Solutions of the central Woods-Saxon potential in $l \neq 0$ case using mathematical modification method

M. R. Pahlavani, J. Sadeghi and M. Ghezelbash

Abstract. In this study the radial part of the Schrödinger equation in presence of the angular momentum ($l \neq 0$) has been solved for the generalized Woods-Saxon potential by using the modification method. This approach is based on the definition of a modified Woods-Saxon potential which is selected that the associated Schrödinger differential equation become comparable with the associated Jacobi differential equation. By using this method, we obtain exactly bound states spectrum and wave function of the generalized Woods-Saxon potential for nonzero angular momentum case.

M.S.C. 2000: 81Q05, 81V35, 83A05.
Key words: Schrödinger equation, angular momentum, Woods-Saxon potential, bound states, special functions.

1 Introduction

Woods and saxon introduced a potential to study elastic scattering of 20 MeV protons by a heavy nuclei [38]. The Woods-saxon potential is a reasonable potential for nuclear shell model and hence attracts lots of attention in nuclear physics [1, 17, 7, 9, 33, 18, 14, 24, 26, 20, 10]. The Woods-Saxon potential plays an essential role in microscopic physics, since it can be used to describe the interaction of a nucleon with the heavy nucleus. Although the non-relativistic Schrödinger equation with this potential has been solved for ground state [16] and the single particle motion in atomic nuclei has been explain quite well, the relativistic effects for a particle under the action of this potential are more important, especially for a strong-coupling system. The Schrödinger equation have been solved for three-body system using adiabatic expansion [28] and cylindrically symmetric static space time [34] used to make differential equation integrable. The relativistic Coulomb and oscillator potential problems, including their bound-state spectra and wave functions, have already been established for a long time [36, 11, 6, 31, 37, 30], and their non-relativistic limits reproduce the usual Schrödinger-Coulomb and Schrödinger-oscillator solutions, respectively.
The behavior of valance electrons are very important to understand the abundance of metallic clusters. Thus, a good description of the motion of these valance electrons

Applied Sciences, Vol.11, 2009, pp. 106-113.

are very useful to study metallic systems. The Woods-Saxon potential is utilized to represent the mean field which is felt by valance electron in Helium model [12]. It is also used in a nonlinear theory of scalar mesons [14]. In addition to these, the three-dimensional Woods-Saxon potential is studied within the context of Supersymmetric Quantum mechanics [22].
The spherical Woods-Saxon potential that was used as a major part of nuclear shell model, was successful to deduce the nuclear energy levels [19]. Also it was used as central part for the interaction of neutron with heavy nucleus [27]. With the help of the axially-deformed Woods-Saxon potential along with the spin-orbit interaction hamiltonian , it is possible to construct the structure of single-particle shell model [25]. The Woods-Saxon potential was used as a part of optical model in elastic scattering of some ions with heavy target at low energies [7]. Recently, A. Calogeracos and his co-workers [8] have been developed a generalized well known theorem of non relativistic scattering in one dimensional potential well in the base of Schrödinger equation to apply in to the Dirac equation. C. Roja and V.M. Villalba [29] developed an approach to obtain the bound states solutions of the one-dimensional Dirac equation for Woods-Saxon potential. For the case of three dimensions, Alhaidari has developed a new two-component approach to Dirac equation for the spherically symmetric potential, and solved a class of shape-invariant potentials that includes Dirac-Morse, Dirac-Rosen-Morse, Dirac-Eckart, Dirac-Scarf, potentials, and obtain their relativistic bound states spectra and related spinors [2, 3, 4, 5].
The Schrödinger equation for the Woods-Saxon potential is exactly solvable for $l = 0$ [13]. Since the woods-saxon potential can not be solved analytically for $l \neq 0$, one often adopts the harmonic oscillator potential or the square well in nuclear shell model for both spherical [21] and deformed nuclei [35] as a good approximation. As an initial approximation, the harmonic type potential used to construct the nuclear interaction hamiltonian. However it is necessary to improve the asymptotic behavior of harmonic oscillator wave function by performing a local scaling transformation [32]. To obtain exact strongly bound level of nucleus, it is necessary to add the Woods-Saxon term $l \neq 0$. The aim of our study is to analyze solutions of Schrödinger equation for the modified form of generalized Woods-Saxon potential with the angular momentum $l \neq 0$.

2 The modified form of Wood-Saxon potential

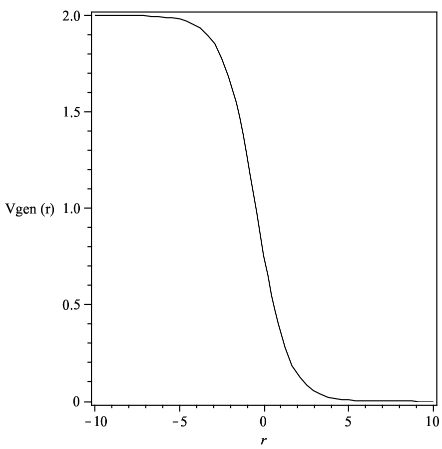
The generalized Woods-Saxon potential can be introduce by the following relation,

$$V_{gen}(r) = \frac{\upsilon_0}{1 + e^{\frac{r-R_0}{a}}} + \frac{\tau}{(1 + e^{\frac{r-R_0}{a}})^2};, \tag{2.1}$$

where $R_0 = r_0 A^{\frac{1}{3}}$ is the radius of the corresponding nuclei with R_0 as a constant and A the mass number of the nucleus. υ_0 is the potential depth and a is a constant that usually adjusted to the experimental value of nuclear interaction barrier and τ is a constant parameter.
In order to modify this potential for $l \neq 0$, it is necessary to add two terms to generalized potential, so we have a modified potential as follows,

$$V_{mod}(r) = \frac{\upsilon_0}{1+e^{\frac{r-R_0}{a}}} + \frac{\tau}{(1+e^{\frac{r-R_0}{a}})^2} + \mu \coth(\frac{r-R_0}{a}) + \eta \coth^2(\frac{r-R_0}{a}); , \tag{2.2}$$

the parameters υ_o, τ, μ and η are real constant values which we will compute them in the next section. It is required to remind the third and forth terms in equation (2.2), in limit $r - R_0 \ll a$ reformed as $\frac{1}{r}$ and $\frac{1}{r^2}$ respectively. These forms are corresponding to the coulombian repulsive potential and it's square. In section 3, by using the associated Jacobi differential equation, we solve analytically the radial part of time-independent Schrödinger equation with angular momentum $l \neq 0$, for this modified shape of generalized Woods-Saxon potential. In Figs. 1 and 2 the generalized and modified Woods-Saxon potential plotted as a function of r and compared with together.



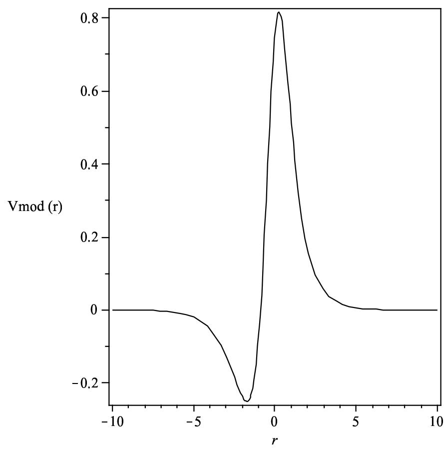


Fig.1. Generalized Woods-Saxon Potential **Fig.2.** Modified Woods-Saxon Potential

3 Solution of modified Woods-Saxon potential

By considering a new parameter as $r \equiv r - R_0$, the time-independent Shrödinger equation for the modified form of generalized spherical Woods-Saxon potential in presence of angular momentum can be written as,

$$\begin{aligned} & -\frac{h^2}{2m}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{L^2}{\hbar^2 r^2}\right)\psi_{n,l}(r) \\ & + \left(\frac{\upsilon_0}{1+e^{\frac{r}{a}}} + \frac{\tau}{(1+e^{\frac{r}{a}})^2} + \mu \coth(\frac{r}{a}) + \eta \coth^2(\frac{r}{a})\right)\psi_{n,l}(r) = E_{n,l}\psi_{n,l}(R);. \end{aligned} \tag{3.1}$$

We compute the parameters υ_0, τ, μ, η and $E_{n,l}$ and also the bound states $\psi_{n,l}(r)$ by comparing this differential equation with the standard associated Jacobi differential equation. So we extend the Woods-Saxon potential to obtain variable r as,

$$r = a(e^{\frac{r}{a}} - 1);, \tag{3.2}$$

this approximation valid for $r \to 0$ in the nuclear size region. By considering wave function as,

$$\psi_{n,l}(r) = \frac{1}{r}\varphi_{n,l}(r);, \tag{3.3}$$

and defining following new parameters $\varepsilon = \frac{8ma^2 E_{n,l}}{\hbar^2}$, $\gamma = \frac{4ma^2\upsilon_0}{\hbar^2}$, $\delta = \frac{2ma^2\tau}{\hbar^2}$, $\rho = \frac{8ma^2\mu}{\hbar^2}$ and $\varrho = \frac{8ma^2\eta}{\hbar^2}$ also variable $x = \tanh(\frac{r}{2a})$, the equation (3.1) can be reduce to,

$$\begin{aligned} &\varphi''_{n,l}(x) - 2x\varphi'_{n,l}(x) + \left(\frac{E_{nl}}{1-x^2} - \frac{\gamma}{1+x} - \frac{\delta(1-x)}{1+x}\right)\varphi_{n,l}(x) - \\ &\left(\frac{\rho}{x^2(1-x^2)} + \frac{\varrho}{x(1-x^2)} + \frac{l(l+1)(1-x)}{x^2(1+x)}\right)\varphi_{n,l}(x) = 0;. \end{aligned} \tag{3.4}$$

The well known associated Jacobi differential equation For real parameters $\alpha, \beta < -1$, and in the interval $x \in (-1, 1)$ can be shown by the following relation [32, 23],

$$\begin{aligned} &(1-x^2)P''^{(\alpha,\beta)}_{n,l}(x) + (\beta - \alpha - (\alpha+\beta+2)x)\, P'^{(\alpha,\beta)}_{n,l}(x) + \\ &\left(n(\alpha+\beta+n+1) - \frac{l(\alpha+\beta+l+(\alpha-\beta)x)}{1-x^2}\right) P^{(\alpha,\beta)}_{n,l}(x) = 0;, \end{aligned} \tag{3.5}$$

where the indices n and l are non-negative integers define in the interval $0 \leq l \leq n < \infty$. The associated Jacobi function $P^{(\alpha,\beta)}_{n,l}(x)$ is the solutions of the differential equation (3.5) and have the following Rodrigues representation,

$$P^{(\alpha,\beta)}_{n,l}(x) = \frac{a_{n,l}(\alpha,\beta)}{(1-x)^{\alpha+\frac{l}{2}}(1+x)^{\beta+\frac{l}{2}}}\left(\frac{d}{dx}\right)^{n-l}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right);. \tag{3.6}$$

Here $a_{n,l}(\alpha,\beta)$ is the normalization coefficient that can be calculate through the following relation for $n \geq l$,

$$a_{n,l}(\alpha,\beta) = \frac{(-1)^l}{2^n}\sqrt{\frac{\Gamma(\alpha+\beta+n+l+1)}{\Gamma(n-l+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}C(\alpha,\beta);, \tag{3.7}$$

in which $C(\alpha,\beta)$ is an arbitrary real constant independent of n and l.
Here we define the $\varphi_{n,l}(x)$ as product of two functions,

$$\varphi_{n,l}(x) = \nu(x)\omega(x);. \tag{3.8}$$

Now we substitute this definition in differential equation (3.4) in order to obtain the following differential equation,

$$\begin{aligned}
&(1-x^2)\nu''(x) + \left(2(1-x^2)\frac{\omega'(x)}{\omega(x)} - 2x\right)\nu'(x) + \\
&\left((1-x^2)\frac{\omega''(x)}{\omega(x)} - 2x\frac{\omega'(x)}{\omega(x)} + \frac{\varepsilon}{1-x^2}\right)\nu(x) - \\
&\left(\frac{\gamma}{1+x} + \delta\frac{1-x}{1+x} + \frac{\rho}{x^2(1-x^2)} + \frac{\varrho}{x(1-x^2)} + \frac{l(l+1)}{x^2}\frac{1-x}{1+x}\right)\nu(x) = 0;.
\end{aligned} \tag{3.9}$$

By comparing equations (3.10) and (3.6), we conclude that $\nu(x)$ correspond to the associated Jacobi function $P_{n,l}^{(\alpha,\beta)}(x)$ and function $\omega(x)$ can be obtain as,

$$\omega(x) = C(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\beta}{2}};, \tag{3.10}$$

here C is the normalization coefficient.
Meanwhile the further comparison between the following equations lead us to compute the parameters ε, γ, δ, ρ and ϱ as,

$$\begin{aligned}
&\varepsilon = l(l+1) - (\alpha+l)^2, \\
&\gamma = n(\alpha+\beta+n+1) - 2(\frac{\beta^2}{4} - \frac{\beta}{2}) - \frac{\alpha-\beta}{2} + l(\alpha+\beta) - \frac{\alpha\beta}{2}, \\
&\delta = -n(\alpha+\beta+n+1) + (\frac{\beta^2}{4} - \frac{\beta}{2}) + (\frac{\alpha^2}{4} - \frac{\alpha}{2}) + \frac{\alpha\beta}{2} + \alpha + \beta, \\
&\rho = \frac{l(l+1)}{2};, \\
&\varrho = l(l+1),
\end{aligned} \tag{3.11}$$

by using the following equations, the parameters υ_0, τ, μ and η in equation (3.1) can be determined from relations $\upsilon_0 = \frac{\hbar^2}{4ma^2}\gamma$, $\tau = \frac{\hbar^2}{2ma^2}\delta$, $\mu = \frac{\hbar^2}{8ma^2}\rho$ and $\eta = \frac{\hbar^2}{4ma^2}\varrho$. Also we obtain the negative energy levels from relation between $E_{n,l}$ and ε as,

$$E_{n,l} = -\frac{\hbar^2}{8ma^2}\left[(\alpha+l)^2 - l(l+1)\right];. \tag{3.12}$$

and the negative energy spectrum can be written as,

$$E_{n,l} = -\frac{\hbar^2}{8ma^2}(\alpha+l)^2;, \tag{3.13}$$

in the limit of zero angular momentum, $l = 0$ we have,

$$E_{n,l} = -\frac{\hbar^2 \alpha^2}{8ma^2}; . \tag{3.14}$$

This relation for negative energy spectrum exactly match with the results of supersymmetry approach [15].
Finally by using equations (3.3), (3.8) and (3.10) the corresponding bound states for these levels can be written as,

$$\psi_{n,l}(r) = \frac{C}{r}\left(1 - \tanh\frac{r}{2a}\right)^{\frac{\alpha}{2}}\left(1 + \tanh\frac{r}{2a}\right)^{\frac{\beta}{2}} P_{n,l}^{(\alpha,\beta)}\left(\tanh\frac{r}{2a}\right); , \tag{3.15}$$

here coefficient C is determined from normalization condition.

4 Conclusions

This research evident that by adding two terms to the generalized Woods-Saxon potential, it is possible to obtain the solutions of the modified shape of this potential for case $l \neq 0$. Then by using the associated Jacobi differential equation, the bound states and also the corresponding wave function can be determined. The obtained results agree with the results calculated in paper [15] in special case $l = 0$. The results can extend for generalized nuclear potential which correspond to modify nucleus in the case of relativistic theory. Also this simple method can be applied for other complicated central potentials.

References

[1] K. Abe & al., *Measurements of the proton and deuteron spin structure function g2 and asymmetry A2*, Phys. Rev. Lett. 76 (1996), 587 - 591.

[2] A. D. Alhaidari, *Solution of the relativistic Dirac-Morse Problem*, Phys. Rev. Lett. 87 (2002), 210405-210408.

[3] A. D. Alhaidari, *Solution of the Dirac equation for potential interaction*, Int. J. Mod. Phys. A 18 (2003), 4955-4964.

[4] A. D. Alhaidari, *Exact L2 series solution of the Dirac-Coulomb problem for all energies*, Ann. Phys. 312(1) (2004), 144-160.

[5] A. D. Alhaidari, *Solution of the relativistic Dirac-Hulthen problem*, J. Phys. A: Math. Gen. 37 (2004), 58055813.

[6] J. Bentez, R. P. Martnez, y. Romero, H.N. Nuez Yepez and A. L. Salas-Brito, *Solution and hidden supersymmetry of a Dirac oscillator*, Phys. Rev. lett. 64 (1990), 1643-1645.

[7] V. Bespalova, E. A. Romanovsky, T. I. Spasskaya, *Nucleonnucleus real potential of WoodsSaxon shape between -60 and +60 MeV for the $40 \leq A \leq 208$ nuclei*, J. Phys. G: Nucl. Part. Phys. 29(6) (2003), 1193-1211.

[8] Alex Calogeracos and Norman Dombey, *Strong Levinson theorem for the Dirac equation*, Phys. Rev. Lett. 93 (2004), 180405-180409.

[9] M. Dasgupta & al., *Effect of breakup on the fusion of ^6Li, ^7Li and ^7Be with heavy nuclei*, Phys. Rev. C 70 (2004), 024606-024625.

[10] A. Diaz-Torres and W. Scheid, *Two center shell model with WoodsSaxon potentials: Adiabatic and diabatic states in fusion*, Nucl. Phys. A 757 (2005), 373-389.

[11] S.-H. Dong and Zhong-Qi Ma, *Exact solutions to the Dirac equation with a Coulomb potential in 2+1 dimensions*, Phys. Lett. A 312 (2003), 78-83.

[12] J. Dudek, K. Pomorski, N. Schunck, and N. Dubray, *Hyperdeformed and megadeformed nuclei*, Eur. Phys. J. A 20 (2004), 15-29.

[13] I. H. Duru, *Path integrals over the SU(2) manifold and related potentials*, Phys. Rev. D 30 (1984), 2121-2127.

[14] H. Erkol, E. Demiralp, *The WoodsSaxon potential with point interactions*, Phys. Lett. A 365 (2007), 55-63.

[15] H. Fakhri, J. Sadeghi, *Supersymmetry approaches to the bound states of the generalized Woods-Saxon potential*, Mod. Phys. Lett. A 19 (2004), 615-625.

[16] S. Flügge, *Practical Quantum Mechanics*, Springer-Verlag, Berlin, 1974.

[17] F. Garcia & al., *Woods-Saxon potential parametrization at large deformations for plutonium odd isotopes*, Eur. Phys. J. A. 6(1) (1999), 49-58.

[18] V. Z. Goldberg & al., *Low-lying levels in ^{15}F and the shell model potential for drip-line nuclei*, Phys. Rev. C 69(3) (2004), 031302-031306.

[19] J. M. G. Gômez, K. Kar, V.K.B. Kota, R.A. Molina and J. Retamosa, *Localization in 2p1f nuclear shell-model wave functions*, Phys. Lett. B 567 (2003), 251-258.

[20] J. Y. Guo and Q.Sheng, *Solution of the Dirac equation for the Woods-Saxon potential with spin and pseudospin symmetry*, Phys. Lett. A 338(2) (2005), 90-96.

[21] I. Hamamoto, *Nilsson diagrams for light neutron-rich nuclei with weakly-bound neutrons*, Phys. Rev. C 76 (2007), 054319-054324.

[22] J. Lin Hwang, *Some geometrical properties of spherical shells implying the systematics of nuclear radii*, Chin. J. Phys. 2(1) (1964), 84-89.

[23] M. A. Jafarizadeh, H. Panahi-Talemi and E. Faizi, *Two- and three-dimensional hamiltonian with shape invariance symmetry*, J. Math. Phys. 42 (2001), 4122-4137.

[24] K. Khounfais, T. Boudjedaa and L. Chetouani, *Scattering matrix for Feshbach-Villars equation for spin 0 and 1/2: Woods-Saxon potential*, Czech. J. Phys. 54 (2004), 697-792.

[25] M. Mirea, *Superasymmetric two-center shell model for spontaneous heavy-ion emission.*, Phys. Rev. C 54 (1996), 302-314.

[26] J. O. Newton & all., *Systematic failure of the Woods-Saxon nuclear potential to describe both fusion and elastic scattering: Possible need for a new dynamical approach to fusion*, Phys. Rev. C 70(2) (2004), 024605-024619.

[27] H. Nicolai, *Supersymmetry and spin systems*, J. Phys. A: Math. Gen. 9 (1976), 1497-1506.

[28] M.R. Pahlavani and S.M. Motevalli,*Adiabatic expansion approximation solutions for the three-body problem* , Applied Sciences (APPS), 10 (2008), 199-206.

[29] C. Rojas, V.M. Villalba , *Scattering of a Klein-Gordon particle by a Woods-Saxon potential*, Phys. Rev. A 71 (2005), 052101-052105.

[30] P. Rozmej and R. Arvieu, *The Dirac oscillator. A relativistic version of the Jaynes-Cummings model*, J. Phys. A 32 (1999), 5367-5382.

[31] K. Saaidi, *Eigenvalue spectrum of a Dirac particle in static and spherical complex potential*, Int. J. The. Phys. 43 (2004), 959-968.

[32] J. Sadeghi, *Factorization method and solution of the non-central modified Kratzer potential*, Acta Physica Polonica A 112 (2007), 23-28.

[33] J. Sadeghi, M. R. Pahlavani, *The hierachy of Hamiltonian for spherical Woods-Saxon potential*, Afr. J. Math. Phys. 1(2) (2004), 195-199.

[34] G. Shabbir and M. Ramzan *Classification of cylindrically symmetric static space-times according to their proper homothetic vector fields*, Applied Sciences (APPS), 9 (2007), 148-154.

[35] M. Stoitsov, P. Ring, D. Vretenar, and G. A. Lalazissis, *Solution of relativistic Hartree-Bogoliubov equations in configurational representation: spherical neutron halo nuclei*, Phys. Rev. C 58 (1998), 2086-2091.

[36] N. V.V.J. Swamy, *Exact solution of the Dirac equation with an equivalent oscillator potential*, Phys. Rev. 180 (1969), 1225-1226.

[37] V. M. Villalba, *Exact solution of the two-dimensional Dirac oscillator*, Phys. Rev. A 49 (1994), 586-587.

[38] R. D. Woods, D. S. Saxon, *Diffuse surface optical model for nucleon-nuclei scattering*, Phys. Rev. 95 (1954), 577-578.

Authors' addresses:

M. R. Pahlavani, J. Sadeghi and M. Ghezelbash
Department of Physics, Faculty of Science, Mazandaran University, P.O.Box 47415-416, Fax.0098-11252-42001, Babolsar, Iran
E-mails: m.pahlavani@umz.ac.ir, pouriya@umz.ac.ir and mohsenarmani@gmail.com