FK Spaces, their duals and the visualisation of neighbourhoods

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Abstract. In this paper, we demonstrate how our software package for visualisation and animations in mathematics can be applied to the representation of neighbourhoods and weak neighbourhoods in certain topologies that arise in the theory of FK spaces and matrix transformations. We also prove some new results that reduce the determination of the β -duals of matrix domains X_{Δ} of the difference operator Δ in FK spaces X with AK to that of the β -dual of X itself.

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1 Introduction

Visualisation and animations are of vital importance in modern mathematical education. They strongly support the understanding of mathematical concepts. We think that the application of most conventional software packages is neither a satisfactory approach for illustrating theoretical concepts nor can it be used as their substitute. The emphasis in the academic mathematical education should be put on teaching the underlying theories.

Thus we developed our own software package ([15, 13, 14, 16]) in Borland PAS-CAL and DELPHI to create our graphics for visualisation and animations, mainly of the results from classical differential geometry. It has applications to physics, crystallography and the engineering sciences. Since the source files are available to the users, it can and has been extended to applications in physics, chemistry, crystallography ([3, 4]), and the engineering sciences. It also has various applications in research.

Our graphics can be exported to several formats such as BMP, PS, PLT, SCR (screen files under DOS), or GCLC, the Geometry Constructions Language Converter developed at Belgrade University ([1, 7], or for further information, url http://www.matfbg.ac.yu/ janicic/gclc). These formats can be converted to a number of other formats by means of any graphics converter software, for instance in Corel Draw to a CDR or GIF file, an EPS file to be included in a $T_{\rm EX}$ or LATEX

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Figure 1: The intersection of a catenoid and a sphere

Animation: href Animat/PRA_00.htmlIntersections

file, or a PNG or PDF file to be included in a T_EX or IAT_EX file which is directly converted into a PDF file by means of PDFIATEX.

We use the software packages Animagic GIF 32 and Image Magic to create an animation in animated GIF format from a number of GIF files of our graphics, and include the animation as an animated GIF image in an HTML file to obtain for instance the following animations

> href Animat/PRA_01.htmlAn isometric map href Animat/PRA_02.htmlAn area preserving map

We emphasize that all the graphics and animations in this paper were created with our software package, and then processed in the way described above; we did not use any other software package.

2 Relative, sup and weak topologies

Before we deal with the proof of the mathematical results that motivated the study of most of the considered topologies, and the representation of neighbourhoods in topologies and weak topologies, we recall a few basic results and definitions.

There are many ways to introduce topologies on a set. A standard way to introduce a topology on a subset of a topological space is to use the relative topology. Sup topologies and their special cases, weak and product topologies, can be used to introduce topologies on sets in a more general case.

Let S be a subset of a topological space (X, \mathcal{T}) . Then the relative topology \mathcal{T}_S of X on S is given by $\mathcal{T}_S = \{O \cap S : O \in \mathcal{T}\}$ (Figures 1 and 2).



Figure 2: The relative topology of the Euclidean metric on a pseudo-sphere

Let (X, \mathcal{T}) be a topological space. A subbase for \mathcal{T} is a collection $\Sigma \subset \mathcal{T}$ such that, for every $x \in X$ and every neighbourhood N of x, there exists a finite subset $\{S_1, \ldots, S_n\} \subset \Sigma$ with $x \in \bigcap_{k=1}^n S_k \subset N$. If $X \neq \emptyset$ and Σ is a collection of sets with $\bigcup \Sigma = X$, then there is a unique topology \mathcal{T}_{Σ} which has Σ as a subbase; \mathcal{T}_{Σ} is the weakest topology with $\Sigma \subset \mathcal{T}_{\Sigma}$, and is called the topology generated by Σ . It consists of \emptyset , X and all unions of finite intersections of members of Σ .

If a set X is given a nonempty collection Φ of topologies and $\Sigma = \{\bigcup \mathcal{T} : \mathcal{T} \in \Phi\}$, then the topology $\bigvee \Phi = \mathcal{T}_{\Sigma}$ is called the sup-topology of Φ ; it is stronger than each $\mathcal{T} \in \Phi$. If X has a countable collection $\{d_n : n \in \mathbb{N}\}$ of semimetrics, then the sup-topology, denoted by $\bigvee d_n$ is semimetrizable, and given by the semimetric

(2.1)
$$d = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{d_n}{1+d_n};$$

if the collection is finite, then $d = \sum d_n$ may be used instead.

Let X be a set, (Y, \mathcal{T}) be a topological space and $g : X \to Y$ be a map. Then $w(X,g) = \bigvee \{g^{-1}(O) : O \in \mathcal{T}\}$ is a topology for X, called the weak topology by g. The map $g : (X, w(X,g)) \to (Y,\mathcal{T})$ is continuous and w(X,g) is the weakest topology on X for which this is true. If $\Sigma(Y)$ is a subbase for \mathcal{T} , then $\Sigma = \{g^{-1}(G) : G \in \Sigma(Y)\}$ is a subbase for w(X,g). If the topology of Y is metrizable and given by the metric d, we may use the concept of the weak topology by g to define a semimetric δ on X by

$$\delta = d \circ g.$$

which is a metric whenever g is one-to-one. A neighbourhood $U_{\delta}(x_0, r)$ of a point x_0 with respect to the weak topology by g is thus given by

$$U_{\delta}(x_0, r) = \{x \in X : \delta(x, x_0) < r\} = \{x \in X : d(g(x), g(x_0)) < r\}$$

More generally, let X be a set, Ψ be a collection of topological spaces, and for each $Y \in \Psi$, we assume given one or more functions $g: X \to Y$. Let the collection

of all these functions be denoted by Φ . Then the topology $\bigvee \{w(X, f) : f \in \Phi\}$ is called the weak topology by Φ , and denoted by $w(X, \Phi)$. Each $f \in \Phi$ is continuous on $(X, w(X, \Phi))$ and $w(X, \Phi)$ is the weakest topology on X such that this is true. If $\Sigma(Y)$ is a subbase for the topology of Y for each $Y \in \Psi$, then $\Sigma = \{f^{-1}(G) : f \in \Phi, f : X \to Y, G \in \Sigma(Y)\}$ generates $w(X, \Phi)$. The weak topology by a sequence (f_n) of maps from a set X to a collection of semimetric spaces is semimetrizable.

The product topology for a product of topological spaces simply is the weak topology by the family of all projections from the product to the factors.

Example 2.1. Let $B = \mathbb{I}N_0$ and $A_n = (\mathbb{C}, |\cdot|)$ for all $n \in \mathbb{I}N_0$ where $|\cdot|$ is the absolute value on the set \mathbb{C} of complex numbers. Then the product $\omega = \mathbb{C}^{\mathbb{I}N_0}$ is the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Its product topology is given by the semimetric

(2.3)
$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x, y \in \omega.$$

If we define the sum and the multiplication by a scalar in the natural way by

$$x + y = (x_k + y_k)_{k=0}^{\infty} \text{ and } \lambda x = (\lambda x_k)_{k=0}^{\infty} (x, y \in \omega; \lambda \in \mathbb{C}),$$

then (ω, d) is a Fréchet space, that is a complete linear metric space, and convergence in (ω, d) and coordinatewise convergence are equivalent; this means $x^{(n)} \to x$ $(n \to \infty)$ if and only if $x_k^{(n)} \to x_k$ $(n \to \infty)$ for every k ([18, Theorem 4.1.1, p. 54]).

3 Certain metrizable linear topological spaces and their dual spaces

Here we consider certain sets of sequences with a metrizable linear topology and their dual spaces. Their common property is that they are continuously embedded in the Fréchet space (ω, d) of Example 2.1.

We write ℓ_{∞} , c, c_0 and ϕ , and bs, cs and ℓ_1 for the sets of all bounded, convergent, null and finite sequences, and for the sets of all bounded, convergent, and absolutely convergent series, and $\ell_p = \{x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for 0 . As usual, <math>e and $e^{(n)}$ $(n \in \mathbb{N}_0)$ are the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Given a sequence $x = (x_k)_{k=0}^{\infty} \in \omega$ and $m \in \mathbb{N}_0$, we write $x^{[m]} = \sum_{n=0}^{m} x_k e^{(k)}$ for the m-section of x.

An *FK* space X is a Fréchet subspace of ω which has continuous coordinates $P_n : X \to \mathbb{C}$ (n = 0, 1, ...) where $P_n(x) = x_n$. An *FK* space $X \supset \phi$ is said to have *AK*, if $x = \lim_{m \to \infty} x^{[m]}$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$. A *BK* space is a normed *FK* space.

The following remark is for the benefit of the interested reader who may not be too familiar with the concept of FK and BK spaces.

Remark 3.1. (a) The letters F, B and K in FK and BK space stand for Fréchet, Banach and Koordinate, the German word for coordinate; AK stands for Abschnittskonvergenz, German for sectional convergence.

(b) The concept of an FK space is fairly general. An example of a Fréchet sequence

space which is not an FK space can be found in [18, Problem 11.3.3, p. 205 and Example 7.5.6, p. 113].

(c) The great importance of FK and BK spaces in the theory of matrix transformations comes from the fact that matrix mappings between FK spaces are continuous ([19, Corollary 11.3.5, p. 204] or [19, Theorem 4.2.8, p. 57]).

(d) The FK topology of an FK space is unique; more precisely, if X and Y are FK spaces with $X \subset Y$, then the topology of X is stronger than that of Y, and they are equal if and only if X is a closed subspace of Y. This means there is at most one way to make a subspace of ω into an FK space ([19, Corollary 4.2.4, p. 56]).

(e) Every Fréchet space with a Schauder basis is congruent to an FK space ([19, Corollary 11.4.1, p. 208]).

Example 3.2. (a) The space (ω, d) with the metric in (2.3) is a locally convex FK space with AK; ϕ has no Fréchet topology ([19, 4.0.2, 4.0.5, p. 51]).

(b) Let $p = (p_k)_{k=0}^{\infty}$ be a positive bounded sequence with $H = \sup_k p_k$. We put $M = \max\{1, H\}$. Then the sets

$$\ell(p) = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^{p_k} < \infty \right\} \text{ and } c_0(p) = \left\{ x \in \omega : \lim_{k \to \infty} |x_k|^{p_k} = 0 \right\}$$

are FK spaces with AK with respect to their natural metrics

$$d_{(p)}(x,y) = \left(\sum_{k=0}^{\infty} |x_k - y_k|^{p_k}\right)^{1/M} \text{ and } d_{0,(p)}(x,y) = \left(\sup_k |x_k - y_k|^{p_k}\right)^{1/M}$$

([8, Theorem 1], [9, p.318] and [11, Theorem 2]). In $\ell_{\infty}(p) = \{x \in \omega : \sup_{k} |x_{k}|^{p_{k}} < 0\}$ ∞ and $c(p) = \{x \in \omega : x - \ell e \in c_0(p) \text{ for some } \ell \in \mathbb{C}\}, d_{0,(p)} \text{ is a linear metric only}$ in the trivial case $\inf_k p_k > 0$, when $\ell_{\infty}(p) = \ell_{\infty}$ and c(p) = c ([17, Theorem 9]). FK metrics for $\ell_{\infty}(p)$ and c(p) using the concepts of co-echelon spaces and the inductive limit topology were given in [2].

(c) The spaces ℓ_p for $1 \leq p < \infty$ are BK spaces with $||x||_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, and p - normed FK spaces for $0 with <math>||x|| = \sum_{k=0}^{\infty} |x_k|^p$ in which case the corresponding topology is not localy convex; c_0 , c and ℓ_{∞} are BK spaces with $||x||_{\infty} = \sup_{k} |x_{k}|, \ell_{p} \text{ and } c_{0} \text{ have } AK, \text{ and } c \text{ and } c_{0} \text{ are closed subspaces of } \ell_{\infty}.$

If X is a linear metric space, the set of all continuous linear functionals on X is denoted by X'; if X is a normed space, we write X^{*} for X' with the norm ||f|| = $\sup_{x \in B_X} |f(x)| \ (f \in X')$ where B_X denotes the closed unit ball in X.

If x and y are sequences, and X and Y are subsets of ω , then we write xy = $(x_k y_k)_{k=0}^{\infty}, x^{-1} * Y = \{a \in \omega : ax \in X\} \text{ and } M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : x \in X\}$ $ax \in Y$ for all $x \in X$ for the multiplier space of X and Y. We use the notations $x^{\alpha} = x^{-1} * \ell_1, x^{\beta} = x^{-1} * cs$ and $x^{\gamma} = x^{-1} * bs$, and $X^{\alpha} = M(X, \ell_1), X^{\beta} = M(X, cs)$ and $X^{\gamma} = M(X, bs)$ for the α -, β - and γ -duals of X. Obviously we have $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. Also the following result holds.

Proposition 3.3. (a) If $X \supset \phi$ is an FK space with AK then $X^{\beta} = X^{\gamma}$ ([19, Theorem 7.2.7 (iii), 106]).

(b) Let X and Y be subsets of ω . If \dagger denotes any of the symbols α , β or γ , then ([19, Theorem 7.2.2, p. 105] and [5, Lemma 2])

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(i)
$$X \subset X^{\dagger\dagger}$$
, (ii) $X^{\dagger} = X^{\dagger\dagger\dagger}$, (iii) $X \subset Y$ implies $Y^{\dagger} \subset X^{\dagger}$.

If I is an arbitrary index set and $\mathcal{X} = \{X_{\iota} : \iota \in I\}$ is a family of subsets of X_{ι} of ω , then

$$(iv) (\bigcup_{\iota \in I} X_{\iota})^{\dagger} = \bigcap_{\iota \in I} X_{\iota}^{\dagger}.$$

The following well–known result shows the close relations between the β – and continuous dual of an FK space.

Proposition 3.4. (cf. [19, Theorem 7.2.9, p. 107]) Let $X \supset \phi$ be an FK space. Then $X^{\beta} \subset X'$ in the sense that each sequence $a \in X^{\beta}$ can be used to represent a function $f_a \in X'$ with $f_a(x) = \sum_{k=0}^{\infty} a_k x_k$ for all $x \in X$, and the map $T : X^{\beta} \to X$ with $T(a) = f_a$ is linear and one-to-one. If X has AK, then T is an isomorphism.

The boundedness of the sequence p is not needed in Part (a) of the next example.

Example 3.5. (a) We have $\ell(p)^{\beta} = \ell_{\infty}(p)$ for $0 < p_k \le 1$ (cf. [17, Theorem 7]), and for $p_k > 1$ and $q_k = p_k/(p_k - 1)$,

$$\ell(p)^{\beta} = \mathcal{M}(p) = \bigcup_{N>1} \left\{ a \in \omega : \sum_{k=0}^{\infty} \left| \frac{a_k}{N} \right|^{q_k} < \infty \right\} \text{ (cf. [10, Theorem 1]);}$$

for all positive sequences ([10, Theorem 6], [5, Theorem 1] and [6, Theorem 2])

$$c_0(p)^{\beta} = \mathcal{M}_0(p) = \bigcup_{N>1} \left\{ a \in \omega : \sum_{k=0}^{\infty} |a_k| N^{-1/p_k} < \infty \right\}, \ c(p)^{\beta} = c_0(p) \cap cs$$

and

$$\ell_{\infty}(p)^{\beta} = \mathcal{M}_{\infty} = \bigcap_{N>1} \left\{ a \in \omega : \sum_{k=0}^{\infty} |a_k| N^{1/p_k} < \infty \right\}.$$

(b) If $1 < \inf_k p_k \le p_k \le \sup_k < \infty$ and $\ell(q)$ has its natural topology given by

$$g_{(p)}(a) = \left(\sum_{k=0}^{\infty} |a_k|^{q_k}\right)^{1/Q} \ (a \in \ell(q)), \ where \ Q = \sup_k q_k,$$

then $\ell(p)'$ and $\ell(q)$ are linearly homeomorphic ([10, Theorem 4]).

The classical special cases of the previous example are well known.

Example 3.6. We have $\ell_p^{\beta} = \ell_{\infty}$ for $0 , <math>\ell_p^{\beta} = \ell_q$ for 1 and <math>q = p/(p-1), $c_0^{\beta} = c^{\beta} = \ell_{\infty}^{\beta} = \ell_1$, $\omega^{\beta} = \phi$ and $\phi^{\beta} = \omega$. Furthermore, ℓ_p^* $(0 and <math>c_0^*$ are norm isomorphic to their β -duals, and $f \in c^*$ if and only if

$$f(x) = \chi \lim_{k \to \infty} x_k + \sum_{k=0}^{\infty} a_k x_k \text{ where } a \in \ell_1 \text{ and } \chi = \chi(f) = f(e) - \sum_{k=0}^{\infty} f(e^{(k)}),$$

and $||f|| = |\lim_{k\to\infty} x_k| + ||a||_1$ ([18, Examples 6.4.2, 6.4.3 and 6.4.4, p. 91]). Finally ℓ_{∞}^* is not isomrphic to any sequence space ([18, Example 6.4.8, p. 93]).

Given any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers and any sequence x, we write A_n for the sequence in the *n*-th row of A, $A_n(x) = \sum_{k=0}^{\infty} a_k x_k$ (n = 0, 1, ...)and $A(x) = (A_n(x))_{n=0}^{\infty}$. If X is a subset of ω then $X_A = \{x \in \omega : A(x) \in X\}$ denotes the matrix domain of A in X. Finally $(X, Y) = \{A : X_A \subset Y\}$ is the class of all matrices that map X into Y, that is $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all n, and $A(x) \in Y$ for all $x \in X$.

An infinite matrix $T = (t_{nk})_{n,k=0}^{\infty}$ is called a triangle, if $t_{nn} \neq 0$ for all n and $t_{nk} = 0$ for k > n.

The following result is well known

Proposition 3.7. ([19, Theorem 4.3.12, p. 63]) Let (X, d) be an FK space, T be a triangle and $Y = X_T$. Then (Y, d_T) is an FK space with

(3.1)
$$d_T(y, y') = d(T(y), T(y')) \text{ for all } y, y' \in Y.$$

Remark 3.8. We observe that the metric d_T in (3.1) yields the weak topology $w(Y, L_T)$ by $L_T : X_T \to X$ on $Y = X_T$, where $L_T(y) = T(y)$ for all $y \in Y$.

Now we confine ourselves to the special case where $T = \Delta$ with $\Delta_{nn} = 1$, $\Delta_{n,n-1} = -1$ and $\Delta_{n,k} = 0$ (otherwise) for $n = 0, 1, \ldots$; we use the convention that any term with a negative subscript is equal to zero. Let Σ be the matrix with $\Sigma_{nk} = 1$ $(0 \le k \le n)$ and $\Sigma_{nk} = 0$ (k > n) for all $n = 0, 1, \ldots$ Then Σ is the inverse of Δ .

If (X, d) is a linear metric space, $x_0 \in X$ and $\rho > 0$, then we denote by $S_{\rho}(x_0) = \{x \in X : d(x, x_0) \leq \rho\}$ the closed ball with radius ρ and centre in x_0 . Let X be an FK space, $X \supset \phi$ and $a \in \omega$. Then we write

$$||a||_{X,D}^* = ||a||^* = \sup_{x \in S_{1/D}(0)} \left| \sum_{k=0}^{\infty} a_k x_k \right| \text{ for } D > 0$$

provided the expression on the right is defined and finite which is the case whenever $a \in X^{\beta}$, by Proposition 3.4. If X is a *BK* space, then we write

$$||a||_X^* = ||a||^* = \sup_{x \in B_X} \left| \sum_{k=0}^\infty a_k x_k \right|.$$

Theorem 3.9. (a) If X is an FK space, then $A \in (X, \ell_{\infty})$ if and only if

(3.2)
$$M(A;D) = \sup_{n} ||A_n||_D^* < \infty \text{ for some} D > 0.$$

If X has AK then $A \in (X, c_0)$ if and only if (3.2) holds and

(3.3)
$$\lim_{n \to \infty} a_{nk} = 0 \text{ for every } k;$$

 $A \in (X, c)$ if and only if (3.2) holds and

(3.4)
$$\lim_{n \to \infty} a_{nk} = \alpha_k \text{ exists for every } k.$$

(b) If X is a BK space then (3.2) reduces to

(3.5)
$$M(A) = \|A\|^* = \sup_{x \in B_X} \|A_n\|^* < \infty.$$

Proof. (a) First we assume that (3.2) holds. Then $A_n \in x^{\beta}$ for all n and $x \in S_{1/D}$, and $A(x) \in \ell_{\infty}$ for all $x \in S_{1/D}(0)$, by the definition of $\|\cdot\|_{D}^{*}$. Since the set $S_{1/D}(0)$ is absorbing by [18, Fact (ix), p. 53], we conclude $A_n \in \bigcap_{x \in X} x^{\beta} = X^{\beta}$ for all n, and $A(x) \in X$ for all $x \in X$.

Conversely, we assume $A \in (X, Y)$. Then $L_A : X \to \ell_{\infty}$ with $L_A(x) = A(x)$ $(x \in X)$ is continuous by [19, Theorem 4.2.8, p. 57]. Hence there exist a neighbourhood N of 0 in X and a real D > 0 such that $S_{1/D}(0) \subset N$ and $||L_A||_{\infty} < 1$ for all $x \in N$. This implies (3.2).

Since c_0 and c are closed subspaces of ω (Example 3.2 (c)), and now X has AK by assumption, the characterisations of the classes (X, c_0) and (X, c) follow from the characterisation of (X, ℓ_{∞}) and [19, 8.3.6, p. 123].

(b) This is an immediate consequence of Part (a).

Theorem 3.10. Let $X \supset \phi$ be an FK space with AK, and the matrix $E = (e_{nk})_{n,k=0}^{\infty}$ be defined by $e_{nk} = 0$ for $0 \leq k \leq n-1$ and $e_{nk} = 0$ for $k \geq n$ (n = 0, 1, ...). Then $(X_{\Delta})^{\beta} = (X^{\beta} \cap M(X_{\Delta}, c_0))_{E}$. Furthermore, if $a \in (X_{\Delta})^{\beta}$, then

(3.6)
$$\sum_{k=0}^{\infty} a_k y_k = \sum_{k=0}^{\infty} E_k(a) \Delta_k(y) \text{ for all } y \in X_{\Delta}.$$

Proof. We put $Y = X_{\Delta}$ and $Z = (X^{\beta} \cap M(Y, c_0))_E$. First we assume $a \in Z$. Then $R = E(a) \in X^{\beta}$ and $R \in M(X, c_0)$. Let $y \in Y$ be given. Then $x = \Delta(y) \in X$, and Abel's summation by parts yields

(3.7)
$$\sum_{k=0}^{n} a_k y_k = \sum_{k=0}^{n+1} R_k x_k - R_{n+1} x_{n+1} \text{ for all } n = 0, 1, \dots$$

Now $R \in X^{\beta}$ and $x \in X$ yield $R \in x^{\beta}$, and $R \in M(X, c_0)$ and $y \in Y$ yield $Ry \in c_0$, hence $a \in y^{\beta}$ by (3.7). Since $y \in Y$ was arbitrary, we have $a \in \bigcap_{y \in Y} y^{\beta} = Y^{\beta}$. Conversely, we assume $a \in Y^{\beta}$. Then $a \in y^{\beta}$ for all $y \in Y$. Since $\Delta(e) = e^{(0)} \in \phi \subset X$, it follows that $e \in Y$, and so $a = ae \in cs$, hence R = E(a) is defined. If $y \in Y$, then $x = \Delta(y) \in X$ and $y = \Sigma(x)$. Now we have for all n

$$\sum_{k=0}^{n} a_k y_k = \sum_{k=0}^{n} a_k \sum_{j=0}^{k} x_j = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} a_k \right) x_j = \sum_{k=0}^{n} \left(\sum_{j=k}^{n} a_j \right) x_k.$$

We define the matrix $A = (a_{nk})_{n,k=0}^{\infty}$ by $a_{nk} = \sum_{j=k}^{n} a_j$ $(0 \le k \le n)$ and $a_{nk} = 0$ (k > n) for all $n = 0, 1, \ldots$ Then $a \in M(Y, c_0)$ implies $A \in (X, c_0)$, and so, by Theorem 3.10 (a), there are a constant C and a real D > 0 such that

$$\left|\sum_{k=0}^{n} \left(\sum_{j=k}^{n} a_{j}\right) x_{k}\right| = \left|\sum_{k=0}^{n} R^{[n]} x_{k}\right| \le C \text{ for all } n \text{ and for all } x \in S_{1/D}(0).$$

We fix $m \in \mathbb{N}_0$. Then for all $x \in S_{1/D}^{[m]}(0) = S_{1/D}(0) \cap \operatorname{span}\{e^{(0)}, e^{(1)}, \dots, e^{(m)}\}$

 $\subset S_{1/D}(0)$

$$\left|\sum_{k=0}^{n} R^{[n]} x_k\right| = \left|\sum_{k=0}^{m} R^{[n]} x_k\right| \le C \text{ for all } n \ge m.$$

Since $a \in cs$, it follows that $|\sum_{k=0}^{m} R_k x_k| = |\sum_{k=0}^{m} (\lim_{n\to\infty} R^{[n]}) x_k| \leq C$, hence $Rx \in bs$ for all $x \in S^{[m]}(0)$. Since X is FK space with AK this implies $R \in X^{\gamma} = X^{\beta}$, by Proposition 3.4 (a). Furthermore, it follows from (3.7), $a \in Y^{\beta}$ and $R \in X^{\beta}$ that $R \in M(X,c)$. We have to show that $R \in M(X,c_0)$. Now we observe that $R_n y_n = R_n \sum_{k=0}^{n} x_k$ for all n. So defining the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ by $b_{nk} = R_n$ $(0 \leq k \leq n)$ and $b_{nk} = 0$ (n > k) for all $n = 0, 1, \ldots$, we have $B \in (X, c) \subset B(X, \ell_{\infty})$, and since $\lim_{n\to\infty} b_{nk} = \lim_{n\to\infty} R_n = 0$ for every k, we obtain $B \in (X, c_0)$ by Theorem 3.9 (a), that is $R \in M(X, c_0)$.

Finally, if $a \in Y^{\beta}$, then $R \in X^{\beta}$ and $R \in M(Y, c_0)$ by the second part of the proof, and (3.6) follows from (3.7).

Remark 3.11. Since the assumption that X has AK was only needed for $X^{\beta} = X^{\gamma}$ in the converse part of the proof of Theorem 3.10, and $\ell_{\infty}^{\beta} = \ell_{\infty}^{\gamma}$, the statement of Theorem 3.10 also holds for $X = \ell_{\infty}$.

Now we apply Theorem 3.10 and Remark 3.11. We write $bv(p) = (\ell(p))_{\Delta}$ and $c_0(p)(\Delta) = (c_0(p))_{\Delta}$.

Example 3.12. Let p be a bounded positive sequence. (a) If $p_k \leq 1$ for all k, then $(bv(p))^{\beta} = cs$; if $1 < p_k$ and $q_k = p_k/(p_k - 1)$ for all k, then $a \in (bv(p))^{\beta}$ if and only if there is an integer N > 1 such that

(3.8)
$$\sum_{k=0}^{\infty} \left| \frac{1}{N} \sum_{j=k}^{\infty} a_j \right|^{q_k} < \infty \text{ and } \sup_n \sum_{k=0}^n \left| \frac{1}{N} \sum_{j=n}^{\infty} a_j \right|^{q_k} < \infty.$$

(b) We have $a \in (c_0(p)(\Delta))^{\beta}$ if and only if there is an integer N > 1 such that

(3.9)
$$\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} a_j \right| N^{-1/p_k} < \infty \text{ and } \sup_n \sum_{k=0}^n \left| \sum_{j=n}^{\infty} a_j \right| N^{-1/p_k} < \infty.$$

Proof. (a) First let $p_k \leq 1$ for all k. Then $bv(p) \subset bv = bv(e)$, and so $(bv(p))^{\beta} \supset bv^{\beta} = cs$ by Proposition 3.3 (b) (iii) and [19, Theorem 7.3.5 (iii), p. 110]. Furthermore, it follows from Theorem 3.10, that if $a \in (bv(p))^{\beta}$, then R exists, and so $a \in cs$. Now let $p_k > 1$ for all k. Then, by Theorem 3.10, $a \in (bv(p))^{\beta}$ if and only if $R = E(a) \in \ell(p)^{\beta}$ and $R \in M(\ell(p), c_0)$. We obtain from Example 3.2 (a), that $R \in \ell(p)^{\beta}$ if and only if the first condition in (3.8) is satisfied. Defining the matrix B as in the converse part of the proof of Theorem 3.10, we see that $R \in M(\ell(p), c_0)$ if and only if $B \in (\ell(p), c_0)$ which is the case by [6, Theorem 1] and [19, 8.3.6, p. 123] if and only if $\sup_n \sum_{k=0}^{\infty} |b_{nk}|^{q_k} N^{-q_k} = \sup_n \sum_{k=0}^{n} (|R_n|/N)^{q_k} < \infty$ for some N > 1, which is the second condition in (3.8), and

(3.10)
$$\lim_{n \to \infty} b_{nk} = 0 \text{ for all } k$$

which is redundant, since $R_n \to 0 \ (n \to \infty)$.

(b) Now $a \in (c_0(p)(\Delta))^{\beta}$ if and only if $R \in c_0(p)^{\beta}$ and $R \in M(c_0(p), c_0)$. We obtain from Example 3.2 (a), that $R \in c_0(p)^{\beta}$ if and only if the first condition in (3.9) is satisfied. Furthermore, $B \in (c_0(p), c_0)$ by [5, Corollary 2] and [19, 8.3.6, p. 123] if and only if $\sup_n \sum_{k=0}^{\infty} |b_{nk}| N^{-1/p_k} < \infty$ for some N > 1, which is the second condition in (3.8), and (3.10) holds which again is redundant.

Now we write $bv_p = (\ell_p)_{\Delta}$ for p > 1, q = p/(p-1), and $c_0(\Delta) = (c_0)_{\Delta}$ and $\ell_{\infty}(\Delta) = (\ell_{\infty})_{\Delta}$.

Example 3.13. (a) If p > 1, then $a \in bv_p^\beta$ if and only if $R \in \ell_q$ and $(nR_n)_{n=0}^\infty \in \ell_\infty$.

(b) We have $a \in (c_0(\Delta))^{\beta}$ if and only if $R \in \ell_1$ and $(nR_n)_{n=0}^{\infty} \in \ell_{\infty}$. (c) We have $a \in (\ell_{\infty}(\Delta))^{\beta}$ if and only if $R \in \ell_1$ and $(nR_n)_{n=0}^{\infty} \in c_0$

Proof. Parts (a) and (b) are immediate consequences of (3.8) and (3.9).

(c) We have $a \in (\ell_{\infty}(\Delta))^{\beta}$ by Remark 3.11 if and only if $R \in \ell_{\infty}^{\beta} = \ell_1$, by Example 3.6, and $R \in M(\ell_{\infty}, c_0)$, which is the case if and only if $B \in (\ell_{\infty}, c_0)$. Now $B \in (\ell_{\infty}, c_0)$ by [19, Theorems 1.7.18 and 1.7.19, pp. 15–17] if and only if $\lim_{n\to\infty} \sum_{k=0}^{\infty} |b_{nk}| = \lim_{n\to\infty} n|R_n| = 0$ for all k, which is the second condition. \Box

4 Neighbourhoods in topolgies and weak topologies

We consider \mathbb{R}^n for given $n \in \mathbb{N}$ as a subset of ω by identifying every point $X = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$ with the real sequence $x = (x_k)_{k=1}^{\infty} \in \omega$ where $x_k = 0$ for all k > n, and introduce on \mathbb{R}^n any of the metrics of Section 3.

We denote by $B_d(r, X_0) = \{X \in \mathbb{R}^n : d(X, X_0) < r\}$ the open ball in (\mathbb{R}^n, d) of radius r > 0 with its centre in X_0 , and consider the cases n = 2 and n = 3 for the graphical representation of neighbourhoods by the boundaries $\partial B_d(X_0)$ of $B_d(r, X_0)$.

4.1 Neighbourhoods in two-dimensional space

The boundaries $\partial B_d(r, X_0)$ of $B_d(r, X_0)$ in \mathbb{R}^2 are given by the zeros of a realvalued function of two variables. Although our software provides an algorithm for this ([3, 4, 12]), it is more convenient and less time consuming if we can find a parametric representation for $\partial B_d(r, X_0)$. For instance, this can be achieved for the metrics d of Example 2.1 and $d_{(p)}$ of Example 3.2 (b).

Example 4.1. (a) We consider \mathbb{R}^2 with the metric $d_{(p)}$ of Example 3.2 (a). (b) Now we represent neighbourhoods in the metric $d_{(p)\circ\Delta}$ of bv(p) and their dual neighbourhoods in the metric $d^*_{(p)\circ\Delta}$ of $(bv(p))^{\beta}$ (Example 3.6; left in Figure 3). (c) Finally, we represent neighbourhoods in the metric

$$d = \frac{d_{(p_1)}}{1 + d_{(p_2)}} + \frac{d_{(p_1)}}{1 + d_{(p_2)}} \quad ((2.1); \text{ right in Figure 3}).$$

Figure 3: Left: $\partial B_{d_{(p)\circ\Delta}}(1, X_0)$ and $\partial B^*_{d_{(p)\circ\Delta}}(1, X_0)$ for p = (1 + 4/(n+1)), 1/(4(n+1)) and (n = 0, 1, 2, 3)Right: $\partial B_{d_{(p)}}(r, X_0)$ for $p_1 = (1, 2), p_2 = (5, 4), r = n/10$ (n = 1, 2, ..., 8), and the metric d of Example 4.1 (c)



Now we represent neighbourhoods in some weak topologies. Again, it is useful to obtain, if possible, parametric representations for the boundaries of the neighbourhood.

Example 4.2. Weak neighbourhoods in the square $[-1,1]^2$ We introduce metric $\delta_{(p)} = d_{(p)} \circ g$ in the square $[-1,1]^2$ by the function $g: [-1,1]^2 \to \mathbb{R}^2$ with $g(x,y) = (\tan(x\pi/2), \tan(y\pi/2))$ (Figure 4).

Example 4.3. Weak topology on a sphere by stereographic projection Let S be the sphere of radius r with its centre in the point M = (0, 0, r) (minus the u^1 -line corresponding to $u^2 = 0$), and $E = \{(\rho, \phi) : \rho > 0 \ \phi \in (0, 2\pi)\}$ denote the xyplane in \mathbb{R}^3 (minus the positive x-axis) with the usual polar coordinates $x = \rho \cos \phi$ and $y = \rho \sin \phi$. Then the stereographic projection $sp : S \to E$ is bijective and we introduce the weak topology w(S, sp) on S (Figures 5 and 6).

4.2 Neighbourhoods in three–dimensional space

Here we consider the case when the boundaries $\partial B_d(r, X_0)$ of neighbourhoods in \mathbb{R}^3 are given by a parametric representation, as in the case of the metric $d_{(p)}$ of Example 3.2 (a). Then the principles of Subsection 4.1 can easily be extended and applied to the representation of neighbourhoods in \mathbb{R}^3 .

Acknowledgement. Work supported by the research projects #1232 and #1646 of the Serbian Ministry of Science, Technology and Environment, and *Multimedia Technolgies in Mathematics and Computer Science Education* of the German Academic Exchange Service (DAAD). Figure 4: Weak neighbourhoods $\partial B_{\delta_{(p)}}(r, X_0)$ in $[-1, 1]^2$ and corresponding neighbourhoods $\partial B_{d_{(p)}}(r, X_0)$ in \mathbb{R}^2 for p = (3, 1/8) (Example 4.1)



Animations: Weak neighbourhoods in href Animat/PRA_03.htmla square href Animat/PRA_04.htmlstrips and squares



Figure 5: The principle of the stereographic projection

Animations: Stereographic projection of a href Animat/PRA_05.html curve; a href Animat/PRA_06.html part of the sphere



Figure 6: Weak neighbourhoods by the stereographic projection

Animation: $\partial B_{d_p}(0,1)$ for href Animat/PRA_07.html
varying p





Figure 10: Left: $\partial B_{\delta_{3/2}}(0.8,0)$; Right: $\partial B_{\delta_3}(0.8,0)$ in (-1,1) with $g = (\tan(2/\pi), \tan(2/\pi), \tan(2/\pi))$



Animation: $\partial B_{\delta_{3/2}}(r,0)$ for href Animat/PRA_08.html
varying r

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