Information geometry of the power inverse Gaussian distribution

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Abstract. The power inverse Gaussian distribution is a common distribution in reliability analysis and lifetime models. In this paper, we give the geometric structures of the power inverse Gaussian manifold from the viewpoint of information geometry. Firstly, we obtain the Fisher information matrix, Riemannian connections, Gaussian curvature of the power inverse Gaussian manifold. Then we consider the dual structure of this manifold and investigate the KullBack divergence. At last we give an immersion from the power inverse Gaussian manifold into an affine space.

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§1. Introduction

It is well known that information geometry has been successfully applied into various fields, such as image processing, statistical inference and control theory. Recently, some scholars studied the probability density function from the viewpoint of information geometry, and use the geometric metrics to give a new description to the statistical distribution (see [4],[6],[7]). Here, the parameters of the probability density function play an important role in statistical manifold and can be regard as the coordinate system of the manifold.

The power inverse Gaussian distribution parameterized by an arbitrarily fixed real number $\lambda \neq 0$ has the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\mu\sigma}\Big(\frac{x}{\mu}\Big)^{-(2+\lambda/2)} \exp\Big\{-\frac{1}{2\lambda^2\sigma^2}\Big(\Big(\frac{x}{\mu}\Big)^{\lambda/2} - \Big(\frac{x}{\mu}\Big)^{-(\lambda/2)}\Big)^2\Big\}, \tag{1.1}$$

where $0 < x < \infty, 0 < \mu < \infty, 0 < \sigma < \infty$. In particular, for $\lambda = 1$ or $\lambda = -1$, the distribution becomes the inverse Gaussian distributions and the reciprocal inverse Gaussian distributions, respectively. Also when $\lambda \to 0$, (1.1) reduces to a log-normal distribution.

In the present paper, we consider the geometric structure of the power inverse Gaussian manifold. Firstly, we give the Fisher information matrix, the Riemannian

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connections, the Gaussian curvature under the coordinate system (μ, σ). Whilst we can see that the power inverse Gaussian distribution is an exponential family distribution, and the corresponding manifold is ± 1-flat, so we construct the dual structure of the power inverse Gaussian manifold. Secondly, we give the Kullback divergence in this manifold, and it is a distance like measure not satisfying the distance axiom. Furthermore we give the relation of Kullback divergence and arc length. At last, we give an immersion from the power inverse Gaussian manifold into the affine space.

§2. The geometric structures of the power inverse Gaussian manifold

Definition 2.1. The set

$$S = \Big\{ f(x;\xi) | \xi = (\xi^1, \xi^2) = (\mu, \sigma) \in R^+ \times R^+ \Big\}, \tag{2.1}$$

is called the power inverse Gaussian manifold, where $f(x;\xi)$ is in the form of (1.1), and $\xi = (\xi^1, \xi^2) = (\mu, \sigma)$ plays the role of the coordinate system.

By a straightforward calculation, we get

$$\begin{aligned} E_f\Big[\Big(\frac{x}{\mu}\Big)^{\lambda} - \Big(\frac{x}{\mu}\Big)^{-\lambda}\Big] &= -\lambda^2\sigma^2, \\ E_f\Big[\Big(\frac{x}{\mu}\Big)^{\lambda} + \Big(\frac{x}{\mu}\Big)^{-\lambda}\Big] &= 2 + \lambda^2\sigma^2, \\ E_f\Big[\Big(\Big(\frac{x}{\mu}\Big)^{\lambda/2} \Big(\frac{x}{\mu}\Big)^{-\lambda/2}\Big)^2\Big] &= \lambda^2\sigma^2. \end{aligned} \tag{2.2}$$

Using

$$g_{ij} = \int \frac{\partial \log f(x)}{\partial \xi^i} \frac{\partial \log f(x)}{\partial \xi^j} f(x) dx, \tag{2.3}$$

and (2.2), we get the Fisher information matrix with respect to the coordinate system (μ, σ),

$$(g_{ij}(\xi)) = \begin{pmatrix} \frac{2+\lambda^2\sigma^2}{2\mu^2\sigma^2} & -\frac{\lambda}{\mu\sigma} \\ -\frac{\lambda}{\mu\sigma} & \frac{2}{\sigma^2} \end{pmatrix}, \tag{2.4}$$

immediately the square of the element of the arc length is given by

$$ds^2 = \frac{2+\lambda^2\sigma^2}{2\mu^2\sigma^2} d\mu^2 - \frac{2\lambda}{\mu\sigma} d\mu d\sigma + \frac{2}{\sigma^2} d\sigma^2. \tag{2.5}$$

From (2.4), we can get the inverse of the Fisher information matrix

$$(g^{ij}(\xi)) = \begin{pmatrix} \mu^2\sigma^2 & \frac{\lambda\mu\sigma^3}{2} \\ \frac{\lambda\mu\sigma^3}{2} & \frac{\sigma^2(2+\lambda^2\sigma^2)}{4} \end{pmatrix}, \tag{2.6}$$

Combining

$$\Gamma_{ijk} = \frac{1}{2}(\partial_1 g_{jk} + \partial_j g_{ki} - \partial_k g_{ij})$$

with (2.4), we obtain the Riemannian connections

$$\begin{aligned} &\Gamma_{111} = -\frac{2+\lambda^2\sigma^2}{2\sigma^2\mu^3}, \quad \Gamma_{112} = \frac{\lambda\sigma^2+1}{\mu^2\sigma^3}, \\ &\Gamma_{121} = -\frac{1}{\mu^2\sigma^3}, \quad \Gamma_{122} = 0, \\ &\Gamma_{221} = \frac{\lambda}{\mu\sigma^2}, \quad \Gamma_{222} = -\frac{2}{\sigma^3}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\Gamma^1_{11} = \frac{\lambda-2}{2\mu}, \quad \Gamma^2_{11} = \frac{\lambda^2\sigma^2+2}{4\mu^2\sigma}, \\ &\Gamma^1_{12} = -\frac{1}{\sigma}, \quad \Gamma^2_{12} = -\frac{\lambda}{2\mu}, \\ &\Gamma^1_{22} = 0, \quad \Gamma^2_{22} = -\frac{1}{\sigma}. \end{aligned} \tag{2.8}$$

Combining

$$R_{ijkl} = (\partial_j \Gamma^s_{ik} - \partial_i \Gamma^s_{jk})g_{sl} + (\Gamma_{jtl}\Gamma^t_{ik} - \Gamma_{itl}\Gamma^t_{jk}),$$

(2.4) with (2.8), by a calculation, we get the nonzero component of the Riemannian curvature tensor

$$R_{1212} = -\frac{1}{\mu^2\sigma^4}. \tag{2.9}$$

Theorem 2.1. *The Gaussian curvature of the power inverse Gaussian manifold is given by*

$$K = -\frac{1}{2}. \tag{2.10}$$

Proof. Since the determinant of the Fisher information matrix is

$$\det(g_{ij}(\xi)) = \frac{2}{\mu^2\sigma^4},$$

and the Gaussian curvature is defined by

$$K = \frac{R_{1212}}{\det(g_{ij})},$$

from (2.9), we obtain Theorem 2.1. □

§3. The dual structures of the power inverse Gaussian manifold

Proposition 3.1. *The power inverse Gaussian distribution is an exponential family distribution.*

Proof. The power inverse Gaussian probability density function (1.1) can be rewritten as

$$
\begin{aligned}
f(x) = \exp\Big\{ & -\frac{1}{2\lambda^2\sigma^2\mu^\lambda}x^\lambda - \frac{\mu^\lambda}{2\lambda^2\sigma^2}x^{-\lambda} - (1+\frac{\lambda}{2})\ln x \\
& - \Big(\ln\sigma - \frac{1}{\lambda^2\sigma^2} - \frac{\lambda}{2}\ln\mu + \frac{1}{2}\ln 2\pi\Big)\Big\}.
\end{aligned} \tag{3.1}
$$

Set

$$
\begin{aligned}
y_1 &= -\frac{x^\lambda}{2\lambda^2}, \qquad \theta_1 = \frac{1}{\sigma^2\mu^\lambda}, \\
y_2 &= -\frac{x^{-\lambda}}{2\lambda^2}, \qquad \theta_2 = \frac{\mu^\lambda}{\sigma^2}, \\
M(y) &= -(\frac{1}{4} + \frac{1}{2\lambda})\ln(\frac{y_1}{y_2}),
\end{aligned} \tag{3.2}
$$

then the potential function $\psi(\theta)$ can be written as

$$
\psi(\theta) = -\frac{\sqrt{\theta_1\theta_2}}{\lambda^2} - \frac{1}{2}\ln\theta_2 + \frac{1}{2}\ln 2\pi. \tag{3.3}
$$

That is, the power inverse Gaussian probability density function can be denoted as

$$
f(y) = \exp\{y_1\theta_1 + y_2\theta_2 - \psi(\theta) + M(y)\},
$$

so it is an exponential family distribution([1]). □

Remark: The power inverse Gaussian manifold is ±1-flat.

Proposition 3.2.

i) $\theta = (\theta_1, \theta_2) = \Big(\frac{1}{\sigma^2\mu^\lambda}, \frac{\mu^\lambda}{\sigma^2}\Big)$, *is the natural coordinate system of the power inverse Gaussian manifold.*

ii) $\psi(\theta) = -\frac{\sqrt{\theta_1\theta_2}}{\lambda^2} - \frac{1}{2}\ln\theta_2 + \frac{1}{2}\ln 2\pi$, *is the potential function with respect to natural coordinate system* θ.

iii) $\eta = (\eta^1, \eta^2) = \Big(-\frac{\mu^\lambda}{2\lambda^2}, -\frac{1+\lambda^2\sigma^2}{2\lambda^2\mu^\lambda}\Big)$, *is the dual coordinate system, and is also called expectation coordinate system.*

iv) $\phi(\eta) = -\frac{1}{2}\ln(4\lambda^2\eta^1\eta^2 - 1) + \frac{1}{2}\ln(-\eta^1) + 2\ln\lambda - \frac{1}{2}\ln\pi e$, *is the dual potential function with respect to the expectation coordinate system.*

Proof. The proof of the (i) and (ii) can be seen in the proof of Proposition 3.1. Now, let us give the proof of (iii) and (iv).

From

$$\eta^i = \frac{\partial \psi(\theta)}{\partial \theta_i},$$

we get the expectation coordinate system of the power inverse Gaussian manifold

$$(\eta^1, \eta^2) = \Big(-\frac{\mu^\lambda}{2\lambda^2}, -\frac{1+\lambda^2\sigma^2}{2\lambda^2\mu^\lambda} \Big).$$

The dual potential function with respect to the expectation coordinate system is

$$\phi(\eta) = \Sigma\theta_i\eta^i - \psi(\theta) = -\frac{1}{2}\ln(4\lambda^2\eta^1\eta^2 - 1) + \frac{1}{2}\ln(-\eta^1) + 2\ln\lambda - \frac{1}{2}\ln \pi e. \qquad \square$$

Under natural coordinate system, the related geometric metrics can be given by

$$\begin{aligned} g_{ij}(\theta) &= \partial_i\partial_j\psi(\theta), \\ T_{ijk}(\theta) &= \partial_i\partial_j\partial_k\psi(\theta) = \partial_k(g_{ij}), \\ \Gamma^{(\alpha)}_{ijk}(\theta) &= \frac{1-\alpha}{2}T_{ijk}(\theta), \\ R^{(\alpha)}_{ijkl} &= \frac{1-\alpha^2}{4}(T_{kmi}T_{jln} - T_{kmj}T_{iln})g^{mn}. \end{aligned} \tag{3.4}$$

Then, by a direct calculation, we get the following geometric metrics of the power inverse Gaussian manifold with respect to the coordinate system (θ_1, θ_2),

$$(g_{ij}(\theta)) = \begin{pmatrix} \frac{\sigma^2\mu^{2\lambda}}{4\lambda^2} & -\frac{\sigma^2}{4\lambda^2} \\ -\frac{\sigma^2}{4\lambda^2} & \frac{2\lambda^2\sigma^4+\sigma^2}{4\lambda^2\mu^{2\lambda}} \end{pmatrix}, \tag{3.5}$$

$$(g^{ij}(\theta)) = \begin{pmatrix} \frac{4\lambda^2\sigma^2+2}{\sigma^4\mu^{2\lambda}} & \frac{2}{\sigma^4} \\ \frac{2}{\sigma^4} & \frac{2\mu^{2\lambda}}{\sigma^4} \end{pmatrix}, \tag{3.6}$$

$$\begin{aligned} T_{111} &= -\frac{3\mu^{3\lambda}\sigma^4}{8\lambda^2}, \\ T_{112} = T_{121} = T_{211} &= \frac{\mu^\lambda\sigma^4}{8\lambda^2}, \\ T_{122} = T_{212} = T_{221} &= \frac{\sigma^4}{8\lambda^2\mu^\lambda}, \\ T_{222} &= -\frac{8\lambda^2\sigma^6 + 3\sigma^4}{8\lambda^2\mu^{3\lambda}}. \end{aligned} \tag{3.7}$$

and

$$
\begin{aligned}
\Gamma^{(\alpha)}_{111} &= -\frac{1-\alpha}{2}\frac{3\mu^{3\lambda}\sigma^4}{8\lambda^2},\\
\Gamma^{(\alpha)}_{112} = \Gamma^{(\alpha)}_{121} = \Gamma^{(\alpha)}_{211} &= \frac{1-\alpha}{2}\frac{\mu^{\lambda}\sigma^4}{8\lambda^2},\\
\Gamma^{(\alpha)}_{122} = \Gamma^{(\alpha)}_{212} = \Gamma^{(\alpha)}_{221} &= \frac{1-\alpha}{2}\frac{\sigma^4}{8\lambda^2\mu^{\lambda}},\\
\Gamma^{(\alpha)}_{222} &= -\frac{1-\alpha}{2}\frac{8\lambda^2\sigma^6+3\sigma^4}{8\lambda^2\mu^{3\lambda}}.
\end{aligned}
\tag{3.8}
$$

From the above, we obtain the nonzero component of the curvature tensor with respect to the natural coordinate system,

$$
R^{(\alpha)}_{1212} = \frac{(1-\alpha^2)\sigma^6}{16\lambda^2},
\tag{3.9}
$$

and the α-Gaussian curvature is

$$
K^{(\alpha)} = -\frac{1-\alpha^2}{2}.
\tag{3.10}
$$

When $\alpha = 0$, it is the Riemannian case, and $K = -\frac{1}{2}$, it is the same as (2.10).

§4. The Kullback divergence

Now, let us give a new distance like measure which is different from the arc length, Kullback divergence. It plays an important role in statistical manifold.

Proposition 4.1. *Let P and Q be two points in the power inverse Gaussian manifold with coordinates (μ_P, σ_P) and (μ_Q, σ_Q), respectively. Then the Kullback divergence between P and Q is given by*

$$
D(P,Q) = \frac{\lambda}{2}\ln\frac{\mu_Q}{\mu_P} + \ln\frac{\sigma_P}{\sigma_Q} + \frac{1}{2\lambda^2\sigma_P^2}\Big(\frac{\mu_Q^{\lambda}}{\mu_P^{\lambda}} + \frac{\mu_P^{\lambda}}{\mu_Q^{\lambda}}\Big) + \frac{1}{2}\frac{\sigma_Q^2}{\sigma_P^2}\frac{\mu_P^{\lambda}}{\mu_Q^{\lambda}} - \frac{1}{\lambda^2\sigma_P^2} - \frac{1}{2}.
\tag{4.1}
$$

Proof. From the dual structure we construct above, we can get the Kullback divergence between P and Q

$$
D(P,Q) = \psi(\theta_P) + \phi(\eta_Q) - \theta_P \cdot \eta_Q.
\tag{4.2}
$$

The natural coordinate system θ_P, the potential function $\psi(\theta_P)$ of the point P, the expectation coordinate system η_Q and the dual potential function $\phi(\eta_Q)$ of the point Q are given by

$$
\begin{aligned}
&\theta_P = (\theta_{1P}, \theta_{2P}) = \Big(\frac{1}{\sigma_P^2 \mu_P^\lambda}, \frac{\mu_P^\lambda}{\sigma_P^2}\Big),\\
&\psi(\theta_P) = -\frac{\sqrt{\theta_{1P}\theta_{2P}}}{\lambda^2} - \frac{1}{2}\ln\theta_{2P} + \frac{1}{2}\ln 2\pi,\\
&\eta_Q = (\eta_Q^1, \eta_Q^2) = \Big(-\frac{\mu_Q^\lambda}{2\lambda^2}, -\frac{1+\lambda^2\sigma_Q^2}{2\lambda^2\mu_Q^\lambda}\Big),\\
&\phi(\eta_Q) = -\frac{1}{2}\ln(4\lambda^2\eta_Q^1\eta_Q^2 - 1) + \frac{1}{2}\ln(-\eta_Q^1) + 2\ln\lambda - \frac{1}{2}\ln\pi e,
\end{aligned}
\tag{4.3}
$$

respectively.

Combining (4.2) with (4.3), by a computation, we get Proposition 3.3. □

Now, let us consider some special cases:

i) When the parameter $\theta^1 = \mu$ is fixed, i.e., $\mu_P = \mu_Q$, we have

$$D(P,Q) = \ln\frac{\sigma_P}{\sigma_Q} + \frac{\sigma_Q^2}{2\sigma_P^2} - \frac{1}{2}. \tag{4.4}$$

From (4.4), we see that when μ is fixed, the Kullback divergence doesn't depend on μ.

ii) When the parameter $\theta^2 = \sigma$ is fixed, i.e., $\sigma_P = \sigma_Q$, we have

$$D(P,Q) = \frac{\lambda}{2}\ln\frac{\mu_Q}{\mu_P} + \frac{1}{2\lambda^2\sigma_P^2}\Big(\frac{\mu_Q^\lambda}{\mu_P^\lambda} + \frac{\mu_P^\lambda}{\mu_Q^\lambda}\Big) + \frac{1}{2}\frac{\mu_P^\lambda}{\mu_Q^\lambda} - \frac{1}{\lambda^2\sigma_P^2} - \frac{1}{2}. \tag{4.5}$$

Next, using (2.5), we obtain the arc length in the 1-dimensional parameter space from the point $P(\mu_P, \sigma_P)$ to $Q(\mu_Q, \sigma_Q)$ as follows:

i) When $\xi^1 = \mu$ is fixed, and $\xi^2 = \sigma$ is free, we get

$$S_\mu = \int_{\xi_P}^{\xi_Q} \frac{\sqrt{2}}{\sigma} d\sigma = \sqrt{2}\left|\ln\frac{\sigma_Q}{\sigma_P}\right|. \tag{4.6}$$

ii) When $\xi^2 = \sigma$ is fixed, and $\xi^1 = \mu$ is free, we get

$$S_\sigma = \int_{\xi_P}^{\xi_Q} \sqrt{\frac{2+\lambda^2\sigma^2}{2\sigma^2}}\frac{d\mu}{\mu} = \sqrt{\frac{2+\lambda^2\sigma^2}{2\sigma^2}}\left|\ln\frac{\mu_Q}{\mu_P}\right|. \tag{4.7}$$

Then from the above results, we get

Proposition 4.2. *Let P and Q be two points in the power inverse Gaussian manifold with coordinates (μ_P, σ_P) and (μ_Q, σ_Q) respectively. Then the Kullback divergence D and the arc length S are connected in the following ways*

i) when μ is fixed,

$$D_\mu = \frac{\sqrt{2}}{S_\mu} + \frac{1}{2}e^{\sqrt{2}S_\mu} - \frac{1}{2}. \tag{4.8}$$

The relation between S_μ and D_μ can be seen in Figure 1 and 2.

ii) when σ is fixed,

(4.9)

$$D_\sigma = \frac{\lambda\sigma}{\sqrt{4+2\lambda^2\sigma^2}} S_\sigma + \frac{1}{2\lambda^2\sigma^2}\left(e^{\frac{\sqrt{2}\lambda\sigma S_\sigma}{\sqrt{2+\lambda^2\sigma^2}}} + e^{-\frac{\sqrt{2}\lambda\sigma S_\sigma}{\sqrt{2+\lambda^2\sigma^2}}}\right) + \frac{1}{2}e^{-\frac{\sqrt{2}\lambda\sigma S_\sigma}{\sqrt{2+\lambda^2\sigma^2}}} - \frac{1}{\lambda^2\sigma^2} - \frac{1}{2}.$$

The relation between S_σ and D_σ can be seen in Figure 3.

§5. Affine immersions

Let M be an m-dimensional manifold, f be an immersion from M to $\mathbb{R}^{m+1}$, and ξ be a transversal vector field along f. We can identify $T_x R^{m+1} \equiv \mathbb{R}^{m+1}$ for $\forall x \in R^{m+1}$. The pair $\{f, \xi\}$ is said to be an affine immersion from M to $\mathbb{R}^{m+1}$, if for each point $P \in M$, the following formula holds

$$T_{f(P)} R^{m+1} = f_*(T_P M) \oplus span\xi_P. \tag{5.1}$$

We denote the standard flat affine connection of $\mathbb{R}^{m+1}$ with D. Identifying the covariant derivative along f with D, we have the following decompositions

$$\begin{aligned} D_X f_* Y &= f_*(\nabla_X Y) + h(X,Y)\xi, \\ D_X \xi &= -f_*(Sh(X)) + \tau(X)\xi. \end{aligned} \tag{5.2}$$

The induced objects ∇, h, Sh and τ are the induced connection, the affine fundamental form, the affine shape operator and the transversal connection form, respectively.

From the fundamental concepts of the affine immersions, we have the following Proposition of the power inverse Gaussian manifold.

Proposition 5.1. *Since (S, h, ∇, ∇^*) is the inverse Guassian manifold, it is dually flat space with a global coordinate system. θ is an affine coordinate system of ∇, and ψ is a θ-potential function. Then the power inverse Gaussian manifold (S, h, ∇) can be immersed into $\mathbb{R}^3$ by the following way*

$$\begin{aligned} &f : S \to \mathbb{R}^3 \\ &\begin{bmatrix} \frac{1}{\sigma^2\mu^\lambda} \\ \frac{\mu^\lambda}{\sigma^2} \end{bmatrix} \to \begin{bmatrix} \frac{1}{\sigma^2\mu^\lambda} \\ \frac{\mu^\lambda}{\sigma^2} \\ -\frac{\sqrt{\theta_1\theta_2}}{\lambda^2} - \frac{1}{2}\ln\theta_2 + \frac{1}{2}\ln 2\pi \end{bmatrix}, \end{aligned} \tag{5.3}$$

which is called a graph immersion from S into $\mathbb{R}^3$. And the transversal vector $\xi = (0,0,1)'$.

6 Figures.

By choosing suitable constants, we draw the pictures below.

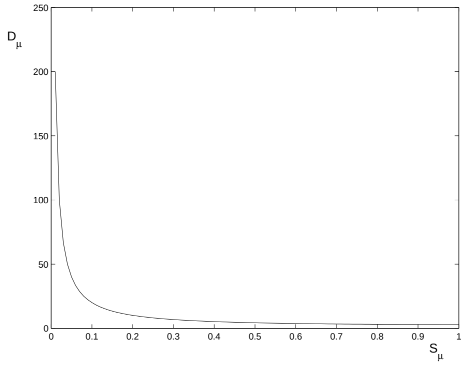

Fig 1. $D_\mu = \frac{\sqrt{2}}{S_\mu} + \frac{1}{2}e^{\sqrt{2}S_\mu} - \frac{1}{2}, \quad 0 < S_\mu < 1.$

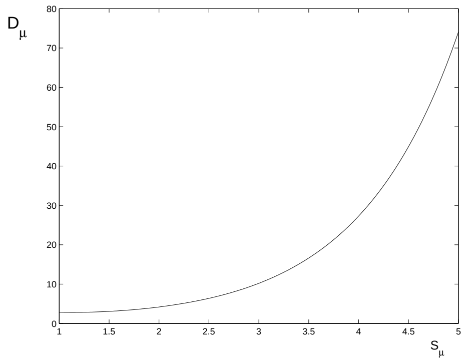

Fig 2. $D_\mu = \frac{\sqrt{2}}{S_\mu} + \frac{1}{2}e^{\sqrt{2}S_\mu} - \frac{1}{2}, \quad S_\mu > 1.$

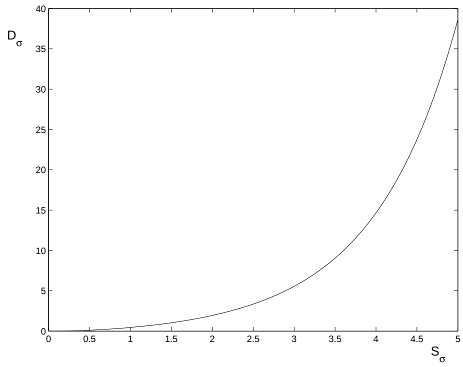

Fig 3. Setting $\lambda = 1$, and $\sigma = \sqrt{2}$, (4.9) becomes to $D_\sigma = \frac{S_\sigma}{2} + \frac{1}{4}e^{S_\sigma} + \frac{3}{4}e^{-S_\sigma} - 1, \ \ S_\sigma > 0.$

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