Homotopy method for solving fuzzy nonlinear equations

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Abstract. In this paper, we introduce the numerical solution for a fuzzy nonlinear systems by homotopy method. The fuzzy quantities are presented in parametric form. Some numerical illustrations are given to show the efficiency of algorithms.

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Key words: homotopy and continuation method, Rung-Kutta method, Fuzzy parametric form, Fuzzy nonlinear equations.

1 Introduction

In recent years, the homotopy method has been used by scientists. Recently, the applications of homotopy theory among scientists were appeared [13, 14, 15, 18, 19], and the homotopy theory becomes a powerful mathematical tool, when it is successfully coupled with perturbation theory, [1, 2, 16, 17].

The numerical solution of a algebraic nonlinear equation like $F(x) = 0$, arises quite often in engineering and the natural sciences. Many engineering design problems that must satisfy specified constraint as a nonlinear equalities or inequalities. One of the major applications of fuzzy number arithmetic is nonlinear equations whose parameters are all or partially represented by fuzzy numbers [4, 11, 21]. Standard analytical techniques like Buckley and Qu method, [5, 6, 7, 8], can not suitable for solving the equations such as

$(i)\quad ax^4 + bx^3 + cx^2 + dx + e = f,$

$(ii)\quad x + \cos(x) = g,$

where x, a, b, c, d, e, f and g are fuzzy numbers. We therefore need to develop the numerical methods to find the roots of these equations, in general as $F(x) = 0$. The Newton's method for solving a fuzzy nonlinear equation is considered in [3]. The advantage of the Newton's method is it's speed of convergence once a sufficiently accurate approximation is known. A weakness of this method is that an accurate initial approximation to the solution is needed to ensure convergence.

In section 2, we recall some fundamental results of fuzzy numbers. In section 3, we propose Homotopy and Continuation Method for solving fuzzy nonlinear systems. In section 4, we illustrate some examples and conclusions in the last section.

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2 Preliminaries

In this section we first present some definitions.

Definition 1. A fuzzy number is a fuzzy set like $u : \mathbb{R} \to I = [0,1]$ which satisfies, [12, 22, 23],

1. u is upper semi-continuous,
2. $u(x) = 0$ outside some interval $[c, d]$,
3. There are real numbers a, b such that $c \le a \le b \le d$ and
   - 3.1. $u(x)$ is monotonic increasing on $[c, a]$,
   - 3.2. $u(x)$ is monotonic decreasing on $[b, d]$,
   - 3.3. $u(x) = 1,\ a \le x \le b$.

The set of all these fuzzy numbers is denoted by E. An equivalent parametric is also given in [20] as follows.

Definition 2. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r),\ 0 \le r \le 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
2. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
3. $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.

Remark 1. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \le r \le 1$.

A popular fuzzy number is the triangular fuzzy number $u = (\alpha, c, \beta)$, with the membership function

$$u(x) = \begin{cases} \dfrac{x-\alpha}{c-\alpha}, & \alpha \le x \le c, \\ \dfrac{x-\beta}{c-\beta}, & c \le x \le \beta, \end{cases}$$

where $c \ne \alpha$, $c \ne \beta$ and hence

$$\underline{u}(r) = \alpha + (c-\alpha)r, \qquad \overline{u}(r) = \beta + (c-\beta)r.$$

Let $TF(\mathbb{R})$ be the set of all triangular fuzzy numbers. The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented as follows.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ we define addition $(u+v)$ and multiplication by real number $k > 0$ as

$$(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \quad (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r),$$

$$(\underline{ku})(r) = k\underline{u}(r), \quad (\overline{ku})(r) = k\overline{u}(r).$$

3 Homotopy or continuation method

Now our aim is to obtain a solution for fuzzy nonlinear equation $F(x) = 0$. The parametric form is as follows:

$$\text{(3.1)} \qquad \begin{cases} \underline{F}(\underline{x}, \overline{x}, r) = 0, \\ \\ \overline{F}(\underline{x}, \overline{x}, r) = 0. \end{cases} \qquad \forall r \in [0,1],$$

In homotopy method, (3.1) is embedded in a one-parameter family of problems using a parameter $\lambda \in [0,1]$. The original problem (3.1) corresponds to $\lambda = 1$ and a problem with a known solution corresponds to $\lambda = 0$. For example for $x_0 \in E$, the set of problems

$$G(\lambda, x) = \lambda F(x) + (1-\lambda)[F(x) - F(x_0)] = 0, \quad 0 \le \lambda \le 1,$$

or in parametric form $\forall r \in [0,1]$

$$\text{(3.2)} \qquad \begin{cases} \underline{G}(\lambda, \underline{x}, \overline{x}, r) = \underline{F}(\underline{x}, \overline{x}, r) + (\lambda - 1)\underline{F}(\underline{x_0}, \overline{x_0}, r) = 0, \\ \overline{G}(\lambda, \underline{x}, \overline{x}, r) = \overline{F}(\underline{x}, \overline{x}, r) + (\lambda - 1)\overline{F}(\underline{x_0}, \overline{x_0}, r) = 0, \end{cases}$$

where $x_0 = (\underline{x_0}, \overline{x_0})$ is an initial approximation of (3.1). It is obvious that

$$G(0, x) = F(x) - F(x_0) = 0, \qquad G(1, x) = F(x) = 0,$$

the changing process of λ from zero to unity is just that of $G(\lambda, x)$ from $F(x) - F(x_0)$ to $F(x)$. In topology, this called deformation, $F(x) - F(x_0)$ and $F(x)$ are called homotopic.

Homotopy or continuation method attempts to determine $x^* = (\underline{x_1}, \overline{x_1})$ (for $\lambda = 1$) by solving the sequence of problems according to $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_m = 1$. The initial approximation to the solution of

$$\text{(3.3)} \qquad \begin{cases} \underline{G}(\lambda_i, \underline{x}, \overline{x}, r) = \underline{F}(\underline{x}, \overline{x}, r) + (\lambda_i - 1)\underline{F}(\underline{x_0}, \overline{x_0}, r) = 0, \\ \overline{G}(\lambda_i, \underline{x}, \overline{x}, r) = \overline{F}(\underline{x}, \overline{x}, r) + (\lambda_i - 1)\overline{F}(\underline{x_0}, \overline{x_0}, r) = 0, \end{cases}$$

would be the solution $x_{\lambda_{i-1}} = (\underline{x_{\lambda_{i-1}}}, \overline{x_{\lambda_{i-1}}})$ to the problem

$$\text{(3.4)} \qquad \begin{cases} \underline{G}(\lambda_{i-1}, \underline{x}, \overline{x}, r) = \underline{F}(\underline{x}, \overline{x}, r) + (\lambda_{i-1} - 1)\underline{F}(\underline{x_0}, \overline{x_0}, r) = 0, \\ \overline{G}(\lambda_{i-1}, \underline{x}, \overline{x}, r) = \overline{F}(\underline{x}, \overline{x}, r) + (\lambda_{i-1} - 1)\overline{F}(\underline{x_0}, \overline{x_0}, r) = 0. \end{cases}$$

In this paper $\forall r \in [0,1]$ and a fixed $\lambda \in [0,1]$, we use Newton's method for solving (3.3) and (3.4). Newton's method to (3.3) generates a sequence

$$\text{(3.5)} \qquad x_{\lambda_i}^{(k)} = x_{\lambda_i}^{(k-1)} - J(x_{\lambda_i}^{(k-1)})^{-1} G(\lambda_i, x_{\lambda_i}^{(k-1)}),$$

which converges rapidly to a solution x_{λ_i} if $x_{\lambda_i}^{(0)} (= x_{\lambda_{i-1}})$ is sufficiently close to x_{λ_i}, where

$$J(x_\lambda) = \begin{bmatrix} \underline{G}_{\underline{x}}(\lambda, \underline{x_\lambda}, \overline{x_\lambda}, r) & \underline{G}_{\overline{x}}(\lambda, \underline{x_\lambda}, \overline{x_\lambda}, r) \\ \\ \overline{G}_{\underline{x}}(\lambda, \underline{x_\lambda}, \overline{x_\lambda}, r) & \overline{G}_{\overline{x}}(\lambda, \underline{x_\lambda}, \overline{x_\lambda}, r) \end{bmatrix},$$

and $\|J(x_\lambda)^{-1}\| \leq M$ for a constant M, [10].

In this paper, computing (3.5) is performed in a two step manner. First, a vector y is found that will satisfy

$$J(x_{\lambda_i}^{(k-1)})y = -G(\lambda_i, x_{\lambda_i}^{(k-1)}),$$

second, the new approximation is obtained by adding y to $x_{\lambda_i}^{(k)}$.

4 Numerical application

Here we consider two examples to illustrating the homotopy method for fuzzy nonlinear equations from Buckley and Qu [5].

Example 1. Consider the fuzzy nonlinear equation

$$(3,4,5)x^2 + (1,2,3)x = (1,2,3).$$

Without any loss of generality, assume that x is positive, then the parametric form of this equation is as follows

$$\begin{cases} (3+r)\underline{x}^2(r) & + & (1+r)\underline{x}(r) & = & (1+r), \\ (5-r)\overline{x}^2(r) & + & (3-r)\overline{x}(r) & = & (3-r). \end{cases}$$

To obtain initial guess we use above system for $r = 0$ and $r = 1$, therefore

$$\begin{cases} 4\underline{x}^2(1) & + & 2\underline{x}(1) & = & 2, \\ 4\overline{x}^2(1) & + & 2\overline{x}(1) & = & 2, \end{cases} \qquad \begin{cases} 3\underline{x}^2(0) & + & \underline{x}(0) & = & 1, \\ 5\overline{x}^2(0) & + & 3\overline{x}(0) & = & 3. \end{cases}$$

Consequently $\underline{x}(0) = 0.4343$, $\overline{x}(0) = 0.5307$ and $\underline{x}(1) = \overline{x}(1) = \frac{1}{2}$. Therefore initial guess is $x_0 = (0.4343, 0.5, 0.5307)$ and hence $x_0 = (\underline{x_0}, \overline{x_0}) = (0.435 + 0.065r, 0.531 - 0.031r)$. The Jacobian matrix is

$$\begin{bmatrix} 2(3+r)\underline{x_\lambda}(r) + (1+r) & 0 \\ 0 & 2(5-r)\overline{x_\lambda}(r) + (3-r) \end{bmatrix}.$$

By $\lambda_i = \lambda_{i-1} + 0.25$ for $i = 1, 2, 3, 4$, we obtain the solution which the maximum error would be less than 10^{-3}, Figures 1 and 2. Now suppose x is negative, we have

$$\begin{cases} (3+r)\overline{x}^2(r) & + & (3-r)\underline{x}(r) & = & (1+r), \\ (5-r)\underline{x}^2(r) & + & (1+r)\overline{x}(r) & = & (3-r). \end{cases}$$

For $r = 0$, we have, $\underline{x}(0) \simeq -0.629$ and $\overline{x}(0) \simeq -0.98$, hence $\underline{x}(0) > \overline{x}(0)$, therefore negative root does not exist.

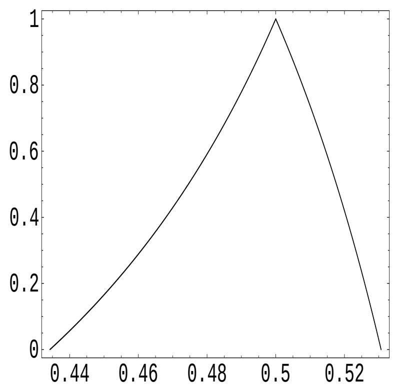

Figure 1 : Standard Analytical Solution

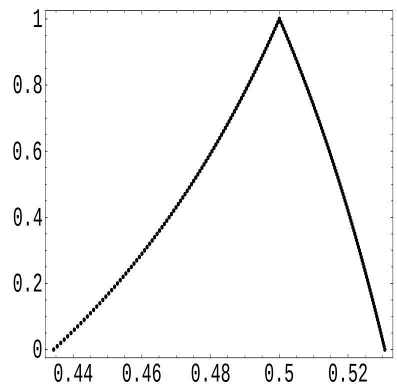

Figure 2 : Solution of Homotopy's Method

Example 2. Consider fuzzy nonlinear equation

$$(1,2,3)x^3 + (2,3,4)x^2 + (3,4,5) = (5,8,13).$$

Without any loss of generality, assume that x is positive, then parametric form of this equation is as follows

$$\begin{cases} (1+r)\underline{x}^3(r) & + & (2+r)\underline{x}^2(r) & + & (3+r) & = & (5+3r), \\ (3-r)\overline{x}^3(r) & + & (4-r)\overline{x}^2(r) & + & (5-r) & = & (13-5r). \end{cases}$$

By solving the above system for $r = 0$ and $r = 1$, we obtain the initial guess $x_0 = (0.76, 0.91, 1.06)$ and hence $x_0 = (\underline{x_0}, \overline{x_0}) = (0.76 + 0.15r, 1.06 - 0.15r)$. The Jacobian matrix is

$$\begin{bmatrix} 3(1+r)\underline{x_\lambda}^2(r) + 2(2+r)\underline{x_\lambda}(r) & 0 \\ 0 & 3(3-r)\overline{x_\lambda}^2(r) + 2(4-r)\overline{x_\lambda}(r) \end{bmatrix}.$$

By $\lambda_i = \lambda_{i-1} + 0.25$ for $i = 1, 2, 3, 4$, we obtain the solution which the maximum error would be less than 10^{-3}, Figures 3 and 4.

5 Conclusions

In this paper, we have suggested numerical solving method for fuzzy nonlinear equations instead of standard analytical techniques which are not suitable everywhere. Initially we wrote fuzzy nonlinear equation in parametric form and then solve it by homotopy method. Finally, examples were presented to illustrate proposed method.

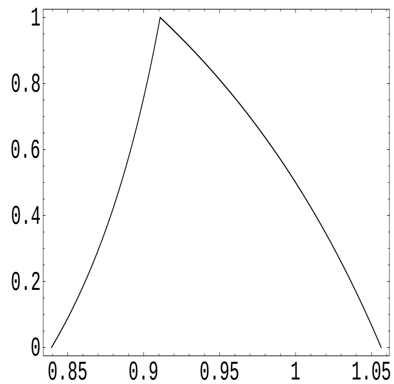

Figure 3 : Standard Analytical Solution

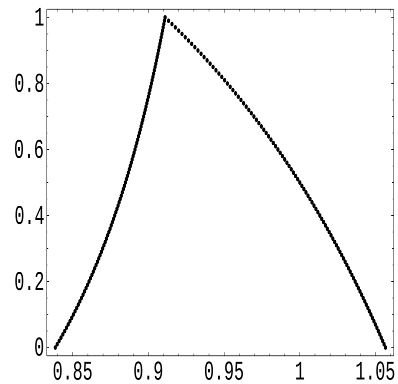

Figure 4 : Solution of Homotopy's Method

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