Optimal control of an inventory system with ameliorating and deteriorating items

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Abstract. This paper is concerned with the development of a production inventory model for both ameliorating and deteriorating items. Given an inventory goal level and a production goal rate set by the production facility, and given penalties for the inventory level and for the production rate to deviate from their respective targets, a system total cost objective function is derived. We seek the optimal production rate, that is the production rate that minimizes this performance measure, while satisfying the system dynamics. The resulting problem is an optimal control problem with mixed inequality constraints, in which the inventory level is the state variable and the production rate is the control variable. The necessary optimality conditions are derived using Pontryagin maximum principle. This paper generalizes some of the models available in the literature.

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1 Introduction

Inventory systems with deteriorating items have received a lot of attention. The recent paper of Goyal and Giri [3] presents a thorough survey of the subject, classifying the literature according to various conditions and constraints and mentioning many applications.

A closely related topic, that of inventory systems with ameliorating items, has received much less attention. The first reference to inventory systems with both ameliorating and deteriorating items seems to be that of Hwang [6]. He assumes that items ameliorate while at a breeding yard, such as fish culture facility, and deteriorate when in the distribution center. Hwang [7] extends his earlier results by developing three models: the economic order quantity (EOQ), the partial selling quantity (PSQ), and the economic order quantity (EPQ). These models are particularly well suited for items that ameliorate and deteriorate at the same time. Fast growing animals such as fish, chickens, ducks, and rabbits are such examples. While in the farm, they will

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Figure 1: Inventory curve for an ameliorating and deteriorating system.

increase in value du to their growth, and once grown, they are used to produce food. For this reason, the inventory of such items grows faster in a first period and then starts to decline, see Figure 1. Mondal *et al.* [10] consider the case where the demand rate depends on the price of the item. Hwang [8] studies a set-covering location problem and determines the number of storage facilities. Moon *et al.* [11, 12] generalize the EOQ model with ameliorating and deteriorating items by allowing shortages and by taking into account the effects of inflation and time value of money. Finally, Law and Wee [9] study the EPQ model with ameliorating and deteriorating items by allowing shortages, by taking into account time discounting, and by incorporating the manufacturer-retail cooperation.

As can be seen, the number of references concerning ameliorating and deteriorating items is very small. Also, all these references use optimization techniques to determine a number or set of numbers, namely, the optimal quantity to order or to produce, or the optimal instants of time when to start or stop the production process. Our approach is going to be quite different. We will be using optimal control theory to determine not a number or set of numbers, but rather a whole function, namely the optimal production rate. Optimal control theory has never been applied in conjunction with ameliorating and deteriorating items. It has been successfully applied in production planning when only deteriorating items are involved, see for example [1, 2, 4, 5] and the references therein. We will present our problem as an optimal control problem with one state variable (the inventory level) and one control variable (the production rate). We will be looking for the necessary optimality conditions using Pontryagin maximum principle, see Pontryagin *et al.* [13].

In the next section, we list the notation and describe the model. In section 3, we derive the necessary and sufficient optimality conditions in the general case and obtain the explicit solution in two special cases. Numerical illustrative examples are provided. The paper is summarized in Section 4 where some directions for further research are also given.

2 Model and Notation

Consider a system where items are subject to both amelioration and deterioration. The length of the planning horizon is T. Assume that in a first phase, say from time 0 to time t_1, the inventory level is increasing, while in the second phase, from from time t_1 to time T, the inventory level is decreasing, see Figure 1. We note that the instant of time t_1 is unknown and needs to be determined. However, the value M of the inventory level at time t_1 is assumed to be known. The following notation will be used to describe the dynamics of the system:

$I(t)$: inventory level function.

$P(t)$: production rate function.

$D(t)$: demand rate function.

I_0 : initial inventory level.

$\theta(t)$: deterioration rate function.

$m(t)$: amelioration rate function.

Also, let $v = m - \theta$. Since m and θ are known functions, so is the function v. The inventory level evolves through time according to the following state equation:

$$\dot{I}(t) = \begin{cases} P(t) + v(t)I(t), & t \in [0, t_1], \\ P(t) - D(t) + v(t)I(t), & t \in [t_1, T]. \end{cases} \tag{2.1}$$

To ensure that the inventory level is increasing from time 0 to time t_1 and decreasing from from time t_1 to time T, we will further set

$$P(t) + v(t)I(t) > 0, \; t \in [0, t_1] \tag{2.2}$$

$$D(t) - P(t) - v(t)I(t) > 0, \; t \in [t_1, T] \tag{2.3}$$

Now to build the objective cost function, we assume that an inventory goal level and a production goal rate are set, and penalties are incurred for deviating from these goals. To explicitly write the objective function, we introduce the following additional notation:

h : holding cost.

K : production cost.

$\hat{I}$: inventory goal level.

$\hat{P}$: production goal rate.

This gives rise to the following performance index that we need to minimize

$$J = \frac{1}{2}\int_0^T \left\{ h\Big[I(t) - \hat{I}\Big]^2 + K\Big[P(t) - \hat{P}\Big]^2 \right\} dt \tag{2.4}$$

subject to (2.1)-(2.3) and the additional nonnegativity constraint

$$P(t) \geq 0, \quad t \in [0, T]. \tag{2.5}$$

The solution to this problem is presented in the next section.

3 Optimal Control

Note that we deal with a problem with mixed inequality constraints, that is constraints involving both control and state variables. The maximum principle for problems with mixed inequality constraints states that there should exist a continuous and piecewise continuously differentiable λ, and a piecewise continuous function μ to define the Hamiltonian function

$$H = -\frac{1}{2}\left[h\left(I - \hat{I}\right)^2 + K\left(P - \hat{P}\right)^2 \right] + \lambda g, \tag{3.1}$$

where

$$g = \begin{cases} P + vI, & t \in [0, t_1], \\ D - P - vI, & t \in [t_1, T], \end{cases}$$

and the Lagrangian function

$$L = -\frac{1}{2}\left[h\left(I - \hat{I}\right)^2 + K\left(P - \hat{P}\right)^2 \right] + \begin{cases} (\lambda + \mu)\, g, & t \in [0, t_1], \\ (\lambda - \mu)\, g, & t \in [t_1, T]. \end{cases} \tag{3.2}$$

The necessary optimality conditions are given by

$$\begin{aligned} H_P &= 0, &(3.3)\\ L_I &= -\dot{\lambda}, &(3.4)\\ L_P &= 0 &(3.5)\\ \mu &\geq 0, \ \ \mu g \geq 0, &(3.6) \end{aligned}$$

These conditions take two different forms depending on whether $t \in [0, t_1]$ or $[t_1, T]$. So, let us start with the case when $t \in [0, t_1]$. In this case, the condition (3.3) is equivalent to

$$P = \hat{P} + \frac{\lambda}{K}. \tag{3.7}$$

The condition (3.4) is equivalent to

$$\dot{\lambda} = h(I - \hat{I}) - (\lambda + \mu)v. \tag{3.8}$$

The condition (3.5) is equivalent to

$$0 = K(P - \hat{P}) - (\lambda + \mu). \tag{3.9}$$

The condition (3.6) with (2.3) imply $\mu = 0$. Therefore, (3.7) and (2.1) when $t \in [0, t_1]$, yield

$$\dot{I} = \hat{P} + \frac{\lambda}{K} + vI. \tag{3.10}$$

Combining (3.10) with (3.8), we get the following second order differential equation

$$\ddot{I} - \left[\frac{h}{K} + v^2 - \dot{v}\right] I = \alpha_1(t), \tag{3.11}$$

where

$$\alpha_1(t) = -\frac{h}{K}\hat{I} + v\hat{P}.$$

Similarly, it can be shown that, when $t \in [t_1, T]$, we get the following second order differential equation

$$\ddot{I} - \left[\frac{h}{K} + v^2 - \dot{v}\right] I = \alpha_2(t), \tag{3.12}$$

where

$$\alpha_2(t) = -\frac{h}{K}\hat{I} - v(D - \hat{P}) - \dot{D}.$$

To determine the optimal inventory level and the optimal production rate, we have to solve the differential equations (3.11) and (3.12). The solutions depend on the shape of the functions m and θ, and thus of the function v. In most cases, it will be impossible to obtain an explicit solution for the differential equations (3.11) and (3.12). Two cases where an explicit solution is available are discussed below. As we will see, even in these very special cases, especially in the second one, the solution is difficult to obtain. In the general case, equations (3.11) and (3.12) are solved numerically.

3.1 The function v is constant

When the function v is constant, the differential equations (3.11) and (3.12), respectively become

$$\ddot{I} - \left[\frac{h}{K} + v^2\right] I = \alpha_1(t), \quad t \in [0, t_1], \tag{3.13}$$

and

$$\ddot{I} - \left[\frac{h}{K} + v^2\right] I = \alpha_2(t), \quad t \in [t_1, T]. \tag{3.14}$$

The characteristic equation of these differential equations has the following solutions

$$r_1 = r = \sqrt{\frac{h}{K} + v^2} \quad \text{and} \quad r_2 = -r.$$

Therefore, the solution of (3.13), (3.14), is given by

$$I(t) = \begin{cases} C_{11}e^{rt} + C_{12}e^{-rt} + Q_1(t), & t \in [0, t_1], \\ C_{21}e^{rt} + C_{22}e^{-rt} + Q_2(t), & t \in [t_1, T], \end{cases} \tag{3.15}$$

where $Q_1(t)$ and $Q_2(t)$ are particular solutions of the differential equations (3.13) and (3.14), respectively. Using the conditions $I(0) = I_0$ and $I(t_1) = M$, one gets

$$C_{1j} = (-1)^{j-1}\frac{Q_1(t_1) - M + [I_0 - Q_1(0)]e^{(-1)^j rt_1}}{e^{-rt_1} - e^{rt_1}}, \quad j = 1, 2.$$

From (3.10) and (3.15), we get

$$\lambda = K \times \begin{cases} C_{11}(r-v)e^{rt} - C_{12}(r+v)e^{-rt} + \dot{Q}_1 - \hat{P} - vQ_1, & t \in [0, t_1], \\ C_{21}(r-v)e^{rt} - C_{22}(r+v)e^{-rt} + \dot{Q}_2 - \hat{P} + D - vQ_2, & t \in [t_1, T]. \end{cases} \tag{3.16}$$

Also, the conditions $I(t_1) = M$ and $\lambda(T) = 0$ give

$$\begin{aligned} C_{21} &= e^{-rt_1}\left[M - C_{22}e^{-rt_1} - Q_2(t_1)\right], \\ C_{22} &= \frac{\gamma}{(v-r)e^{-rT} - e^{(T-2t_1)r}}, \end{aligned}$$

where

$$\gamma = [Q_2(t_1) - M](r - v)e^{(T-t_1)r} - \dot{Q}_2(T) + \hat{P} - D(T) + vQ_2(T).$$

Once we get the inventory level, the production rate P can be found by substituting (3.16) into (3.7), as

$$P(t) = \hat{P} + \begin{cases} C_{11}(r-v)e^{rt} - C_{12}(r+v)e^{-rt} + \dot{Q}_1 - \hat{P} - vQ_1, & t \in [0, t_1], \\ C_{21}(r-v)e^{rt} - C_{22}(r+v)e^{-rt} + \dot{Q}_2 - \hat{P} + D - vQ_2, & t \in [t_1, T]. \end{cases} \tag{3.17}$$

The function Q_1 and Q_2 will be determined once the function D is known. We show below an illustrative example where the function D is known and constant.

Example 3.1. To illustrate the solution procedure, let us consider a simple example in which the demand rate is constant $D(t) = D$ and thus $\dot{D} = 0$. Then, it is not difficult to see that Q_1 and Q_2 are now constant and given by

$$Q_1 = \frac{h\hat{I} - vK\hat{P}}{h + Kv^2} \quad \text{and} \quad Q_2 = \frac{h\hat{I} + vK(D - \hat{P})}{h + Kv^2}.$$

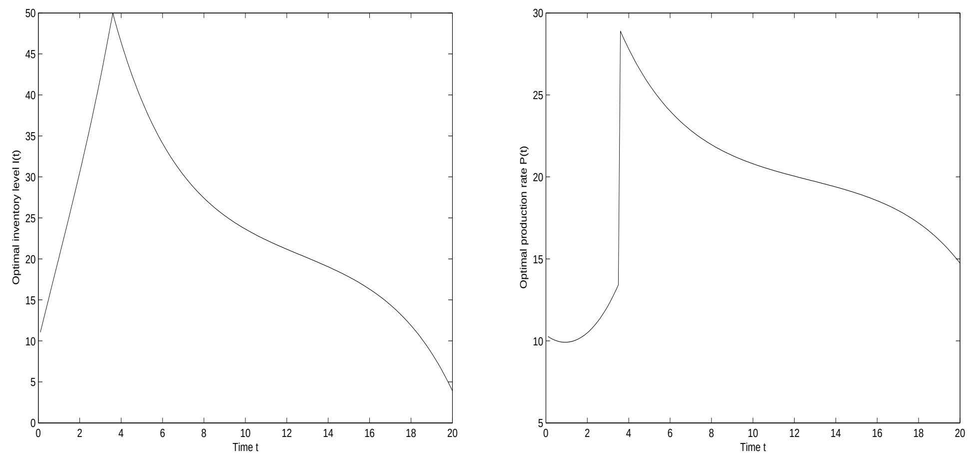

Figure 2: Optimal inventory level (left) and production rate (right).

The constants $C_{ij}, i, j = 1, 2$ are now easily obtained to yield an explicit expression for $I(t)$ as given by (3.15). Note that the unknown t_1 is determined by equating the expressions of $I(t)$ at that instant of time, that is, by solving the nonlinear equation

$$C_{11}e^{rt_1} + C_{12}e^{-rt_1} + Q_1 = C_{21}e^{rt_1} + C_{22}e^{-rt_1} + Q_2.$$

To further illustrate, let us take a numerical example in which the constant demand rate is $D = 20$. Assume a planning horizon of length $T = 20$. The deterioration and amelioration rates are $\theta = 0.001$ and $m = 0.01$ respectively, so that the function v is constant and worth $v = 0.009$. Also, take goals $\hat{I} = 20$ and $\hat{P} = 15$, initial and maximum inventory levels $I_0 = 10$ and $M = 50$, and unit costs $h = 1.5$ and $K = 15$. The optimal inventory level and the optimal production rate in this case are depicted in Figure 2 below.

3.2 The function $\frac{h}{K} + v^2 - \dot{v}$ is constant

When the function $\frac{h}{K} + v^2 - \dot{v}$ is constant, the shape of solution will vary according to whether this constant is positive or negative. We will illustrate the solution procedure in both cases.

Case 1: The constant $\frac{h}{K} + v^2 - \dot{v}$ is positive, i.e.,

$$\frac{h}{K} + v^2 - \dot{v} = k_1^2. \tag{3.18}$$

Now, the differential equations (3.11) and (3.12), respectively become

$$\ddot{I} - k_1^2 I = \alpha_1(t), \quad t \in [0, t_1], \tag{3.19}$$

and

$$\ddot{I} - k_1^2 I = \alpha_2(t), \quad t \in [t_1, T]. \tag{3.20}$$

Note that we need to solve the differential equation (3.18) to determine v first, before a solution to equations (3.19) and (3.20) is attempted. The solution of this equation (3.18) depends on the sign of the constant $k_1^2 - \frac{h}{K}$. Let us distinguish the two cases.

(a) The constant $k_1^2 - \frac{h}{K}$ is positive. Let $k_1^2 - \frac{h}{K} = a^2$. Equation (3.18) can be written

$$\frac{dv}{(v+a)(v-a)} = dt,$$

and it has the solution

$$v(t) = \frac{1}{a} \cdot \frac{1 - e^{2at}}{1 + e^{2at}}. \tag{3.21}$$

- Solve (3.19): This is a nonhomogeneous second-order linear differential equation. It has a solution of the form

$$I(t) = c_1 e^{k_1 t} + c_2 e^{-k_1 t} + Q(t),$$

where $Q(t)$ is a particular solution of (3.19) whose right-hand side is $\alpha_1(t) = -\frac{h\hat{I}}{K} + \frac{\hat{P}}{a}\left(\frac{1-e^{2at}}{1+e^{2at}}\right)$. This particular solution can be written as $Q(t) = Q_1(t) + Q_2(t)$ where $Q_1(t) = -\frac{h\hat{I}}{Kk_1^2}$. To determine $Q_2(t)$, we use the method of variation of parameters, see for example Zill and Cullen [14]. We briefly recall that to apply this method to solve a second-order linear differential equation put in the standard form

$$\ddot{y} + P(x)\dot{y} + Q(x)y = f(x),$$

a fundamental set of solutions y_1 and y_2 of the associated homogeneous equation is required. Then, look for two functions u_1 and u_2 such that the particular solution take the form $u_1(x)y_1(x) + u_2(x)y_2(x)$. To determine u_1 and u_2, one has to compute the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix}.$$

Then, find u_1 and u_2 by integrating

$$\dot{u}_1 = \frac{W_1}{W} \quad \text{and} \quad \dot{u}_2 = \frac{W_2}{W}$$

where W_1 and W_2 are the determinants

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & \dot{y}_2 \end{vmatrix} \quad \text{and} \quad W_2 = \begin{vmatrix} y_1 & 0 \\ \dot{y}_1 & f(x) \end{vmatrix}.$$

In our case, $y_1 = e^{k_1 t}$ and $y_2 = e^{-k_1 t}$. Also, $W = -2k_1$, $W_1 = -\frac{\hat{P}}{a}\left(\frac{1-e^{2at}}{1+e^{2at}}\right)e^{-k_1 t}$, and $W_2 = \frac{\hat{P}}{a}\left(\frac{1-e^{2at}}{1+e^{2at}}\right)e^{k_1 t}$. So we look for $Q_2(t)$ in the form $Q_2(t) = V_1(t)e^{k_1 t} + V_2(t)e^{-k_1 t}$, where V_1 and V_2 are determined by integrating the derivatives $\dot{V}_1$ and $\dot{V}_2$ given by

$$\dot{V}_1 = \frac{\hat{P}}{2ak_1}\left(\frac{1-e^{2at}}{1+e^{2at}}\right)e^{-k_1 t} \tag{3.22}$$

$$\dot{V}_2 = -\frac{\hat{P}}{2ak_1}\left(\frac{1-e^{2at}}{1+e^{2at}}\right)e^{k_1 t}. \tag{3.23}$$

Once V_1 and V_2 have been calculated, we have

$$I(t) = [c_1 + V_1(t)]\,e^{k_1 t} + [c_2 + V_2(t)]\,e^{-k_1 t} - \frac{h\hat{I}}{Kk_1^2}. \tag{3.24}$$

The constants c_1 and c_2 are determined using the conditions $I(0) = I_0$ and $I(t_1) = M$. Also, from (3.7) and (3.10), we have

$$P(t) = [v(t) + k_1]\,[c_1 + V_1(t)]\,e^{k_1 t} + [v(t) - k_1]\,[c_2 + V_2(t)]\,e^{-k_1 t} - \frac{h\hat{I}v(t)}{Kk_1^2}, \tag{3.25}$$

where $v(t)$ is given by (3.21) and $V_1(t)$ and $V_2(t)$ satisfy (3.22) and (3.23), respectively.

- Solve (3.20): This is also a nonhomogeneous second-order linear differential equation. It has a solution of the form

$$I(t) = c_1 e^{k_1 t} + c_2 e^{-k_1 t} + Q(t),$$

where $Q(t)$ is a particular solution of (3.20) whose right-hand side is $\alpha_2(t) = -\frac{h\hat{I}}{K} - \dot{D} - \frac{D-\hat{P}}{a}\left(\frac{1-e^{2at}}{1+e^{2at}}\right)$. Proceeding as before, we find that $I(t)$ is again given by (3.24) and $P(t)$ by (3.25), except that now V_1 and V_2 are determined by integrating the derivatives $\dot{V}_1$ and $\dot{V}_2$ given by

$$\dot{V}_1 = -\frac{1}{2k_1}\left[\frac{D-\hat{P}}{a}\left(\frac{1-e^{2at}}{1+e^{2at}}\right) + \dot{D}\right]e^{-k_1 t} \tag{3.26}$$

$$\dot{V}_2 = \frac{1}{2k_1}\left[\frac{D-\hat{P}}{a}\left(\frac{1-e^{2at}}{1+e^{2at}}\right) + \dot{D}\right]e^{k_1 t}. \tag{3.27}$$

To determine the constants c_1 and c_2 we need to use the conditions $I(t_1) = M$ and $\lambda(T) = 0$. The second condition can be used once we note that from (3.10), we have

$$\lambda(t) = K\left\{[k_1 - v(t)]\,[c_1 + V_1(t)]\,e^{k_1 t} - [k_1 + v(t)]\,[c_2 + V_2(t)]\,e^{-k_1 t}\right\} - \frac{h\hat{I}v(t)}{k_1^2},$$

where $v(t)$ is given by (3.21) and $V_1(t)$ and $V_2(t)$ satisfy (3.26) and (3.27), respectively.

(b) The constant $k_1^2 - \frac{h}{K}$ is negative. Let $k_1^2 - \frac{h}{K} = -a^2$. Equation (3.18) now has the solution

$$v(t) = \sqrt{a}\tan(\sqrt{a}t). \tag{3.28}$$

Only the right-hand side of equations (3.19) and (3.20). Therefore, we still have $I(t)$ given by (3.24) and $P(t)$ given by (3.25) while V_1 and V_2 will differ according to whether we are dealing with (3.19) or (3.20).

- Solve (3.19): V_1 and V_2 are determined by integrating the derivatives $\dot{V}_1$ and $\dot{V}_2$ given by

$$\dot{V}_1 = \frac{\hat{P}\sqrt{a}}{2k_1}\tan(\sqrt{a}t)e^{-k_1 t} \tag{3.29}$$

$$\dot{V}_2 = -\frac{\hat{P}\sqrt{a}}{2k_1}\tan(\sqrt{a}t)e^{k_1 t}. \tag{3.30}$$

- Solve (3.20): V_1 and V_2 are determined by integrating the derivatives $\dot{V}_1$ and $\dot{V}_2$ given by

$$\dot{V}_1 = -\frac{1}{2k_1}\left[\left(D - \hat{P}\right)\sqrt{a}\tan(\sqrt{a}t) - \dot{D}\right]e^{-k_1 t} \tag{3.31}$$

$$\dot{V}_2 = \frac{1}{2k_1}\left[\left(D - \hat{P}\right)\sqrt{a}\tan(\sqrt{a}t) - \dot{D}\right]e^{k_1 t}. \tag{3.32}$$

The constants c_1 and c_2 are determined as in the case (a).

Case 2: The constant $\frac{h}{K} + v^2 - \dot{v}$ is negative, i.e.,

$$\frac{h}{K} + v^2 - \dot{v} = -k_1^2. \tag{3.33}$$

Now, the differential equations (3.11) and (3.12), respectively become

$$\ddot{I} + k_1^2 I = \alpha_1(t), \quad t \in [0, t_1], \tag{3.34}$$

and

$$\ddot{I} + k_1^2 I = \alpha_2(t), \quad t \in [t_1, T]. \tag{3.35}$$

As in the previous case, we need to solve the differential equation (3.33) to determine v first, before a solution to equations (3.34) and (3.35) is attempted. The differential equation (3.33) can be written

$$v^2 - \dot{v} = -b^2, \tag{3.36}$$

where $b^2 = k_1^2 + \frac{h}{K} > 0$. The solution of this equation is the same as in (b) of case 1 above. It is given by

$$v(t) = \sqrt{b}\tan(\sqrt{b}t). \tag{3.37}$$

We now proceed to the solution of the differential equations (3.34) and (3.35).

- Solve (3.34): This is a nonhomogeneous second-order linear differential equation. It has a solution of the form

$$I(t) = c_1 \sin(k_1 t) + c_2 \cos(k_1 t) + Q(t),$$

where $Q(t)$ is a particular solution of (3.34) whose right-hand side is $\alpha_1(t) = -\frac{h\hat{I}}{K} + \hat{P}\sqrt{b}\tan(\sqrt{b}t)$. As in the previous case, the particular solution can be written as $Q(t) = Q_1(t) + Q_2(t)$ where $Q_1(t) = -\frac{h\hat{I}}{Kk_1^2}$ and $Q_2(t)$ is determined by the method of variation of parameters. Now, $y_1 = \sin(k_1 t)$ and $y_2 = \cos(k_1 t)$. Also, $W = -k_1$, $W_1 = -\hat{P}\sqrt{b}\tan(\sqrt{b}t)\cos(k_1 t)$, and $W_2 = \hat{P}\sqrt{b}\tan(\sqrt{b}t)\sin(k_1 t)$. So we look for $Q_2(t)$ in the form $Q_2(t) = V_1(t)\sin(k_1 t) + V_2(t)\cos(k_1 t)$, where V_1 and V_2 are determined by integrating the derivatives $\dot{V}_1$ and $\dot{V}_2$ given by

$$\dot{V}_1 = \frac{\hat{P}\sqrt{b}}{k_1}\tan(\sqrt{b}t)\cos(k_1 t), \tag{3.38}$$

$$\dot{V}_2 = -\frac{\hat{P}\sqrt{b}}{k_1}\tan(\sqrt{b}t)\sin(k_1 t). \tag{3.39}$$

Once V_1 and V_2 have been calculated, we have

$$I(t) = (c_1 + V_1)\sin(k_1 t) + (c_2 + V_2)\cos(k_1 t) - \frac{h\hat{I}}{Kk_1^2}. \tag{3.40}$$

The constants c_1 and c_2 are determined using the conditions $I(0) = I_0$ and $I(t_1) = M$. Also, from (3.7) and (3.10), we have

$$\begin{aligned} P(t) = \; & [k_1(c_2 + V_2) + v(c_1 + V_1)]\sin(k_1 t) - \\ & - [k_1(c_1 + V_1) - v(c_2 + V_2)]\cos(k_1 t) + \tfrac{h\hat{I}v}{Kk_1^2}, \end{aligned} \tag{3.41}$$

where $v(t)$ is given by (3.37) and $V_1(t)$ and $V_2(t)$ satisfy (3.38) and (3.39), respectively.

- Solve (3.35): This is also a nonhomogeneous second-order linear differential equation. It has a solution of the form

$$I(t) = c_1 \sin(k_1 t) + c_2 \cos(k_1 t) + Q(t),$$

where $Q(t)$ is a particular solution of (3.35) whose right-hand side is $\alpha_2(t) = -\frac{h\hat{I}}{K} - \dot{D} - \left(D - \hat{P}\right)\sqrt{b}\tan(\sqrt{b}t)$. Proceeding as before, we find that $I(t)$ is again given by (3.40) and $P(t)$ by (3.41), except that now V_1 and V_2 are determined by integrating the derivatives $\dot{V}_1$ and $\dot{V}_2$ given by

$$\dot{V}_1 = -\frac{1}{k_1}\left[\left(D - \hat{P}\right)\sqrt{b}\tan(\sqrt{b}t) + \dot{D}\right]\cos(k_1 t), \tag{3.42}$$

$$\dot{V}_2 = \frac{1}{k_1}\left[\left(D - \hat{P}\right)\sqrt{b}\tan(\sqrt{b}t) + \dot{D}\right]\sin(k_1 t). \tag{3.43}$$

To determine the constants c_1 and c_2 we need to use the conditions $I(t_1) = M$ and $\lambda(T) = 0$. The second condition can be used once we note that from (3.10), we have

$$\begin{aligned} \lambda(t) = \quad & K\left\{[k_1(c_1+V_1) - v(c_2+V_2)]\cos(k_1 t)\right. \\ & \left. - [k_1(c_2+V_2) + v(c_1+V_1)]\sin(k_1 t) - \hat{P}\right\} + \tfrac{h\hat{I}v}{k_1^2}, \end{aligned}$$

where $v(t)$ is given by (3.37) and $V_1(t)$ and $V_2(t)$ satisfy (3.42) and (3.43), respectively.

4 Conclusion

To summarize, we have considered in this paper a production inventory model for both ameliorating and deteriorating items. Instead of the usual optimization approach found in the literature, we have used a more general optimal control approach to the problem. We assumed that the production facility sets an inventory goal level and a production goal rate and penalties are incurred when the inventory level and the production rate deviate from their respective goals. The necessary optimality conditions are derived using Pontryagin maximum principle. In order to obtain the optimal production rate, nonhomogeneous differential equations with variable coefficients need to be solved. Except in some very special cases, these equations need to be solved numerically. This paper generalizes some of the models available in the literature but it can be further generalized by incorporating other features known in inventory theory. Another direction of research would be to consider a stochastic analog of equation (2.1) in which the stochastic state equation of the model is expressed as an Itô stochastic differential equation.

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