The CC-version of Stewart's Theorem

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Abstract. E. F. Krause [8] asked the question of how to develop a metric which would be similar to the movement made by playing Chinese Checker. Latter G. Chen [1] developed the Chinese Checker metric for plane. In this work, we give a CC-analog of the Theorem of Stewart and the median property.

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1 Introduction

The CC-plane geometry is a Minkowski geometry of dimension two with the distance function

$$d_c(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \left(\sqrt{2} - 1\right) \min\{|x_1 - x_2|, |y_1 - y_2|\}$$

where $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$. That is, Chinese Checker plane $\mathbb{R}_c^2$ is almost the same as the Euclidean analytical plane $\mathbb{R}^2$. The points and lines are the same, and angles are measured in the same way. However, distance function is different. According to definition of d_c-distance the shortest path between the points P_1 and P_2 is the union of a vertical or horizontal line segment and a line segment with the slope 1 or -1. CC-analogues of some of the topics that include the concept of CC-distance have been studied by some authors [1], [2], [3], [11], [12] and [5] which generalizes the CC-distance; the group of isometries of the CC-plane has been given in [6]; and two different CC-analogues of the Pythagoras' theorem have been introduced in [4]. In this work, we give the CC-versions of Stewart's theorem and the median property.

2 Preliminaries

Let $\mathbb{R}_c^2$ denote the CC-plane. The following proposition and corollaries give some results of $\mathbb{R}_c^2$.

Proposition 2.1. *Every Euclidean translation is an isometry of* $\mathbb{R}_c^2$.

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Corollary 2.1. *If A, B, X are any three collinear points in the analytical plane $\mathbb{R}^2$, then*

$$d(X, A) = d(X, B) \text{ if and only if } d_c(X, A) = d_c(X, B)$$

where d stands for the Euclidean distance.

Corollary 2.2. *If A, B and X are any three distinct collinear points in $\mathbb{R}_c^2$, then* $d_c(X, A)/d_c(X, B) = d(X, A)/d(X, B)$.

Proof of the above assertions are given in [6].

We need the following definitions given in [7] and [9]:

Let ABC be any triangle in $\mathbb{R}_c^2$. Clearly, there exists a pair of lines passing through every vertex of the triangle, each of which is parallel to a coordinate axis. A line l is called a *base line* of ABC iff

1) l passes through a vertex,

2) l is parallel to a coordinate axis,

3) l intersects the opposite side (as a line segment) to the vertex in Condition 1. Clearly, at least one of the vertices of a triangle always has one or two base lines. Such a vertex of a triangle is called a *basic vertex.* A *base segment* is a line segment on a base line, which is bounded by a basic vertex and its opposite side.

Finally, we consider the following separation of $\mathbb{R}_c^2$ to eight regions S_i (i=0, 1, . . . , 7) such that

$$\begin{aligned}
S_0 &= \{(x, y)|x \geq y \geq 0\} \\
S_1 &= \{(x, y)|y \geq x \geq 0\} \\
S_2 &= \{(x, y)|y \geq |x| \geq 0 \ , \ x < 0\} \\
S_3 &= \{(x, y)|\,|x| \geq y \geq 0 \ , \ x < 0\} \\
S_4 &= \{(x, y)|x \leq y \leq 0\} \\
S_5 &= \{(x, y)|y \leq x \leq 0\} \\
S_6 &= \{(x, y)|\,|y| \geq x \geq 0 \ , \ y < 0\} \\
S_7 &= \{(x, y)|x \geq |y| \geq 0 \ , \ y < 0\}
\end{aligned}$$

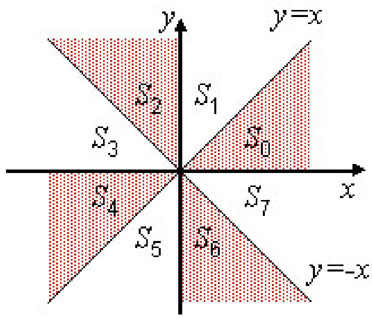

Figure 1

as shown in Figure 1. In what follows S_{i+1}, S_{i+2}, S_{i+3} and S_{i+4} stand for $S_{i+1(\text{mod } 8)}$, $S_{i+2(\text{mod } 8)}$, $S_{i+3(\text{mod } 8)}$ and $S_{i+4(\text{mod } 8)}$, respectively.

3 A CC-Version of the Stewart's Theorem

It is known for any triangle ABC in the Euclidean plane that if $X \in BC$ and $a = d(B, C)$, $b = d(A, C)$, $c = d(A, B)$, $p = d(B, X)$, $q = d(C, X)$, $x = d(A, X)$, then

$$x^2 = \frac{b^2 p + c^2 q}{p + q} - pq$$

which is known as Stewart's theorem.

The next theorem gives a CC-version of the Stewart's theorem.

Theorem 3.1. *Let the sides of a triangle ABC in the $\mathbb{R}_c^2$ have lengths* $\mathbf{a}$=$d_c(B,C)$, $\mathbf{b}$=$d_c(A,C)$ *and* $\mathbf{c}$=$d_c(A,B)$. *If* $X \in BC$ *and* $\mathbf{p}$=$d_c(B,X)$, $\mathbf{q}$=$d_c(C,X)$ *and* $\mathbf{x}$=$d_c(A,X)$, *then* $\mathbf{x}=(\mathbf{bp}+\mathbf{qc}\text{-}\Delta)/(\mathbf{p}+\mathbf{q})$, *where* Δ *is as in the following table:*

Δ	Number of base lines	AB lies in	AX lies in	AC lies in	
0	0	S_i	S_i	S_i	$\forall i$
$\sqrt{2}\,w\lvert\gamma-\delta\rvert\mathbf{p}$	0	S_i	S_i	S_{i+1}	$\forall i$
$\sqrt{2}\,w\lvert\theta-\beta\rvert\mathbf{q}$	0	S_i	S_{i+1}	S_{i+1}	$\forall i$
$2w\mathbf{q}\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+1}	$\forall i$
$2w\mathbf{p}\min\{\gamma,\delta\}$	1	S_i	S_i	S_{i+1}	$\forall i$
$\sqrt{2}\,w\lvert\gamma-\delta\rvert\mathbf{p}+2w\mathbf{q}\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+2}	$i\in\{1,3,5,7\}$
$\sqrt{2}\,\mathbf{p}\max\{\gamma-w\delta,\ \delta-w\gamma\}$	1	S_i	S_i	S_{i+2}	$i\in\{1,3,5,7\}$
$\sqrt{2}\,\mathbf{q}\min\{\theta+w\beta,\ \beta+w\theta\}$	1	S_i	S_{i+2}	S_{i+2}	$i\in\{1,3,5,7\}$
$2w\mathbf{p}\min\{\gamma,\delta\}+\sqrt{2}\,w\lvert\theta-\beta\rvert\mathbf{q}$	1	S_i	S_{i+1}	S_{i+2}	$i\in\{0,2,4,6\}$
$\sqrt{2}\,\mathbf{q}\max\{\theta-w\beta,\ \beta-w\theta\}$	1	S_i	S_{i+2}	S_{i+2}	$i\in\{0,2,4,6\}$
$\sqrt{2}\,\mathbf{p}\min\{\gamma+w\delta,\ \delta+w\gamma\}$	1	S_i	S_i	S_{i+2}	$i\in\{0,2,4,6\}$
$\sqrt{2}\,w\lvert\gamma-\delta\rvert\mathbf{p}+\sqrt{2}\,\mathbf{q}\max\{\theta-w\beta,\ \beta-w\theta\}$	1	S_i	S_{i+2}	S_{i+3}	$\forall i$
$\sqrt{2}\,\mathbf{p}\max\{\gamma-w\delta,\ \delta--w\gamma\}+\sqrt{2}\,w\mathbf{q}\lvert\theta-\beta\rvert$	1	S_i	S_{i+1}	S_{i+3}	$\forall i$
$2\mathbf{q}\max\{\theta,\beta\}$	1	S_i	S_{i+3}	S_{i+3}	$\forall i$
$2\mathbf{p}\max\{\gamma,\delta\}$	1	S_i	S_i	S_{i+3}	$\forall i$
$\sqrt{2}\,\mathbf{p}\min\{\gamma+w\delta,\ \delta+w\gamma\}+2w\mathbf{q}\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+3}	$\forall i$
$\sqrt{2}\,(\gamma+\delta)\mathbf{p}$	2	S_i	S_i	S_{i+3}	$\forall i$
$2w\mathbf{p}\min\{\gamma,\delta\}+\sqrt{2}\,\mathbf{q}\min\{\theta+w\beta,\ \beta+w\theta\}$	2	S_i	S_{i+2}	S_{i+3}	$\forall i$
$\sqrt{2}\,(\theta+\beta)\mathbf{q}$	2	S_i	S_{i+3}	S_{i+3}	$\forall i$
$2\mathbf{p}\max\{\gamma,\delta\}+2w\mathbf{q}\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+4}	$i\in\{1,3,5,7\}$
$\sqrt{2}\,\mathbf{p}\max\{\gamma-w\delta,\ \delta-w\gamma\}+\sqrt{2}\,\mathbf{q}\min\{\theta+w\beta,\ \beta+w\theta\}$	2	S_i	S_{i+2}	S_{i+4}	$i\in\{1,3,5,7\}$
$\sqrt{2}\,w\lvert\gamma-\delta\rvert\mathbf{p}+\sqrt{2}\,(\theta+\beta)\mathbf{q}$	2	S_i	S_{i+3}	S_{i+4}	$i\in\{1,3,5,7\}$
$2w\mathbf{p}\min\{\gamma,\delta\}+2\mathbf{q}\max\{\theta,\beta\}$	2	S_i	S_{i+3}	S_{i+4}	$i\in\{0,2,4,6\}$
$\sqrt{2}\,\mathbf{p}\min\{\gamma+w\delta,\ \delta+w\gamma\}+\sqrt{2}\,\mathbf{q}\max\{\theta-w\beta,\ \beta-w\theta\}$	2	S_i	S_{i+2}	S_{i+4}	$i\in\{0,2,4,6\}$
$\sqrt{2}\,(\gamma+\delta)\mathbf{p}+\sqrt{2}\,w\lvert\theta-\beta\rvert\mathbf{q}$	2	S_i	S_{i+1}	S_{i+4}	$i\in\{0,2,4,6\}$
$2\mathbf{pb}$	2	S_i	S_i	S_{i+4}	$\forall i$
$2\mathbf{qc}$	2	S_i	S_{i+4}	S_{i+4}	$\forall i$

$B=(b_1,b_2)$, $C=(c_1,c_2)$, $\lvert b_1\rvert=\theta$, $\lvert b_2\rvert=\beta$, $\lvert c_1\rvert=\gamma$, $\lvert c_2\rvert=\delta$ and $w=\sqrt{2}-1$.

Proof. Without loss of generality, the vertex A of the triangle ABC in $\mathbb{R}_c^2$ can be chosen at origin by Proposition 2.1. Let B=(b_1,b_2), C=(c_1,c_2), $X=(x_1,x_2)$,

$|b_1| = \theta$, $|b_2| = \beta$, $|c_1| = \gamma$ and $|c_2| = \delta$. Thus $\mathbf{b}$=$\max\{\gamma,\delta\}+(\sqrt{2}-1)\min\{\gamma,\delta\}$, $\mathbf{c}$=$\max\{\theta,\beta\}+(\sqrt{2}-1)\min\{\theta,\beta\}$ and $\mathbf{x}=\max\{|x_1|,|x_2|\}+(\sqrt{2}-1)\min\{|x_1|,|x_2|\}$. Three main cases are possible for the base line through the vertex A:

Case I: Let ABC be a triangle which has no base line through the vertex A. Since vertex A is at origin, AB and AC are in same quadrant. So one can easily obtain

$$\mathbf{q}(\gamma-\theta)=(\mathbf{p}+\mathbf{q})(\gamma-x_1) \quad , \quad \mathbf{p}(\delta-\beta)=(\mathbf{p}+\mathbf{q})(x_2-\beta),$$

by the Corollaries 2.2 and Corollaries 2.3. Thus $x_1 = \frac{\mathbf{p}\gamma+\mathbf{q}\theta}{\mathbf{p}+\mathbf{q}}$ and $x_2 = \frac{\mathbf{p}\delta+\mathbf{q}\beta}{\mathbf{p}+\mathbf{q}}$. According to the positions of AB, AC and AX, one can obtain $d_c(A,X)=\mathbf{x}$ as follows:

If the AB, AC and AX are in S_i as in Figure 2, then $\mathbf{x}$=$\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}$.
If AB and AX are in S_i and AC is in S_{i+1} as in Figure 2, then

$$\mathbf{x}=\frac{(\mathbf{b}-(2-\sqrt{2})|\gamma-\delta|)\mathbf{p}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}(\sqrt{2}-1)|\gamma-\delta|\mathbf{p}}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+1} as in Figure 2, then

$$\mathbf{x}=\frac{\mathbf{bp}+(\mathbf{c}-(2-\sqrt{2})|\theta-\beta|)\mathbf{q}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}(\sqrt{2}-1)|\theta-\beta|\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

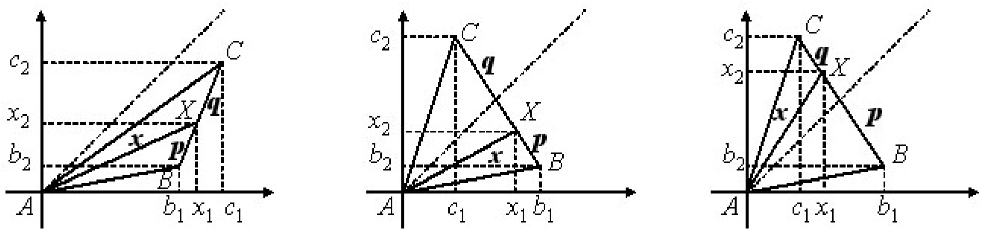

Figure 2

Case II: Let ABC be a triangle which has only one base line through the vertex A. Since vertex A is at origin, AB and AC are in a neighbor quadrant. That is, $b_1c_1<0$ or $b_2c_2<0$. According to the positions of AB, AC and AX, one can obtain

$$x_1=\frac{\mathbf{p}\gamma+\mathbf{q}\theta}{\mathbf{p}+\mathbf{q}} \quad \text{and} \quad x_2=\frac{\mathbf{p}\delta-\mathbf{q}\beta}{\mathbf{p}+\mathbf{q}} \quad \text{or} \quad x_2=\frac{-\mathbf{p}\delta+\mathbf{q}\beta}{\mathbf{p}+\mathbf{q}}$$

or

$$x_1=\frac{\mathbf{p}\gamma-\mathbf{q}\theta}{\mathbf{p}+\mathbf{q}} \quad \text{or} \quad x_1=\frac{-\mathbf{p}\gamma+\mathbf{q}\theta}{\mathbf{p}+\mathbf{q}} \quad \text{and} \quad x_2=\frac{\mathbf{p}\delta+\mathbf{q}\beta}{\mathbf{p}+\mathbf{q}}.$$

Now using these values, $d_c(A,X)=\mathbf{x}$ is obtained as follows:
If AB is in S_i and AX and AC are in S_{i+1}, then

$$\mathbf{x}=\frac{\mathbf{bp}+\mathbf{q}(\mathbf{c}-2(\sqrt{2}-1)\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2(\sqrt{2}-1)\min\{\theta,\beta\}\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB and AX are in S_i and AC is in S_{i+1} as in Figure 3, then

$$\mathbf{x}=\frac{(\mathbf{b}-2(\sqrt{2}-1)\min\{\gamma,\delta\})\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{2(\sqrt{2}-1)\min\{\gamma,\delta\}\mathbf{p}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+2} for $i\in\{1,3,5,7\}$, respectively, then

$$\mathbf{x}=\frac{(\mathbf{b}-2(\sqrt{2}-1)|\delta-\gamma|)\mathbf{p}+\mathbf{q}(\mathbf{c}-2(\sqrt{2}-1)\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}(\sqrt{2}-1)|\delta-\gamma|\mathbf{p}+2(\sqrt{2}-1)\min\{\theta,\beta\}\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB and AX are in S_i and AC is in S_{i+2} for $i\in\{1,3,5,7\}$, then

$$\mathbf{x}=\frac{(\mathbf{b}-\sqrt{2}\max\left\{\delta-(\sqrt{2}-1)\gamma,\gamma-(\sqrt{2}-1)\delta\right\})\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\max\{\delta-w\gamma,\gamma-w\delta\}\mathbf{p}}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+2} for $i\in\{1,3,5,7\}$, then

$$\mathbf{x}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}\min\left\{\theta+(\sqrt{2}-1)\beta,\beta+(\sqrt{2}-1)\theta\right\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\min\{\theta+w\beta,\beta+w\theta\}\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+2} for $i\in\{0,2,4,6\}$ as in Figure 3, respectively, then

$$\mathbf{x}=\frac{(\mathbf{b}-2(\sqrt{2}-1)\min\{\gamma,\delta\})\mathbf{p}+\mathbf{q}(\mathbf{c}-(2-\sqrt{2})|\beta-\theta|)}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{2w\min\{\gamma,\delta\}\mathbf{p}+\sqrt{2}w|\beta-\theta|\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB is in S_i and AX and AC are in S_{i+2} for $i\in\{0,2,4,6\}$, then

$$\mathbf{x}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}\max\left\{\theta-(\sqrt{2}-1)\beta,\beta-(\sqrt{2}-1)\theta\right\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\mathbf{q}\max\{\theta-w\beta,\beta-w\theta\}}{\mathbf{p}+\mathbf{q}}.$$

If AB and AX are in S_i and AC is in S_{i+2} for $i\in\{0,2,4,6\}$, then

$$\mathbf{x}=\frac{(\mathbf{b}-\sqrt{2}\min\left\{\delta+(\sqrt{2}-1)\gamma,\gamma+(\sqrt{2}-1)\delta\right\})\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\min\{\delta+w\gamma,\gamma+w\delta\}\mathbf{p}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+2}, S_{i+3}, respectively, then

$$\begin{aligned}\mathbf{x}&=\frac{(\mathbf{b}-(2-\sqrt{2})|\delta-\gamma|)\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}\max\left\{\theta-(\sqrt{2}-1)\beta,\beta-(\sqrt{2}-1)\theta\right\})}{\mathbf{p}+\mathbf{q}}\\&=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}w|\delta-\gamma|\mathbf{p}+\sqrt{2}\mathbf{q}\max\{\theta-w\beta,\beta-w\theta\}}{\mathbf{p}+\mathbf{q}}.\end{aligned}$$

If AB, AX and AC in S_i, S_{i+1}, S_{i+3} as in Figure 3, respectively, then

$$\begin{aligned}\mathbf{x}&=\frac{(\mathbf{b}-\sqrt{2}\max\left\{\delta-(\sqrt{2}-1)\gamma,\gamma-(\sqrt{2}-1)\delta\right\})\mathbf{p}+\mathbf{q}(\mathbf{c}-(2-\sqrt{2})|\beta-\theta|)}{\mathbf{p}+\mathbf{q}}\\&=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\mathbf{p}\max\{\delta-w\gamma,\gamma-w\delta\}+\sqrt{2}w\mathbf{q}|\beta-\theta|}{\mathbf{p}+\mathbf{q}}.\end{aligned}$$

If AB is in S_i and AX and AC are in S_{i+3}, then

$$\mathbf{x}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}(\mathbf{c}-2\max\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{2\max\{\theta,\beta\}\mathbf{q}}{\mathbf{p}+\mathbf{q}}.$$

If AB and AX are in S_i and AC is in S_{i+3}, then

$$\mathbf{x}=\frac{(\mathbf{b}-2\max\{\gamma,\delta\})\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{b}\mathbf{p}+\mathbf{q}\mathbf{c}}{\mathbf{p}+\mathbf{q}}-\frac{2\mathbf{p}\max\{\gamma,\delta\}}{\mathbf{p}+\mathbf{q}}.$$

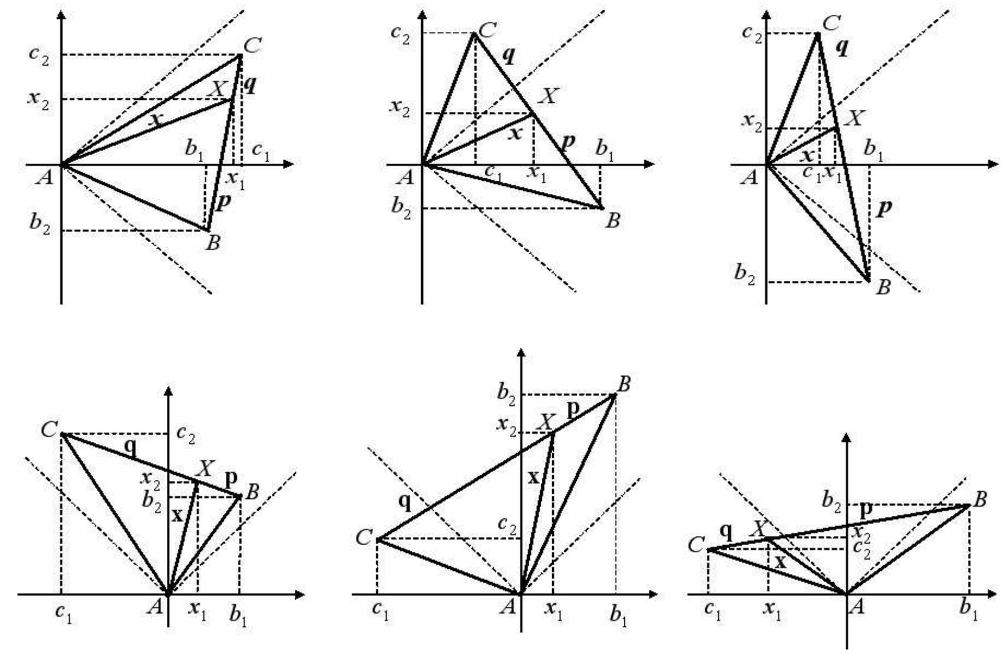

Figure 3

Case III: Let ABC is a triangle which has two base line through the vertex A. Since the vertex A is at origin, AB and AC are in opposite quadrants. According to the positions of AB, AC and AX, one can obtain

$$x_1 = \frac{-\mathbf{p}\gamma + \mathbf{q}\theta}{\mathbf{p}+\mathbf{q}} \text{ or } x_1 = \frac{\mathbf{p}\gamma - \mathbf{q}\theta}{\mathbf{p}+\mathbf{q}} \text{ and } x_2 = \frac{-\mathbf{p}\delta + \mathbf{q}\beta}{\mathbf{p}+\mathbf{q}} \text{ or } x_2 = \frac{\mathbf{p}\delta - \mathbf{q}\beta}{\mathbf{p}+\mathbf{q}} .$$

Using these values of x_1 and x_2, $d_c(A,X) = \mathbf{x}$ is obtained as follows:

If AB, AX and AC are in S_i, S_{i+1}, S_{i+3} as in Figure 4, respectively, then

$$\begin{aligned} \mathbf{x} &= \frac{(\mathbf{b}-\sqrt{2}\min\left\{\delta+(\sqrt{2}-1)\gamma,\gamma+(\sqrt{2}-1)\delta\right\})\mathbf{p}+\mathbf{q}(\mathbf{c}-2(\sqrt{2}-1)\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{\sqrt{2}\mathbf{p}\min\{\delta+w\gamma,\gamma+w\delta\}+2w\mathbf{q}\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}. \end{aligned}$$

If AB and AX are in S_i and AC is in S_{i+3}, then

$$\mathbf{x} = \frac{(\mathbf{b}-\sqrt{2}(\delta+\gamma))\mathbf{p}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} = \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{\sqrt{2}\mathbf{p}(\delta+\gamma)}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+2}, S_{i+3}, respectively, then

$$\begin{aligned} \mathbf{x} &= \frac{(\mathbf{b}-2(\sqrt{2}-1)\min\{\gamma,\delta\})\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}\min\left\{\theta+(\sqrt{2}-1)\beta,\beta+(\sqrt{2}-1)\theta\right\})}{\mathbf{p}+\mathbf{q}} \\ &= \frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}} - \frac{2w\mathbf{p}\min\{\gamma,\delta\}+\sqrt{2}\mathbf{q}\min\{\theta+w\beta,\beta+w\theta\}}{\mathbf{p}+\mathbf{q}}. \end{aligned}$$

If AB is in S_i and AX and AC are in S_{i+3}, then

$$\mathbf{x}=\frac{\mathbf{bp}+\mathbf{q}(\mathbf{c}-\sqrt{2}(\theta+\beta))}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\mathbf{q}(\theta+\beta)}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+1}, S_{i+4} for $i\in\{1,3,5,7\}$, respectively, then

$$\mathbf{x}=\frac{(\mathbf{b}-2\max\{\delta,\gamma\})\mathbf{p}+\mathbf{q}(\mathbf{c}-2(\sqrt{2}-1)\min\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2\mathbf{p}\max\{\delta,\gamma\}+2w\mathbf{q}\min\{\theta,\beta\}}{\mathbf{p}+\mathbf{q}}.$$

If AB, AX and AC are in S_i, S_{i+2}, S_{i+4} for $i\in\{1,3,5,7\}$ as in Figure4, respectively, then

$$\begin{aligned}\mathbf{x}&=\frac{(\mathbf{b}-\sqrt{2}\max\left\{\delta-(\sqrt{2}-1)\gamma,\gamma-(\sqrt{2}-1)\delta\right\})\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}\min\left\{\theta+(\sqrt{2}-1)\beta,\beta+(\sqrt{2}-1)\theta\right\})}{\mathbf{p}+\mathbf{q}}\\&=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\mathbf{p}\max\{\delta-w\gamma,\gamma-w\delta\}+\sqrt{2}\mathbf{q}\min\{\theta+w\beta,\beta+w\theta\}}{\mathbf{p}+\mathbf{q}}.\end{aligned}$$

If AB, AX and AC are in S_i, S_{i+3}, S_{i+4} for $i\in\{1,3,5,7\}$, respectively, then

$$\mathbf{x}=\frac{(\mathbf{b}-(2-\sqrt{2})|\gamma-\delta|)\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}(\theta+\beta))}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}w\mathbf{p}|\gamma-\delta|+\sqrt{2}\mathbf{q}(\theta+\beta)}{\mathbf{p}+\mathbf{q}}.$$

If AB and AC are in S_i, S_{i+4} and AX is in S_{i+3}, S_{i+2}, S_{i+1} for $i\in\{1,3,5,7\}$, then

$$\begin{aligned}\mathbf{x}&=\frac{(\mathbf{b}-2(\sqrt{2}-1)\min\{\gamma,\delta\})\mathbf{p}+\mathbf{q}(\mathbf{c}-2\max\{\theta,\beta\})}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2w\mathbf{p}\min\{\gamma,\delta\}+2\mathbf{q}\max\{\theta,\beta\}}{\mathbf{p}+\mathbf{q}},\\\mathbf{x}&=\frac{(\mathbf{b}-\sqrt{2}\min\left\{\delta+(\sqrt{2}-1)\gamma,\gamma+(\sqrt{2}-1)\delta\right\})\mathbf{p}+\mathbf{q}(\mathbf{c}-\sqrt{2}\max\left\{\theta-(\sqrt{2}-1)\beta,\beta-(\sqrt{2}-1)\theta\right\})}{\mathbf{p}+\mathbf{q}}\\\mathbf{x}&=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\mathbf{p}\min\{\delta+w\gamma,\gamma+w\delta\}+\sqrt{2}\mathbf{q}\max\{\theta-w\beta,\beta-w\theta\})}{\mathbf{p}+\mathbf{q}},\\\mathbf{x}&=\frac{(\mathbf{b}-\sqrt{2}(\gamma+\delta))\mathbf{p}+\mathbf{q}(\mathbf{c}-(2-\sqrt{2})|\theta-\beta|)}{\mathbf{p}+\mathbf{q}}=\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{\sqrt{2}\mathbf{p}(\gamma+\delta)+\sqrt{2}w\mathbf{q}|\theta-\beta|}{\mathbf{p}+\mathbf{q}},\end{aligned}$$

respectively.

If AB and AC are in S_i, S_{i+4}, respectively, and AX is in S_i or S_{i+4}, then

$$\mathbf{x}=\frac{|\mathbf{bp}-\mathbf{qc}|}{\mathbf{p}+\mathbf{q}}=\begin{cases}\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2\mathbf{pb}}{\mathbf{p}+\mathbf{q}}, & AX \text{ in } S_i\\\frac{\mathbf{bp}+\mathbf{qc}}{\mathbf{p}+\mathbf{q}}-\frac{2\mathbf{qc}}{\mathbf{p}+\mathbf{q}}, & AX \text{ in } S_{i+4}.\end{cases}$$

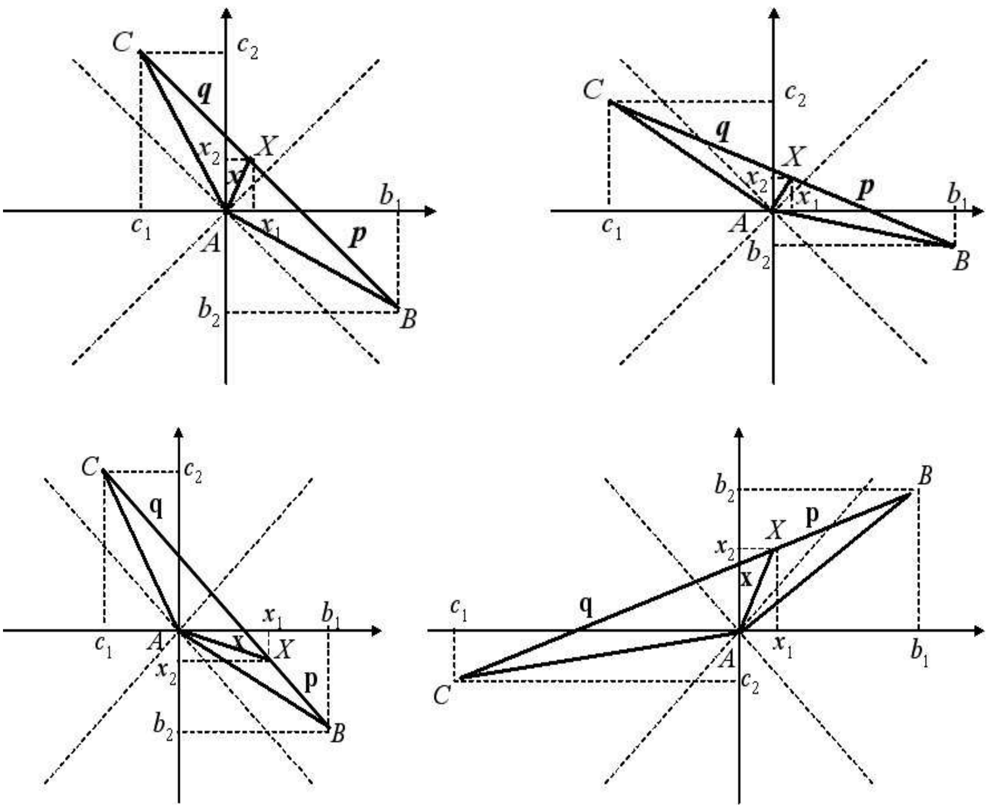

Figure 4

□

If X is the midpoint of BC of any triangle ABC in the Euclidean plane with $\mathbf{a}$=$d(B,C)$, $\mathbf{b}$=$d(A,C)$, $\mathbf{c}$=$d(A,B)$ and V_a=$d(A,X)$, then

$$2V_a^2 = \mathbf{b}^2 + \mathbf{c}^2 - \mathbf{a}^2/2$$

which is known as the *Median property*. The following corollary gives a CC-version of this property, for $\mathbf{p} = \mathbf{q}$ in Theorem 3.1:

Corollary 3.1. *Let the side of a triangle ABC in $\mathbb{R}_c^2$ have lengths $\mathbf{a}$=$d_c(B,C)$, $\mathbf{b}$=$d_c(A,C)$. If X is the midpoint of BC and V_a=$d_c(A,X)$, then $2V_a$ can be given as in following table:*

$2V_a$	Number of base lines	AB lies in	AX lies in	AC lies in	
$\mathbf{b}+\mathbf{c}$	0	S_i	S_i	S_i	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,w\lvert\gamma-\delta\rvert$	0	S_i	S_i	S_{i+1}	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,w\lvert\theta-\beta\rvert$	0	S_i	S_{i+1}	S_{i+1}	$\forall i$
$\mathbf{b}+\mathbf{c}-2w\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+1}	$\forall i$
$\mathbf{b}+\mathbf{c}-2w\min\{\gamma,\delta\}$	1	S_i	S_i	S_{i+1}	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,w\lvert\gamma-\delta\rvert-2w\min\{\theta,\beta\}$	1	S_i	S_{i+1}	S_{i+2}	$i\in\{1,3,5,7\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\max\{\gamma-w\delta,\ \delta-w\gamma\}$	1	S_i	S_i	S_{i+2}	$i\in\{1,3,5,7\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\min\{\theta+w\beta,\ \beta+w\theta\}$	1	S_i	S_{i+2}	S_{i+2}	$i\in\{1,3,5,7\}$
$\mathbf{b}+\mathbf{c}-2w\min\{\gamma,\delta\}-\sqrt{2}\,w\lvert\theta-\beta\rvert$	1	S_i	S_{i+1}	S_{i+2}	$i\in\{0,2,4,6\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\max\{\theta-w\beta,\ \beta-w\theta\}$	1	S_i	S_{i+2}	S_{i+2}	$i\in\{0,2,4,6\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\min\{\gamma+w\delta,\ \delta+w\gamma\}$	1	S_i	S_i	S_{i+2}	$i\in\{0,2,4,6\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,w\lvert\gamma-\delta\rvert-\sqrt{2}\max\{\theta-w\beta,\ \beta-w\theta\}$	1	S_i	S_{i+2}	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\max\{\gamma-w\delta,\ \delta-w\gamma\}-\sqrt{2}\,w\lvert\theta-\beta\rvert$	1	S_i	S_{i+1}	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-2\max\{\theta,\beta\}$	1	S_i	S_{i+3}	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-2\max\{\gamma,\delta\}$	1	S_i	S_i	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\min\{\gamma+w\delta,\ \delta+tw\gamma\}-2w\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,(\gamma+\delta)$	2	S_i	S_i	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-2w\min\{\gamma,\delta\}-\sqrt{2}\min\{\theta+w\beta,\ \beta+w\theta\}$	2	S_i	S_{i+2}	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,(\theta+\beta)$	2	S_i	S_{i+3}	S_{i+3}	$\forall i$
$\mathbf{b}+\mathbf{c}-2\max\{\gamma,\delta\}-2w\min\{\theta,\beta\}$	2	S_i	S_{i+1}	S_{i+4}	$i\in\{1,3,5,7\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\max\{\gamma-w\delta,\ \delta-w\gamma\}-\sqrt{2}\min\{\theta+w\beta,\ \beta+w\theta\}$	2	S_i	S_{i+2}	S_{i+4}	$i\in\{1,3,5,7\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,w\lvert\gamma-\delta\rvert-\sqrt{2}\,(\theta+\beta)$	2	S_i	S_{i+3}	S_{i+4}	$i\in\{1,3,5,7\}$
$\mathbf{b}+\mathbf{c}-2w\min\{\gamma,\delta\}-2\max\{\theta,\beta\}$	2	S_i	S_{i+3}	S_{i+4}	$i\in\{0,2,4,6\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\min\{\gamma+w\delta,\ \delta+w\gamma\}-\sqrt{2}\max\{\theta-w\beta,\ \beta-w\theta\}$	2	S_i	S_{i+2}	S_{i+4}	$i\in\{0,2,4,6\}$
$\mathbf{b}+\mathbf{c}-\sqrt{2}\,(\gamma+\delta)-\sqrt{2}\,w\lvert\theta-\beta\rvert$	2	S_i	S_{i+1}	S_{i+4}	$i\in\{0,2,4,6\}$
$\lvert\mathbf{b}-\mathbf{c}\rvert$	2	S_i	S_i	S_{i+4}	$\forall i$
$\lvert\mathbf{b}-\mathbf{c}\rvert$	2	S_i	S_{i+4}	S_{i+4}	$\forall i$

$B=(b_1,b_2)$, $C=(c_1,c_2)$, $\lvert b_1\rvert=\theta$, $\lvert b_2\rvert=\beta$, $\lvert c_1\rvert=\gamma$, $\lvert c_2\rvert=\delta$ and $w=\sqrt{2}-1$.

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