check that ψ is an homomorphism. Since $A(XAY) = (AXA)Y = AY$, we obtain $XAY \sim Y$. Then $\chi(XAY) = \chi(Y)$. Therefore

$$\begin{aligned}\psi(\chi(X)\chi(Y)) &= \psi(\chi(X * Y)) = \psi(\chi(XAY)) = \psi(\chi(Y)) = AY = (AXA)Y = \\ &= (AX)(AY) = \psi(\chi(X))\,\psi(\chi(Y)).\end{aligned}$$

Now, since $AX = AY$, it follows that $\chi(X) = \chi(Y)$. We conclude that for any $AX \in \Pi$ there is a unique $\chi(X) \in \frac{A^{\{1\}}}{\sim}$ such that $\psi(\chi(X)) = AX$ which means that ψ is a bijection. □

3.2 The set $A^{\{1\}}$ of semigroups

Denote by $M^{\{1\}}_{m\times n}(K)$ the set $M^{\{1\}}_{m\times n}(K) = \left\{A^{\{1\}}/A \in M_{m\times n}(K)\right\}$.

In this subsection, we will establish a relation between $M_{m\times n}(K)$ and $M^{\{1\}}_{m\times n}(K)$ and deduce some properties of $M^{\{1\}}_{m\times n}(K)$ like isomorphism of semigroups, intersection of them and relation order. For this main, the following Lemma will be useful.

Lemma 3.1. *[1], [5]. Let A and B be two equivalent matrices, such that $B = Q^{-1}AP$. Then, for every Y in $B^{\{1\}}$ there exists a unique X in $A^{\{1\}}$ such that $Y = P^{-1}XQ$.*

3.2.1 Isomorphism between semigroups

Theorem 3.3. *Let A and B be two equivalent matrices. Then $\left(A^{\{1\}},*\right)$ and $\left(B^{\{1\}},*\right)$ are isomorphic.*

Proof. By using the previous Lemma, we can define a map φ from $A^{\{1\}}$ on $B^{\{1\}}$ as follows $\varphi(X) = P^{-1}XQ$. Then φ^{-1} is the inverse map from $B^{\{1\}}$ on $A^{\{1\}}$ given by $\varphi^{-1}(X) = PXQ^{-1}$. In addition, for every X and Y in $A^{\{1\}}$, we have

$$\begin{aligned}\varphi(X*Y) &= \varphi(XAY) = P^{-1}(XAY)Q = (P^{-1}XQ)(Q^{-1}AP)(P^{-1}YQ) = \\ &= \varphi(X)B\varphi(Y) = \varphi(X)*\varphi(Y).\end{aligned}$$

Also, we have for every X and Y in $B^{\{1\}}$,

$$\varphi^{-1}(X*Y) = \varphi^{-1}(XBY) = \varphi^{-1}(X)A\varphi^{-1}(Y) = \varphi^{-1}(X)*\varphi^{-1}(Y).$$

Then the map φ is an isomorphism. □

We remark that $\varphi(A^{+}) = P^{-1}A^{+}Q = B^{+}$ only if P and Q are orthogonal.

Lemma 3.2. [4] *Let A and B be two matrices. Then the following statements are equivalent:*
a) $rank(A) + rank(B - A) = rank(B)$.
b) Every $\{1\}-$ inverse of B is a $\{1\}-$ inverse of both A and $B - A$.
c) $R(A) \cap R(B) = \{0\}$ and $R(A^t) \cap R(B^t) = \{0\}$.

Theorem 3.4. *There is a one-to-one correspondence between $M_{m\times n}(K)$ and $M^{\{1\}}_{m\times n}(K)$ maps 0 to $M_{n\times m}(K)$ and preserves isomorphisms between semigroups.*

Proof. Let ψ be a map from $M_{m\times n}(K)$ onto $M^{\{1\}}_{m\times n}(K)$ defined for every $A \in M_{m\times n}(K)$ by $\psi(A) = A^{\{1\}}$. Since $0X0 = 0$ for any $X \in M_{n\times m}(K)$, we get $0^{\{1\}} = M_{n\times m}(K)$. Thus $\psi(0) = M_{n\times m}(K)$. According to Lemma 3.2, if $A^{\{1\}} = B^{\{1\}}$, we have $rank(A) + rank(B - A) = rank(B)$ and $rank(B) + rank(A - B) = rank(A)$. Thus we have $rank\,(A - B) = 0 = rank\,(B - A)$. Therefore $A = B$. Now, let A and $B \in M_{m\times n}(K)$ such that $B = Q^{-1}AP$. According to Lemma 3.1 and Theorem 3.3 , we have $B^{\{1\}} = \left\{P^{-1}XQ/X \in A^{\{1\}}\right\} = \varphi\left(A^{\{1\}}\right)$. Hence we have $\psi(B) = \varphi(\psi(A))$. □

3.2.2 Intersection of semigroups

Theorem 3.5. *a) For every matrix $A \in M_{m\times n}(K)$ there exists a matrix $A' \in M_{m\times n}(K)$ such that $A^{\{1\}} \cap A'^{\{1\}} \neq \varnothing$.*

b) For any matrices $A, B \in M_{m\times n}(K)$ there exists an isomorphism φ from $A^{\{1\}}$ on $\varphi\left(A^{\{1\}}\right)$ such that $\varphi\left(A^{\{1\}}\right) \cap B^{\{1\}} \neq \varnothing$.

Proof. a) Let $rankA = r \leq \min(m,n)$. It is sufficient to prove that for $A \in M_{m\times n}(K)$ with $rank(A) = r$, there exists a matrix $A' \in M_{m\times n}(K)$ such that $rank\,(A + A') = rank(A) + rank(A') = \min(m,n)$. Applying Lemma 3.2, we have every $\{1\}-$ inverse of $(A + A')$ is a $\{1\}-$ inverse of both A and A'. Thus $A^{\{1\}} \cap A'^{\{1\}} \neq \varnothing$. Let P and

Q be two nonsingular matrices such that $A = Q^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}P$. Then it is sufficient to take $A' = Q^{-1}\begin{pmatrix} 0 & 0 \\ 0 & w \end{pmatrix}P$ with $w \in M_{(m-r)\times(n-r)}(K)$ and

$$rank(w) = \min(m, n) - r.$$

b) Let $rank(A) = r$, $rank(B) = s$. According to 1, there exist matrices A' and $B' \in M_{m\times n}(K)$ of ranks $\min(m,n) - r$ and $\min(m,n) - s$ such that $rank\,(A + A') = rank(A) + rank(A') = \min(m,n)$ and $rank\,(B + B') = rank(B) + rank(B') = \min(m,n)$. It follows that $(A + A')^{\{1\}} \subset A^{\{1\}} \cap A'^{\{1\}}$, $(B + B')^{\{1\}} \subset B^{\{1\}} \cap B'^{\{1\}}$. Since $A + A'$, $B + B'$ have the same rank, they are equivalent. So, there exists an isomorphism φ from $A^{\{1\}}$ on $\varphi\left(A^{\{1\}}\right)$ such that $\varphi\left((A + A')^{\{1\}}\right) = (B + B')^{\{1\}}$. Hence we have

$$\varphi\left((A + A')^{\{1\}}\right) \subset \varphi\left(A^{\{1\}} \cap A'^{\{1\}}\right) = \varphi\left(A^{\{1\}}\right) \cap \varphi\left(A'^{\{1\}}\right)$$

and $\varphi\left((A + A')^{\{1\}}\right) \subset B^{\{1\}} \cap B'^{\{1\}}$. Finally, we have $\varphi\left(A^{\{1\}}\right) \cap B^{\{1\}} \neq \varnothing$. $\square$

The natural question arises whether it is possible to conclude that for any matrices A, $B \in M_{m\times n}(K)$ we have $A^{\{1\}} \cap B^{\{1\}} \neq \varnothing$. The answer is negative. In fact if we take $B = \alpha A$ for a scalar α which is different of 1 and 0, then if $X \in A^{\{1\}} \cap B^{\{1\}}$, we have $\alpha A = \alpha A X \alpha A = \alpha^2 A$. Thus $\alpha \in \{0, 1\}$ which contradicts our assumption. Consequently we have $A^{\{1\}} \cap B^{\{1\}} = \varnothing$.

3.2.3 Partial order in $M_{m\times n}^{\{1\}}(K)$

Definition 3.1. ([5]) A minus partial order, denoted by $\prec^-$, is defined as follows for A, $B \in M_{m\times n}(K)$, then $A \prec^- B$ if $rank(B) = rank(A) + rank\,(B - A)$.

Theorem 3.6. *1) The inclusion is a partial order in $M_{m\times n}^{\{a\}}(K)$ induced by the minus order in the reverse order.*
b) Let $m_0 = \min(m, n)$. For any matrix $A \in M_{m\times n}(K)$ of rank r there exists a sequence of matrices $A = A_r \prec^- A_{r+1} \prec^- .... \prec^- A_{m_0}$ in $M_{m\times n}(K)$ such that $rank\,(A_r) = rank\,(A) = r, rank A_{r+i} = r + i$ for $i = 1, ...m_0 - r$. Thus, there exists a sequence $A_{m_0}^{\{1\}} \subset \cdots \subset A_r^{\{1\}} = A^{\{1\}}$ and $A_{m_0}^{\{1\}}$ is the last term.

Proof. a) For $A, B \in M_{m\times n}(K)$, if $A \prec^- B$, then $rank(B) = rank(A) + rank\,(B - A)$. By Lemma 3.2 it follows that $B^{\{1\}} \subset A^{\{1\}}$. Hence we have the partial order $\subset$ in $M_{m\times n}^{\{1\}}(K)$. b) Let $\{v_1, v_2, ...v_r\}$ be a basis of $R(A)$ and $\{v_{r+1}, v_{r+2}, ...v_m\}$ be such together with the basis of $R(A)$ form a basis for K^m. Let $\{e_1, e_2, ...e_n\}$ be a basis for K^n such that $A_r e_j = A e_j = v_j$ for $j = 1, ...r$ and $A_r e_j = 0$ for $j = r + 1, ...m$ and for $i = 1, ...m_0 - r$, $A_{r+i} e_j = v_j$ for $j = 1, ...r + i$ and $A_{r+i} e_j = 0$ for $j = r + i + 1, ...m$. In these bases the matrices $A = A_r$ and A_{r+i} are of the form $A = A_r = Q^{-1}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}P$, $A_{r+i} = Q^{-1}\begin{pmatrix} I_{r+i} & 0 \\ 0 & 0 \end{pmatrix}P$ for $i = 1, ...m_0 - r$. Then we have $rank\,(A_r) = rank(A) = r$, and

$$rank\,(A_{r+i}) = rank\,(A_r) + i = r + i = rank(A_r) + rank\,(A_{r+i} - A_r),$$

for $i = 1, ...m_0 - r$. Thus $A = A_r \prec^- A_{r+1} \prec^- ..... \prec^- A_{m_0}$. By a), we have $A_{m_0}^{\{1\}} \subset .... \subset A_r^{\{1\}} = A^{\{1\}}$. $A_{m_0}^{\{1\}}$ is the last term because A_{m_0} is of maximal rank. $\square$

4 Conclusions

In the present paper we give the set of generalized inverses of a matrix a structure of a semigroup and factorize its elements to simple ones. Unfortunately, this semigroup is not commutative. To get a nice structural result, we define an equivalence relation and we obtain an isomorphism between the quotient semigroup and the semigroup of projectors. Furthermore we establish a one-to-one correspondence between the set of matrices and the set of associated semigroups. This correspondence preserves isomorphisms between semigroups and maps the zero matrix (which is the zero of the additive group of matrices) into the entire set of matrices (which is the unity related to intersection of sets). Finally, we have proved that for any matrix, there exists a sequence of semigroups ordered by inclusion.

References

[1] R.B. Bapat, *Linear Algebra and Linear Models*, Springer 2000.

[2] A. Ben-Israel and T.N.E. Greville, *Generalized inverses, theory and applications*, Kreiger Eds. 1980.

[3] O. Demirel, E. Soyturk, *On the ordered sets in n-dimensional real inner product spaces*, Appl. Sci. 10 (2008), 66-72.

[4] J.A. Fill and D.E. Fishkind, *The Moore-Penrose Generalized Inverse for Sums of Matrices*, SIAM J. Matrix Anal. Appl. 21 (1999), 629-635.

[5] M.Z. Nashed, *Generalized Inverses, Theory and Applications*, Academic Press 1976.

[6] L. Noui, *A note on consecutive ones in a binary matrix*, Appl. Sci. 9 (2007), 141-147.

[7] Z. Xiong, B. Zheng, *The reverse order laws for* $\{1, 2, 3\}-$ *and* $\{1, 2, 4\}-$ *inverses of a two-matrix product*, Appl. Math. Lett. 21, 7 (2008), 649-655.

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