Some fixed points theorems in generalized D^*-metric spaces

C. T. Aage and J. N. Salunke

Abstract. In this paper we introduce the notion of generalized D^*-metric space and prove some fixed point theorems in complete generalized D^*-metric spaces.

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1 Introduction

Huang and Zhang [11] generalized the notion of metric spaces, replacing the real numbers by an ordered Banach space and defined cone metric spaces. They have proved Banach contraction mapping theorem and some other fixed point theorems of contractive type mappings in cone metric spaces. Subsequently, Rezapour and Hamlbarani [17], Ilic and Rakocevic [9], contributed some fixed point theorems for contractive type mappings in cone metric spaces.

Let E be a real Banach space and P a subset of E. P is called a cone if and only if:
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in intP$, where $\mathrm{int}P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \tag{1.1}$$

The least positive number K satisfying the above is called the normal constant of P [11]. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$.

The cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. Rezapour and Hamlbarani [17] proved every regular cone is normal and there are normal cone with normal constant $M \geq 1$.

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Gahler [7, 8] introduced the notions of 2-metric space and Dhage [5, 6] defined D-metric spaces as a generalization of metric spaces. In 2003, Zead Mustafa and Brailey Sims[13] introduced a new structure of generalized metric spaces, which are called G-metric spaces. Recently Aage and Salunke[2] generalized G-metric space by replacing R by real Banach space in G-metric spaces. In 2007 Shaban Sedghi et al [18] modify the D-metric space and defined D^*-metric spaces. Now in this paper we generalized D^*-metric spaces by introducing generalized D^*-metric space by replacing R by a real Banach space in D^*-metric spaces.

In the following we always suppose that E is a Banach space, P is a cone in E with $\mathrm{int}P \neq \{0\}$ and $\leq$ is a partial ordering with respect to P.

The concept of generalized D^*-metric space is defined as follows

Definition 1.1. Let X be a non empty set. A generalized D^*-metric on X is a function, $D^* : X^3 \to E$, that satisfies the following conditions for all $x, y, z, a \in X$:
(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$,
Then the function D^* is called a generalized D^*-metric and the pair (X, D^*) is called a generalized D^*-metric space

Example 1.2. Let $E = R^2, P = \{(x, y) \in E : x, y \geq 0\}, X = R$ and $D^* : X \times X \times X \to E$ defined by $D^*(x, y, z) = (|x-y|+|y-z|+|x-z|, \alpha(|x-y|+|y-z|+|x-z|))$, where $\alpha \geq 0$ is a constant. Then (X, D^*) is a generalized D^*- metric space.

Proposition 1.3. *If (X, D^*) be generalized D^* metric space, Then for all $x, y, z \in X$, we have $D^*(x, x, y) = D^*(x, y, y)$.*

Proof. Let $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and similarly $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$. Hence we have $D^*(x, x, y) = D^*(x, y, y)$. □

Definition 1.4. Let (X, D^*) be a generalized D^*- metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $m, n > N$, $D^*(x_m, x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $x_n \to x (n \to \infty)$.

Lemma 1.5. *Let (X, D^*) be a generalized D^*- metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $D^*(x_m, x_n, x) \to 0$ $(m, n \to \infty)$.*

Proof. Let $\{x_n\}$ be a sequence in generalized D^*-metric space X converge to $x \in X$ and $\epsilon > 0$ be any number. Then for any $c \in E$, with $0 \ll c$ there is a positive integer N such that $m, n > N$ implies

$$D^*(x_m, x_n, x) \ll c \Rightarrow \|D^*(x_m, x_n, x)\| \leq K\|c\|,$$

since $0 \leq D^*(x_m, x_n, x) < c$ and K is normal constant. Choose c such that $K\|c\| < \epsilon$. Then $\|D^*(x_m, x_n, x)\| < \epsilon$ for all $m, n > N$ and hence $D^*(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Conversely let $D^*(x_m, x_n, x) \to 0$ as $m, n \to \infty$ (in $P \subset E$). So for any $c \in E$ with $0 \ll c$ (i.e. $c \in intP$), we have $\|c\| > 0$, let $r = dist(c, \partial P) = inf\{\|c - t\| : t \in \partial P\}$, where ∂P denotes the boundary of P, for this given $r, 0 < r \leq \|c\|$, there exist a positive integer N such that $m, n > N$ implies that $\|G(x_m, x, x)\| < \frac{r}{2} < \|c\|$ and for any $t \in \partial P$,

$$\|(c - D^*(x_m, x_n, x)) - t\| \geq \|c - t\| - \|D^*(x_m, x_n, x)\| > r - \frac{r}{2} = \frac{r}{2}$$

which proves that $c - D^*(x_m, x_n, x) \in intP$ i.e. $G(x_m, x_n, x) \ll c$. $\square$

Remark. If $\{u_n\}$ is a sequence in $P \subset E$ and $u_n \to u$ in E the $u \in P$ as P is a closed subset of E. From this $u_n \geq 0 \Rightarrow u \geq 0$. Thus if $u_n \leq v_n$ in P then $\lim u_n \leq \lim v_n$, provided limit exist.

Lemma 1.6. *Let (X, D^*) be a generalized D^*-metric space then the following are equivalent.*
(i) *$\{x_n\}$ is D^*-convergent to x.*
(ii) $D^*(x_n, x_n, x) \to 0$, as $n \to \infty$
(iii) $D^*(x_n, x, x) \to 0$, as $n \to \infty$.

Proof. (i) $\Rightarrow$ (ii). by Lemma 1.5.
$(ii) \to$ (i). Assume (ii), i.e. $D^*(x_n, x_n, x) \to 0$ as $n \to \infty$ i.e. for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $D^*(x_n, x_n, x) \ll c/2$,

$$\begin{aligned} D^*(x_m, x_n, x) &\leq D^*(x_m, x, x) + D^*(x, x_n, x_n) \\ &\leq D^*(x, x_m, x_m) + D^*(x, x_n, x_n) \\ &\ll c \quad \text{for all } m, n > N. \end{aligned}$$

Hence $\{x_n\}$ is D^*-convergent to x.
$(ii) \Leftrightarrow (iii)$. Assume (ii), i.e. $D^*(x_n, x_n, x) \to 0$ as $n \to \infty$ i.e. for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $D^*(x_n, x_n, x) \ll c$,

$$D^*(x_n, x, x) = D^*(x, x_n, x_n) = D^*(x_n, x_n, x) \ll c.$$

Hence $D^*(x_n, x, x) \to 0$ as $n \to \infty$, since c is arbitrary.
$(iii) \Rightarrow (ii)$. Assume that (iii) i.e. $D^*(x_n, x, x) \to 0$ as $n \to \infty$. Then for any $c \in E$ with $0 \ll c$, there is an N such that $n > N$ implies $D^*(x_n, x, x) \ll \frac{c}{2}$. Hence $m, n > N$ gives $D^*(x_m, x_n, x) \leq D^*(x_m, x, x) + D^*(x, x_n, x_n) \ll c$. Thus $\{x_n\}$ is D^*-convergent to x. $\square$

Lemma 1.7. *Let (X, D^*) be a generalized D^*- metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$, if exists, is unique.*

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $m, n > N$, $D^*(x_m, x_n, x) \ll c$. We have

$$\begin{aligned} 0 \leq D^*(x, x, y) &\leq D^*(x, x, x_n) + D^*(x_n, y, y) \\ &= D^*(x_n, x_n, x) + D^*(x_n, x_n, y) \\ &\ll 2c. \text{ for } \quad \text{all } n > N. \end{aligned}$$

Hence $\|D^*(x, x, y)\| \leq 2K\|c\|$. Since c is arbitrary, $D^*(x, x, y) = 0$, therefore $x = y$.$\square$

Definition 1.8. Let (X, D^*) be a generalized D^*- metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $m, n, l > N$, $D^*(x_m, x_n, x_l) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.9. Let (X, D^*) be a generalized D^*-metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete generalized D^*-metric space.

Lemma 1.10. *Let (X, D^*) be a generalized D^*-metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $m, n, l > N$, $D^*(x_m, x_n, x) \ll c/2$ and $D^*(x_l, x_l, x) \ll c/2$. Hence

$$D^*(x_m, x_n, x_l) \leq D^*(x_m, x_n, x) + D^*(x, x_l, x_l) \leq c.$$

Therefore $\{x_n\}$ is a Cauchy sequence. □

Lemma 1.11. *Let (X, D^*) be a generalized D^* metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $D^*(x_m, x_n, x_l) \to 0$ $(m, n, l \to \infty)$.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in generalized D^*-metric space (X, D^*) and $\epsilon > 0$ be any real number. Then for any $c \in E$ with $0 \ll c$, there exist a positive integer N such that $m, n, l > N$ implies $D^*(x_m, x_n, x_l) \ll c \Rightarrow 0 \leq D^*(x_m, x_n, x_l) < c$ i.e. $\|D^*(x_m, x_n, x_l)\| \leq K\|c\|$, where K is a normal constant of P in E. Choose c such that $K\|c\| < \epsilon$. Then $\|D^*(x_m, x_n, x_l)\| < \epsilon$ for all $m, n, l > N$, showing that $D^*(x_m, x_n, x_l) \to 0$ as $m, n, l \to \infty$.

Conversely let $D^*(x_m, x_n, x_l) \to 0$ as $m, n, l \to \infty$. For any $c \in E$ with $0 \ll c$ we have $K\|c\| > 0$ ($\|c\| > 0$, as $c = c - 0 \in IntP$ and $K \geq 1$). For given $K\|c\|$ there is a positive integer N such that $m, n, l > N \Rightarrow \|D^*(x_m, x_n, x_l)\| < K\|c\|$. This proves that $D^*(x_m, x_n, x_l) \ll c$ for all $m, n, l > N$ and hence $\{x_n\}$ is a Cauchy sequence. □

Definition 1.12. Let (X, D^*), (X', D'^*) be generalized D^*-metric spaces, then a function $f : X \to X'$ is is said to be D^*-continuous at a point $x \in X$ if and only if it is D^*-sequentially continuous at x, that is, whenever $\{x_n\}$ is D^*-convergent to x we have $\{fx_n\}$ is D^*-convergent to fx.

Lemma 1.13. *Let (X, D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be three sequences in X and $x_n \to x$, $y_n \to y, z_n \to z$ $(n \to \infty)$. Then $D^*(x_n, y_n, z_n) \to D^*(x, y, z)$ $(n \to \infty)$.*

Proof. For every $\epsilon > 0$, choose $c \in E$ with $0 \ll c$ and $\|c\| < \dfrac{\epsilon}{6K+3}$. From $x_n \to x, y_n \to y$ and $z_n \to z$, there is N such that for all $n > N, D^*(x_n, x_n x) \ll c$, $D^*(y_n, y_n, y) \ll c$ and $D^*(z_n, z_n, z) \ll c$. We have

$$\begin{aligned} D^*(x_n, y_n, z_n) &\leq D^*(x_n, y_n, z) + D^*(z, z_n, z_n) = D^*(z, x_n, y_n) + D^*(z_n, z_n, z) \\ &\leq D^*(z, x_n, y) + D^*(y, y_n, y_n) + D^*(z_n, z_n, z) \\ &= D^*(y, z, x_n) + D^*(y_n, y_n, y) + D^*(z_n, z_n, z) \\ &\leq D^*(y, z, x) + D^*(x, x_n, x_n) + D^*(y_n, y_n, y) + D^*(z_n, z_n, z) \\ &\leq 3c + D^*(x, y, z). \end{aligned}$$

Similarly, we infer $D^*(x,y,z) \leq D^*(x_n,y_n,z_n) + 3c$. Hence

$$0 \leq D^*(x,y,z) + 3c - D^*(x_n,y_n,z_n) \leq 6c$$

and

$$\begin{aligned} \|D^*(x_n,y_n,z_n) - D^*(x,y,z)\| &\leq \|D^*(x,y,z) + 3c - D^*(x_n,y_n,z_n)\| + \|3c\| \\ &\leq (6K+3)\|c\| < \epsilon, \text{ for all } n > N. \end{aligned}$$

Therefore $D^*(x_n,y_n,z_n) \to D^*(x,y,z)$ $(n \to \infty)$. □

Remark. If $x_n \to x$ in generalized D^*-metric space X, then every subsequence of $\{x_n\}$ converges to x in X. Let $\{x_{k_n}\}$ be any subsequence of $\{x_n\}$ and $x_n \to x$ in X then $D^*(x_m,x_n,x) \to 0$ as $m,n \to \infty$ and also $D^*(x_{k_m},x_{k_n},x) \to 0$ as $m,n \to \infty$ since $k_n \geq n$ for all n.

Definition 1.14. [10]Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Proposition 1.15. *[3]Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

2 Main results

The first main result is

Theorem 2.1. *Let (X,D^*) be generalized D^*-metric space, P be a normal cone with normal constant K and let $S,T : X \to X$, be two mappings which satisfy the following conditions,*
(i) $T(X) \subset S(X)$,
(ii) $T(X)$ or $S(X)$ is complete, and
(iii)

$$\begin{aligned} D^*(Tx,Ty,Tz) \leq{}& aD^*(Sx,Sy,Sz) + bD^*(Sx,Tx,Tx) + cD^*(Sy,Ty,Ty) \\ &+ dD^*(Sz,Tz,Tz) \end{aligned} \tag{2.1}$$

for all $x,y,z \in X$, where $a,b,c,d \geq 0, a+b+c+d < 1$. Then S and T have a unique point of coincidence in X. Moreover if S and T are weakly compatible, Then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary, there exist $x_1 \in X$ such that $Tx_0 = Sx_1$, in this way we have sequence $\{Sx_n\}$ with $Tx_{n-1} = Sx_n$. Then from the inequality (2.1), we have

$$\begin{aligned} D^*(Sx_n,Sx_{n+1},Sx_{n+1}) &= D^*(Tx_{n-1},Tx_n,Tx_n) \\ &\leq aD^*(Sx_{n-1},Sx_n,Sx_n) + bD^*(Sx_{n-1},Tx_{n-1},Tx_{n-1}) \\ &\quad + cD^*(Sx_n,Tx_n,Tx_n) + dD^*(Sx_n,Tx_n,Tx_n) \\ &= aD^*(Sx_{n-1},Sx_n,Sx_n) + bD^*(Sx_{n-1},Sx_n,Sx_n) \\ &\quad + cD^*(Sx_n,Sx_{n+1},Sx_{n+1}) + dD^*(Sx_n,Sx_{n+1},Sx_{n+1}) \\ &= (a+b)D^*(Sx_{n-1},Sx_n,Sx_n) + (c+d)D^*(Sx_n,Sx_{n+1},Sx_{n+1}) \end{aligned}$$

This implies

$$D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq qD^*(Sx_{n-1}, Sx_n, Sx_n)$$

where $q = \frac{(a+b)}{1-(c+d)}$, then $0 \leq q < 1$. By repeated application of above inequality we have,

$$D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq q^n D^*(Sx_0, Sx_1, Sx_1) \tag{2.2}$$

Then, for all $n, m \in \mathbf{N}$,$n < m$ we have by repeated use the rectangle inequality and equality (2.2) that

$$\begin{aligned} D^*(Sx_n, Sx_m, Sx_m) &\leq D^*(Sx_n, Sx_n, Sx_{n+1}) + D^*(Sx_{n+1}, Sx_{n+1}, Sx_{n+2}) \\ &\quad + D^*(Sx_{n+2}, Sx_{n+2}, Sx_{n+3}) + \cdots + D^*(Sx_{m-1}, Sx_{m-1}, Sx_m) \\ &\leq D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_{n+1}, Sx_{n+2}, Sx_{n+2}) \\ &\quad + D^*(Sx_{n+2}, Sx_{n+3}, Sx_{n+3}) + \cdots + D^*(Sx_{m-1}, Sx_m, Sx_m) \\ &\leq (q^n + q^{n+1} + \cdots + q^{m-1}) D^*(Sx_0, Sx_1, Sx_1) \\ &\leq \frac{q^n}{1-q} D^*(Sx_0, Sx_1, Sx_1). \end{aligned}$$

From (1.1) we infer

$$\|D^*(Sx_n, Sx_m, Sx_m)\| \leq \frac{q^n}{1-q} K \|D^*(Sx_0, Sx_1, Sx_1)\|$$

which implies that $D^*(Sx_n, Sx_m, Sx_m) \to 0$, as $n, m \to \infty$, since $\frac{q^n}{1-q} K \|D^*(Sx_0, Sx_1, Sx_1)\| \to 0$ as $n, m \to \infty$.

For $n, m, l \in \mathbf{N}$, and

$$D^*(Tx_n, Sx_m, Sx_l) \leq D^*(Sx_n, Sx_m, Sx_m) + D^*(Sx_m, Sx_l, Sx_l),$$

from (1.1)

$$\|D^*(Tx_n, Sx_m, Sx_l)\| \leq K[\|D^*(Sx_n, Sx_m, Sx_m)\| + \|D^*(Sx_m, Sx_l, Sx_l)\|]$$

taking limit as $n, m, l \to \infty$, we get $D^*(Sx_n, Sx_m, Sx_l) \to 0$. So $\{Sx_n\}$ is D^*-Cauchy a sequence, since $S(X)$ is D^*-complete, there exists $u \in S(X)$ such that $\{Sx_n\} \to u$ as $n \to \infty$, there exist $p \in X$ such that $Sp = u$. If $T(X)$ is complete, then there exist $u \in T(X)$ such that $Sx_n \to u$, as $T(X) \subset S(X)$, we have $u \in S(X)$.Then there exist $p \in X$ such that $Sp = u$. We claim that $Tp = u$,

$$\begin{aligned} D^*(Tp, u, u) &= D^*(Tp, Tp, u) \\ &\leq D^*(Tp, Tp, Tx_n) + D^*(Tx_n, u, u) \\ &\leq aD^*(Sp, Sp, Sx_n)) + bD^*(Sp, Tp, Tp) + cD^*(Sp, Tp, Tp) \\ &\quad + dD^*(Sx_n, Tx_n, Tx_n) + D^*(Sx_{n+1}, u, u) \\ &\leq aD^*(u, u, Sx_n) + bD^*(u, Tp, Tp) + cD^*(u, Tp, Tp) \\ &\quad + dD^*(Sx_n, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_{n+1}, u, u) \end{aligned}$$

This implies that

$$D^*(Tp,Tp,u) \leq \frac{1}{1-(b+c)}\{aD^*(u,u,Sx_n)+dD^*(Sx_n,Sx_{n+1},Sx_{n+})$$
$$+D^*(Sx_{n+1},u,u)\}$$

from (1.1)

$$\|D^*(Tp,Tp,u)\| \leq K\frac{1}{1-(b+c)}\{a\|D^*(u,u,Sx_n)\|+d\|D^*(Sx_n,Sx_{n+1},Sx_{n+1})\|$$
$$+\|D^*(Sx_{n+1},u,u)\|\}$$

as $n \to \infty$, right hand side approaches to zero. Hence $\|D^*(Tp,Tp,u)\| = 0$ and $Tp = u$. i.e. $Tp = Sp$ and p is a point of coincidence point of S and T. Now we show that S and T have a unique point of coincidence. For this, assume that there exists a point q in$X$ such that $Sq = Tq$. Now

$$\begin{aligned} D^*(Tp,Tp,Tq) &\leq aD^*(Sp,Sp,Sq)+bD^*(Sp,Tp,Tp)+cD^*(Sp,Tp,Tp) \\ &\quad +dD^*(Sq,Tq,Tq) \\ &= aD^*(Tp,Tp,Tq)+bD^*(Tp,Tp,Tp)+cD^*(Tp,Tp,Tp) \\ &\quad +dD^*(Tq,Tq,Tq) \\ &= aD^*(Tp,Tp,Tq) \end{aligned}$$

we have $D^*(Tp,Tp,Tq) \leq aD^*(Tp,Tp,Tq)$, i.e. $(a-1)D^*(Tp,Tp,Tq) \in P$, but $(a-1)D^*(Tp,Tp,Tq) \in -P$, since $k-1<0$. Hence $(a-1)D^*(Tp,Tp,Tq) = 0$, this implies that $D^*(Tp,Tp,Tq) = 0$ i.e. $Tp = Tq$. Thus p is a unique point of coincidence of S and T. By Proposition 1.15, S and T have a unique common fixed point. □

Theorem 2.2. *Let (X,D^*) be complete generalized D^*-metric spaces, P be a normal cone with normal constant K and let $T : X \to X$, be a mapping satisfies the following conditions*

$$\begin{aligned} D^*(Tx,Ty,Tz) &\leq aD^*(x,y,z)+bD^*(x,Tx,Tx) \\ &\quad +cD^*(y,Ty,Ty)+dD^*(z,Tz,Tz) \end{aligned} \tag{2.3}$$

for all $x,y,z \in X$, where $a,b,c,d \geq 0, a+b+c+d<1$. Then T have a unique fixed point in X.

Proof. The proof uses Theorem 2.1 by replacing S by identity mapping. □

Theorem 2.3. *Let (X,D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K and let S and T, be two self mappings of X which satisfy the following conditions*
(i) $T(X) \subset S(X)$,
(ii) $T(X)$ or $S(X)$ is D^*-complete,
(iii)

$$\begin{aligned} D^*(Tx,Ty,Tz) &\leq a[D^*(Sx,Ty,Ty)+D^*(Sy,Tx,Tx)] \\ &\quad +b[D^*(Sy,Tz,Tz)+D^*(Sz,Ty,Ty)] \\ &\quad +c[D^*(Sx,Tz,Tz)+D^*(Sz,Tx,Tx)] \end{aligned}$$

(2.4)

for all $x, y, z \in X$, where $a, b, c \geq 0, 2a + 2b + 2c < 1$. Then S and T have a unique point of coincidence in X. Moreover if S and T are weakly compatible, then S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary, there exist $x_1 \in X$ such that $Tx_0 = Sx_1$, in this way we have sequence $\{Tx_n\}$ with $Tx_n = Sx_{n+1}$. Then from inequality (2.4), we have

$$\begin{aligned} D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) &= D^*(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq a[D^*(Sx_{n-1}, Tx_n, Tx_n) + D^*(Sx_n, Tx_{n-1}, Tx_{n-1})] \\ &\quad + b[D^*(Sx_n, Tx_n, Tx_n) + D^*(Sx_n, Tx_n, Tx_n)] \\ &\quad + [D^*(Sx_{n-1}, Tx_n, Tx_n) + D^*(Sx_n, Tx_{n-1}, Tx_{n-1})] \\ &= a[D^*(Sx_{n-1}, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_n, Sx_n, Sx_n)] \\ &\quad + b[D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_n, Sx_{n+1}, Sx_{n+1})] \\ &\quad + c[D^*(Sx_{n-1}, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_n, Sx_n, Sx_n)] \\ &= (a + c)(D^*(Sx_{n-1}, Sx_n, Sx_n) + D^*(Sx_n, Sx_{n+1}, Sx_{n+1})) \\ &\quad + 2bD^*(Sx_n, Sx_{n+1}, Sx_{n+1})\}, \end{aligned}$$

This implies that

$$D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq qD^*(Sx_{n-1}, Sx_n, Sx_n)$$

where $q = \frac{(a+c)}{1-(a+2b+c)}$, then $0 \leq q < 1$ and by repeated application of above inequality, we have,

$$D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq q^n D^*(Sx_0, Sx_1, Sx_1). \tag{2.5}$$

Then, for all $n, m \in N, m < n$, we have, by repeated use of the rectangle inequality,

$$\begin{aligned} D^*(Sx_m, Sx_n, Sx_n) &\leq D^*(Sx_m, Sx_{m+1}, Sx_{m+1}) + D^*(Sx_{m+1}, Sx_n, Sx_{n)} \\ &\leq D^*(Sx_m, Sx_{m+1}, Sx_{m+1}) + D^*(Sx_{m+1}, Sx_{m+2}, Sx_{m+2}) \\ &\quad + D^*(Sx_{m+2}, Sx_n, Sx_n) \\ &\leq D^*(Sx_m, Sx_{m+1}, Sx_{m+1}) + D^*(Sx_{m+1}, Sx_{m+2}, Sx_{m+2}) \\ &\quad + \cdots + D^*(Sx_{n-1}, Sx_n, Sx_n) \\ &\leq (q^m + q^{m+1} + \cdots + q^{n-1})D^*(Sx_0, Sx_1, Sx_1) \\ &\leq \frac{q^m}{1-q} D^*(Sx_0, Sx_1, Sx_1), \end{aligned}$$

from (1.1)

$$\|D^*(Sx_m, Sx_n, Sx_n)\| \leq \frac{q^m}{1-q} K\|D^*(Sx_0, Sx_1, Sx_1)\| \to 0 \text{ as } m, n \to \infty.$$

since $0 \leq q < 1$. So $\{Sx_n\}$ is D^*-Cauchy sequence. By the completeness of $S(X)$, there exists $u \in S(X)$ such that $\{Sx_n\}$ is D^*-convergent to u. Then there is $p \in X$, such that $Sp = u$. If $T(X)$ is complete, then there exist $u \in T(X)$ such that $Sx_n \to u$, as $T(X) \subset S(X)$, we have $u \in S(X)$.Then there exist $p \in X$ such that $Sp = u$. We claim that $Tp = u$,

$$\begin{aligned}
D^*(Tp,Tp,u) &\leq D^*(Tp,Tp,Tx_n) + D^*(Tx_n,u,u) \\
&\leq a[D^*(Sp,Tp,Tp) + D^*(Sp,Tp,Tp)] \\
&\quad + b[D^*(Sp,Tx_n,Tx_n) + D^*(Sx_n,Tp,Tp)] \\
&\quad + c[D^*(Sp,Tx_n,Tx_n) + D^*(Sx_n,Tp,Tp)] + D^*(Tx_n,u,u) \\
&\leq a[D^*(u,Tp,Tp) + D^*(u,Tp,Tu))] \\
&\quad + b[D^*(u,Sx_{n+1},Sx_{n+1},Sx_{n+1}) + D^*(Tp,Tp,u) + D^*(u,Sx_n,Sx_n)] \\
&\quad + c[D^*(u,Sx_{n+1},Sx_{n+1}) + D^*(Tp,Tp,u) + D^*(u,Sx_n,Sx_n)] \\
&\quad + D^*(Sx_{n+1},u,u)
\end{aligned}$$

This implies that

$$\begin{aligned}
D^*(Tp,Tp,u) \leq \frac{1}{1-(2a+b+c)}\big\{&(b+c)[D^*(u,Sx_{n+1},Sx_{n+1}) + D^*(u,Sx_n,Sx_n)] \\
&+ D^*(Sx_{n+1},u,u)\big\}
\end{aligned}$$

from (1.1)

$$\begin{aligned}
\|D^*(Tp,Tp,u)\| \leq K\frac{1}{1-(2a+b+c)}\big\{&(b+c)[\|D^*(u,Sx_{n+1},Sx_{n+1})\| \\
&+ \|D^*(u,Sx_n,Sx_n)\|] + \|D^*(Sx_{n+1},u,u)\|\big\}
\end{aligned}$$

the right hand side approaches to zero as $n \to \infty$. Hence $\|D^*(Tp,Tp,u)\| = 0$ and $Tp = u$. Hence $Tp = Sp$ and p is a point of coincidence point of S and T. Now we show that S and T have a unique point of coincidence. For this, assume that there exists a point q in$X$ such that $Sq = Tq$. Now

$$\begin{aligned}
D^*(Tp,Tp,Tq) &\leq a[D^*(Sp,Tp,Tp) + D^*(Sp,Tp,Tp)] \\
&\quad + b[D^*(Sp,Tq,Tq) + D^*(Sq,Tp,Tp)] \\
&\quad + c[D^*(Sp,Tq,Tq) + D^*(Sq,Tp,Tp)] \\
&= a[D^*(Tp,Tp,Tp) + D^*(Tp,Tp,Tp)] \\
&\quad + b[D^*(Tp,Tq,Tq) + D^*(Tq,Tp,Tp)] \\
&\quad + c[D^*(Tp,Tq,Tq) + D^*(Tq,Tp,Tp)] \\
&= b[D^*(Tp,Tp,Tq) + D^*(Tp,Tp,Tq)] \\
&\quad + c[D^*(Tp,Tp,Tq) + D^*(Tp,Tp,Tq)] \\
&= (2b+2c)D^*(Tp,Tp,Tq)
\end{aligned}$$

$$D^*(Tp,Tp,Tq) \leq (2b+2c)D^*(Tp,Tp,Tq).$$

This implies $((2b+2c)-1)D^*(Tp,Tp,Tq) \in P$ and $((2b+2c)-1)D^*(Tp,Tp,Tq) \in -P$, since $(2b+2c)-1 < 0$. As $P \cap -P = \{0\}$, we have $((2b+2c)-1)D^*(Tp,Tp,Tq) = 0$, i.e. $D^*(Tp,Tp,Tq) = 0$. Hence $Tp = Tq$. Also $Sp = Sq$, since $Tp = Sp$. Hence p is a unique point of coincidence of S and T. By Proposition 1.15, p is a unique common fixed point of S and T in X. □

Corollary 2.4. *Let (X, D^*) be a complete generalized D^*-metric space,P be a normal cone with normal constant K and let $T : X \to X$ be a mappings satisfy the condition*

$$\begin{aligned} D^*(Tx, Ty, Tz) \leq\ & a[D^*(x, Ty, Ty) + D^*(y, Tx, Tx)] \\ & + b[D^*(y, Tz, Tz) + D^*(z, Ty, Ty)] \\ & + c[D^*(x, Tz, Tz) + D^*(z, Tx, Tx)] \end{aligned}$$

for all $x, y, z \in X$, where $a, b, c \geq 0, 2a+2b+2c < 1$. Then T has a unique fixed point in X.

Proof. The proof follows from Theorem 2.3 by replacing S by identity mapping.□

Theorem 2.5. *Let (X, D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K and let $S, T : X \to X$ be two mappings which satisfy, the following conditions*
(i) $T(X) \subset S(X)$,
(ii) $T(X)$ or $S(X)$ is complete,
(iii)

$$\begin{aligned} D^*(Tx, Ty, Ty) \leq\ & a[D^*(Sy, Ty, Ty) + D^*(Sx, Ty, Ty)] \\ & + bD^*(Sy, Tx, Tx) \end{aligned} \tag{2.6}$$

for all $x, y, z \in X$, where $a, b \geq 0, 3a + b < 1$, Then S and T have a unique point of coincidence in X. Moreover if S and T are weakly compatible, then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary, there exist $x_1 \in X$ such that $Tx_0 = Sx_1$, in this way we have sequence $\{Tx_n\}$ with $Tx_n = Sx_{n+1}$. Then from inequality (2.6), we have

$$\begin{aligned} D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) &= D^*(Tx_{n-1}, Tx_n, Tx_n) \\ &\leq a[D^*(Sx_n, Tx_n, Tx_n) + D^*(Sx_{n-1}, Tx_n, Tx_n)] \\ &\quad + D^*(Sx_n, Tx_{n-1}, Tx_{n-1}) \\ &= a[D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_{n-1}, Sx_{n+1}, Sx_{n+1})] \\ &\quad + bD^*(Sx_n, Sx_n, Sx_n) \\ &\leq a[D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) + D^*(Sx_{n-1}, Sx_n, Sx_n) \\ &\quad + D^*(Sx_n, Sx_{n+1}, Sx_{n+1})] \end{aligned}$$

This implies that

$$D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq rD^*(Sx_{n-1}, Sx_n, Sx_n) \tag{2.7}$$

where $r = \frac{a}{1-2a}$, then $0 \leq r < 1$. Then repeating application of (2.7), we get

$$D^*(Sx_n, Sx_{n+1}, Sx_{n+1}) \leq r^n D^*(Sx_0, Sx_1, Sx_1)$$

Then, for all $n, m \in N, n > m$ we have, by repeated use of the rectangle inequality,

$$\begin{aligned} D^*(Sx_m, Sx_n, Sx_n) &\leq D^*(Sx_m, Sx_{m+1}, Sx_{m+1}) + D^*(Sx_{m+1}, Sx_{m+2}, Sx_{m+2}) \\ &\quad + \cdots + D^*(Sx_{n-1}, Sx_n, Sx_n) \\ &\leq (r^m + r^{m+1} + \cdots + r^{n-1}) D^*(Sx_0, Sx_1, Sx_1) \\ &\leq \frac{r^m}{1-r} D^*(Sx_0, Sx_1, Sx_1). \end{aligned}$$

From (1.1)

$$\|D^*(Sx_m, Sx_n, Sx_n)\| \leq \frac{r^m}{1-r} K \|D^*(Sx_0, Sx_1, Sx_1)\| \to 0 \quad \text{as } m, n \to \infty.$$

since $0 \leq r < 1$. So $\{Sx_n\}$ is D^*-Cauchy sequence. By the completeness of $S(X)$, there exists $u \in S(X)$ such that $\{Sx_n\}$ is D^*-convergent to u. Then there is $p \in X$, such that $Sp = u$. If $T(X)$ is complete, then there exist $u \in T(X)$ such that $Sx_n \to u$, as $T(X) \subset S(X)$, we have $u \in S(X)$.Then there exists $p \in X$ such that $Sp = u$. We claim that $Tp = u$,

$$\begin{aligned} D^*(Tp, Tu, u) &\leq D^*(Tp, Tp, Sx_{n-1}) + D^*(Sx_{n-1}, u, u) \\ &\leq a[D^*(Sp, Tp, Tp) + D^*(Sp, Tp, Tp)] \\ &\quad + bD^*(Sp, Tp, Tp) + D^*(Sx_{n-1}, u, u) \\ &= a[D^*(u, Tp, Tp) + D^*(u, Tp, Tp)] \\ &\quad + bD^*(u, Tp, Tp) + D^*(Sx_{n-1}, u, u) \\ &= a[D^*(Tp, Tp, u) + D^*(Tp, Tp, u)] \\ &\quad + bD^*(Tp, Tp, u) + D^*(Sx_{n-1}, u, u) \end{aligned}$$

This implies that

$$D^*(Tp, u, u) \leq \frac{1}{1-(2a+b)} D^*(Sx_{n-1}, u, u)$$

from (1.1)

$$\|D^*(Tp, u, u)\| \leq K \frac{1}{1-(2a+b)} \|D^*(Sx_{n-1}, u, u)\|$$

right hand side approaches to zero as $n \to \infty$. Hence $\|G^*(Tp, Tp, u)\| = 0$ and $Tp = u$ and $Sp = Tp$ i.e. p is a coincidence point of S and T. Now we show that S and T have a unique point of coincidence. For this, assume that there exists a point q in$X$ such that $Sq = Tq$. Now

$$\begin{aligned} D^*(Tp, Tq, Tq) &\leq a[D^*(Sq, Tq, Tq) + D^*(Sp, Tq, Tq)] + bD^*(Sq, Tp, Tp) \\ &= a[D^*(Tq, Tq, Tq) + D^*(Tp, Tq, Tq)] + bD^*(Tq, Tp, Tp)] \\ &= aD^*(Tp, Tq, Tq) + bD^*(Tp, Tq, Tq) = (a+b)D^*(Tp, Tq, Tq). \end{aligned}$$

This implies $((a+b)-1)D^*(Tp, Tq, Tq) \in P$ and $((a+b)-1)D^*(Tp, Tq, Tq) \in -P$, since $D^*(Tp, Tq, Tq) \in P$ and $(a+b)-1 < 0$. As $P \cap -P = \{0\}$, we have $((a+b)-1)D^*(Tp, Tq, Tq) = 0$, i.e. $D^*(Tp, Tq, Tq) = 0$. Hence $Tp = Tq$. Also $Sp = Sq$, since $Sp = Tp$. Hence p is a unique point of coincidence of S and T. By Proposition 1.15, p is a unique common fixed point of S and T in X. □

Corollary 2.6. *Let (X, D^*) be a complete generalized D^*-metric space, P be a normal cone with normal constant K and let $T : X \to X$ be a mapping which satisfies the following condition,*

$$\begin{aligned} D^*(Tx,Ty,Ty) \leq a[D^*(y,Ty,Ty) + D^*(x,Ty,Ty)] \\ + bD^*(y,Tx,Tx)] \end{aligned} \tag{2.8}$$

for all $x, y \in X$, where $a, b \geq 0, 3a + b < 1$, Then T has a unique fixed point in X.

Proof. The proof follows from the previous theorem and the same argument used in corollary 2.4. □

Theorem 2.7. *Let (X, D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K and let $S, T : X \to X$, be two mappings which satisfy the following conditions*
(i) *$T(X) \subset S(X)$,*
(ii) $T(X)$ or $S(X)$ is complete,
(iii)

$$\begin{aligned} D^*(Tx,Ty,Tz) \leq a[D^*(Sz,Tx,Tx) + D^*(Sy,Tx,Tx)] \\ + b[D^*(Sy,Tz,Tz) + D^*(Sx,Tz,Tz)] \\ + c[D^*(Sx,Ty,Ty) + D^*(Sz,Ty,Ty)] \end{aligned} \tag{2.9}$$

for all $x, y, z \in X$, where $a, b, c \geq 0, 3a + 2b + 3c < 1$, Then S and T have a unique point of coincidence in X. Moreover if S and T are weakly compatible, then S and T have a unique common fixed point in X.

Proof. Setting $z = y$ in condition (2.9), reduces it to condition (2.6), and the proof follows from Theorem 2.5. □

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Authors' addresses:

C. T. Aage and J. N. Salunke
School of Mathematical Sciences,
North Maharashtra University, Jalgaon, India.
E-mail: caage17@gmail.com.