

# On the curvature of $K$ -contact Riemannian manifolds with constant $\Phi$ -sectional curvature with a submersion of geodesic fibres

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## Abstract

We give the curvature tensor of  $K$ -contact Riemannian manifolds of constant  $\phi$ -sectional curvature.

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**Key words:**  $K$ -contact Riemannian manifold, almost Kähler manifold, constant  $\phi$ -sectional curvature, Riemannian flow, submersion with geodesic fibres.

## 1 Introduction

In a Sasakian manifold of pointwise constant  $\phi$ -sectional curvature, it is well-known that the curvature tensor is given by the  $\phi$ -sectional curvature ([4] p.250). On the other hand, in [2] T. Sato gave the curvature of almost Kähler manifold of point wise constant holomorphic sectional curvature.

In this paper, by using the result of Sato, we consider a  $K$ -contact Riemannian manifold of pointwise constant  $\phi$ -sectional curvature with a submersion of geodesic fibres. Consequently we give the curvature tensor of such a class of  $K$ -contact Riemannian manifolds.

## 2 Almost Kähler manifolds and $K$ -contact Riemannian manifolds

Let  $(M^{2n+1}, D)$  be a  $(2n+1)$ -dimensional *contact manifold* and fix a contact 1-form  $\eta$  such that  $D = \ker \eta$ . If  $\omega$  is the restriction of  $d\eta$  to  $D$ ,  $\omega$  gives the structure of a symplectic vector bundle on  $D$ . *Almost complex structures*  $J$  on  $D$  that is compatible with  $\omega$  satisfy

$$J^2 = -1, \quad d\eta(JX^D, JY^D) = d\eta(X^D, Y^D), \quad d\eta(JX^D, Y^D) \geq 0$$

for any smooth sections  $X^D, Y^D$  of  $D$ . By setting  $g_D(X^D, Y^D) = d\eta(JX^D, Y^D)$  and  $\Omega(X^D, Y^D) = g_D(X^D, JY^D)$ , one notice that  $J$  defines a Riemannian metric  $g_D$  on  $D$  and that  $\Omega$  is a Kähler form. Then one easily sees that  $g_D$  satisfies  $g_D(JX^D, JY^D) = g_D(X^D, Y^D)$ , i.e., the transverse structure  $(J, g_D)$  on  $M$  is an *almost Hermitian structure*.

Here, we define the section  $\phi$  of  $\text{End } TM$  by  $\phi = J$  on  $D$  and  $\phi\xi = 0$ , where  $\xi$  is the Reeb vector field associated to  $\eta$ . Moreover we can also extend the transverse metric  $g_D$  to a metric  $g$  on all of  $M$  by

$$g(X, Y) = g_D(X^D, Y^D) + \eta(X)\eta(Y) = d\eta(\phi X, Y) + \eta(X)\eta(Y),$$

for all vector fields  $X, Y$  on  $M$ . Then we have  $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ .

A contact manifold  $M(\phi, \xi, \eta, g)$  with a fixed contact form  $\eta$  together with a vector field  $\xi$ , a section  $\phi$  of  $\text{End } TM$ , and a Riemannian metric  $g$  which satisfy the conditions

$$\eta(\xi) = 1, \quad \phi^2 = -I + \xi \otimes \eta, \quad \phi\xi = 0, \quad \eta(\phi X) = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad g(X, \phi Y) = d\eta(X, Y)$$

is known as a *contact metric structure* on  $M$  ([1], see also [4] p.256).

A contact metric manifold  $M^{2n+1}(\phi, \xi, \eta, g)$  for which  $\xi$  is Killing is called a *K-contact Riemannian manifold*. We also remind that on a  $K$ -contact Riemannian manifold it is valid that

$$(2.1) \quad \nabla_X^M \xi = -\phi X, \quad (\text{in particular } \nabla_\xi^M \xi = 0)$$

$$(2.2) \quad R^M(X, \xi)\xi = X - \eta(X)\xi,$$

$$(2.3) \quad \nabla_\xi^M \phi = 0,$$

where  $\nabla^M$  and  $R^M$  are the Levi-civita connection and the curvature tensor with respect to  $g$  on  $M$  respectively.

Moreover, we have, by differentiating  $\phi^2 = -I + \xi \otimes \eta$  covariantly and using (2.1),

$$(2.4) \quad (\nabla_Y^M \phi)\phi X + \phi(\nabla_Y^M \phi)X = -g(\phi Y, X)\xi - \eta(X)\phi Y.$$

In particular, it follows that  $(\nabla_X^M \phi)\xi = -X + \eta(X)\xi$ .

A contact metric manifold  $M^{2n+1}(\phi, \xi, \eta, g)$  with the normality (i.e.,  $\tilde{N}_\phi = N_\phi + 2d\eta \otimes \xi = 0$ , where  $N_\phi(X, Y) = [\phi X, \phi Y] - [X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ ) is called a *Sasakian manifold*.

The sectional curvature  $K(X, \phi X)$  on a contact metric manifold is independent of choice the tangent vector  $0 \neq X \perp \xi$  at a point  $m$  on  $M$  is called a pointwise *constant  $\phi$ -sectional curvature* at a point  $m$ .

From now on, we assume that  $M$  is a  $K$ -contact Riemannian manifold. Let us consider the foliation (flow)  $F_\xi$  ([3], p.69) generated by  $\xi$ . Since  $\xi$  is a non singular Killing vector field,  $F_\xi$  define a Riemannian flow and  $g$  bundle-like ([3], p.46).

Now we suppose that there exist a *submersion with geodesic fibres*  $\pi: M \rightarrow (M, F_\xi)$ ,  $\pi_*\phi = J\pi_*$

By restricting  $\nabla^M$  to  $D$  and taking the horizontal projection to  $D$  we get an induced connection  $\nabla^D$  on  $D$  defined by

$$\nabla_X^D Y^D = \begin{cases} (\nabla_X^M Y^D)^h & \text{if } X \text{ is a vector field of } D \\ [\xi, Y^D]^h = 0 & \text{if } X = \xi, \text{ (see the last line p.441 in [4])} \end{cases}$$

where  $Y^D$  is a vector field of  $D$  and superscript  $h$  denotes the projection onto  $D$  ([3] p.21 (3.3)). For this connection  $\nabla^D$  we have the following proposition.

**Propositon 2.1.** *Let  $M$  be a  $K$ -contact Riemannian manifold with a submersion of geodesic fibres  $\pi: M \rightarrow (M, F_\xi)$ . Then  $(M, F_\xi)$  is a  $2n$ -dimensional almost Kähler manifold ([4], p.128) with respect to  $\nabla^D$ .*

*Proof.* Putting  $X = X^D + \eta(X)\xi$  and  $Y = Y^D + \eta(Y)\xi$ , we calculate  $\nabla_X^M(\phi Y) = \nabla_X^M(\phi Y^D) = \nabla_X^M(JY^D)$ . Then we have

$$\begin{aligned} (2.5) \quad & (\nabla_X^M \phi)Y + \phi(\nabla_X^M Y) = \nabla_{X^D}^M(JY^D) + \eta(X)\nabla_\xi^M(JY^D) \\ & = (\nabla_{X^D}^M(JY^D))^h + \eta(\nabla_{X^D}^M(JY^D))\xi + \eta(X)\nabla_\xi^M(JY^D) \\ & = \nabla_{X^D}^D(JY^D) + \eta(\nabla_{X^D}^M(JY^D))\xi + \eta(X)\nabla_\xi^M(JY^D) \\ & = (\nabla_{X^D}^D J)Y^D + J(\nabla_{X^D}^D Y^D) + \eta(\nabla_{X^D}^M(JY^D))\xi + \eta(X)\nabla_\xi^M(JY^D) \\ & = (\nabla_{X^D}^D J)Y^D + \phi(\nabla_{X^D}^M Y^D - \eta(\nabla_{X^D}^M Y^D)\xi) + \eta(\nabla_{X^D}^M(JY^D))\xi + \eta(X)\nabla_\xi^M(JY^D). \end{aligned}$$

On the other hand, we find

$$\begin{aligned} (2.6) \quad & \nabla_{X^D}^M Y^D = \nabla_{X-\eta(X)\xi}^M(Y - \eta(Y)\xi) = \nabla_X^M Y - \eta(X)\nabla_\xi^M Y - g(\nabla_X^M Y, \xi)\xi \\ & \quad - g(Y, \nabla_X^M \xi)\xi - g(Y, \xi)\nabla_X^M \xi + \eta(X)g(\nabla_\xi^M Y, \xi)\xi. \end{aligned}$$

By (2.6) and putting  $X^D = X - \eta(X)\xi$ , the right hand side of (2.5) implies

$$(\nabla_{X^D}^D J)Y^D + \phi(\nabla_X^M Y - \eta(X)\nabla_\xi^M Y - \eta(Y)\nabla_X^M \xi) + \eta(\nabla_{X-\eta(X)\xi}^M(JY^D))\xi + \eta(X)\nabla_\xi^M(\phi Y).$$

Thus, from (2.1), (2.3) and (2.4), we get

$$(2.7) \quad (\nabla_X^M \phi)Y = (\nabla_{X^D}^D J)Y^D - \eta(Y)X + g(X, Y)\xi.$$

However, since  $M$  is  $K$ -contact, we notice that  $d^2\eta = 0$  implies

$$(2.8) \quad g(X, (\nabla_Z^M \phi)Y) + g(Z, (\nabla_Y^M \phi)X) + g(Y, (\nabla_X^M \phi)Z) = 0.$$

By substituting (2.7) into (2.8) and putting  $X = X^D + \eta(X)\xi$ ,  $Y = Y^D + \eta(Y)\xi$  and  $Z = Z^D + \eta(Z)\xi$ , we get

$$g_D(X^D, (\nabla_{Z^D}^D J)Y^D) + g_D(Z^D, (\nabla_{Y^D}^D J)X^D) + g_D(Y^D, (\nabla_{X^D}^D J)Z^D) = 0.$$

This implies that the Kähler form  $\Omega$  is closed i.e.,  $(M, F_\xi)$  is an almost Kähler manifold with respect to  $\nabla^D$ .

Let  $N_J$  be the Nijenhuis tensor of an almost Kähler manifold  $((M, F_\xi), J, g_D)$ . Then  $N_J$  is

$$N_J(X^D, Y^D) = [JX^D, JY^D] - [X^D, Y^D] - J[JX^D, Y^D] - J[X^D, JY^D]$$

for any vector fields  $X^D$  and  $Y^D$  on  $(M, F_\xi)$ . Then  $N_J$  satisfies that

$$(2.9) \quad N_J(JX^D, Y^D) = N_J(X^D, JY^D) = -JN_J(X^D, Y^D)$$

for any vector fields  $X^D$  and  $Y^D$  on  $(M, F_\xi)$ .

From now on we assume that  $((M, F_\xi), J, g_D)$  is an almost Kähler manifold. Let  $((M, F_\xi), J, g_D)$  be a  $2n$ -dimensional almost Kähler manifold of pointwise constant holomorphic curvature  $H^D = H^D(p)$ ,  $(p \in (M, F_\xi))$ . Then the curvature tensor  $R^D$  on  $(M, F_\xi)$  satisfies the following ([2]):

$$(2.1) \quad \begin{aligned} g(R^D(X^D, Y^D)Z^D, W^D) &= \frac{H^D}{4} \{g(X^D, W^D)g(Y^D, Z^D) \\ &\quad -g(X^D, Z^D)g(Y^D, W^D) + g(JX^D, W^D)g(JY^D, Z^D) \\ &\quad -g(JX^D, Z^D)g(JY^D, W^D) - 2g(JX^D, Y^D)g(JZ^D, W^D)\} \\ &\quad + \frac{1}{16} \{g(N_J(X^D, Z^D), N_J(Y^D, W^D)) - g(N_J(X^D, W^D), N_J(Y^D, Z^D)) \\ &\quad + 2g(N_J(X^D, Y^D), N_J(Z^D, W^D))\} + \frac{1}{96}g(Q(X^D, Y^D)Z^D, W^D), \end{aligned}$$

where  $Q$  is as follows:

$$\begin{aligned} \tilde{Q} = g(Q(X^D, Y^D)Z^D, W^D) &= -13 \{g(N_J(JZ^D, W^D), (\nabla_{X^D}^D J)Y^D) \\ &\quad + g(JY^D, N_J((\nabla_{X^D}^D J)Z^D, W^D)) + g(JY^D, (\nabla_{X^D}^D N_J)(JZ^D, W^D)) \\ &\quad - g(N_J(JZ^D, W^D), (\nabla_{Y^D}^D J)X^D) - g(JX^D, N_J((\nabla_{Y^D}^D J)Z^D, W^D)) \\ &\quad - g(JX^D, (\nabla_{Y^D}^D N_J)(JZ^D, W^D)) + g(N_J(JX^D, Y^D), (\nabla_{Z^D}^D J)W^D) \\ &\quad + g(JW^D, N_J((\nabla_{Z^D}^D J)X^D, Y^D)) + g(JW^D, (\nabla_{Z^D}^D N_J)(JX^D, Y^D)) \\ &\quad - g(N_J(JX^D, Y^D), (\nabla_{W^D}^D J)Z^D) - g(JZ^D, N_J((\nabla_{W^D}^D J)X^D, Y^D)) \\ &\quad - g(JZ^D, (\nabla_{W^D}^D N_J)(JX^D, Y^D))\} \\ &\quad + 3 \{g(N_J(Z^D, (\nabla_{X^D}^D J)W^D), Y^D) + g((\nabla_{X^D}^D N_J)(Z^D, JW^D), Y^D) \\ &\quad - g(N_J(Z^D, (\nabla_{Y^D}^D J)W^D), X^D) - g((\nabla_{Y^D}^D N_J)(Z^D, JW^D), X^D) \\ &\quad + g(N_J(X^D, (\nabla_{Z^D}^D J)Y^D), W^D) + g((\nabla_{Z^D}^D N_J)(X^D, JY^D), W^D) \\ &\quad - g(N_J(X^D, (\nabla_{W^D}^D J)Y^D), Z^D) - g((\nabla_{W^D}^D N_J)(X^D, JY^D), Z^D)\} \\ &\quad - \frac{13}{2} \{g(N_J(JY^D, W^D), (\nabla_{X^D}^D J)Z^D) + g(JZ^D, N_J((\nabla_{X^D}^D J)Y^D, W^D)) \\ &\quad + g(JZ^D, (\nabla_{X^D}^D N_J)(JY^D, W^D)) - g(N_J(JY^D, W^D), (\nabla_{Z^D}^D J)X^D) \\ &\quad - g(JX^D, N_J((\nabla_{Z^D}^D J)Y^D, W^D)) - g(JX^D, (\nabla_{Z^D}^D N_J)(JY^D, W^D)) \\ &\quad + g(N_J(JX^D, Z^D), (\nabla_{Y^D}^D J)W^D) + g(JW^D, N_J((\nabla_{Y^D}^D J)X^D, Z^D)) \\ &\quad + g(JW^D, (\nabla_{Y^D}^D N_J)(JX^D, Z^D)) - g(N_J(JX^D, Z^D), (\nabla_{W^D}^D J)Y^D) \\ &\quad - g(JY^D, N_J((\nabla_{W^D}^D J)X^D, Z^D)) - g(JY^D, (\nabla_{W^D}^D N_J)(JX^D, Z^D)) \\ &\quad - g(N_J(JY^D, Z^D), (\nabla_{X^D}^D J)W^D) - g(JW^D, N_J((\nabla_{X^D}^D J)Y^D, Z^D)) \\ &\quad - g(JW^D, (\nabla_{X^D}^D N_J)(JY^D, Z^D)) + g(N_J(JY^D, Z^D), (\nabla_{W^D}^D J)X^D) \\ &\quad + g(JX^D, N_J((\nabla_{W^D}^D J)Y^D, Z^D)) + g(JX^D, (\nabla_{W^D}^D N_J)(JY^D, Z^D)) \} \end{aligned}$$

$$\begin{aligned}
& -g(N_J(JX^D, W^D), (\nabla_{Y^D}^D J)Z^D) - g(JZ^D, N_J((\nabla_{Y^D}^D J)X^D, W^D)) \\
& -g(JZ^D, (\nabla_{Y^D}^D N_J)(JX^D, W^D)) + g(N_J(JX^D, W^D), (\nabla_{Z^D}^D J)Y^D) \\
& +g(JY^D, N_J((\nabla_{Z^D}^D J)X^D, W^D)) + g(JY^D, (\nabla_{Z^D}^D N_J)(JX^D, W^D)) \} \\
& +\frac{3}{2} \{ g(N_J(Y^D, (\nabla_{JX^D}^D J)W^D), Z^D) + g((\nabla_{JX^D}^D N_J)(Y^D, JW^D), Z^D) \\
& -g(N_J(Y^D, (\nabla_{JZ^D}^D J)W^D), X^D) - g((\nabla_{JZ^D}^D N_J)(Y^D, JW^D), X^D) \\
& +g(N_J(X^D, (\nabla_{Y^D}^D J)Z^D), W^D) + g((\nabla_{Y^D}^D N_J)(X^D, JZ^D), W^D) \\
& -g(N_J(X^D, (\nabla_{W^D}^D J)Z^D), Y^D) - g((\nabla_{W^D}^D N_J)(X^D, JZ^D), Y^D) \\
& -g(N_J(Y^D, (\nabla_{X^D}^D J)Z^D), W^D) - g((\nabla_{X^D}^D N_J)(Y^D, JZ^D), W^D) \\
& +g(N_J(Y^D, (\nabla_{W^D}^D J)Z^D), X^D) + g((\nabla_{W^D}^D N_J)(Y^D, JZ^D), X^D) \\
& -g(N_J(X^D, (\nabla_{Y^D}^D J)W^D), Z^D) - g((\nabla_{Y^D}^D N_J)(X^D, JW^D), Z^D) \\
& +g(N_J(X^D, (\nabla_{JZ^D}^D J)W^D), Y^D) + g((\nabla_{JZ^D}^D N_J)(X^D, JW^D), Y^D) \} \\
& +2 \{ g(N_J(\nabla_{X^D}^D J)Z^D, JW^D), Y^D) + g(N_J(JZ^D, (\nabla_{X^D}^D J)W^D), Y^D) \\
& +g((\nabla_{X^D}^D N_J)(JZ^D, JW^D), Y^D) + g(N_J(Z^D, W^D), (\nabla_{JX^D}^D J)Y^D) \\
& +g((\nabla_{JX^D}^D N_J)(Z^D, W^D), JY^D) + g(X^D, (\nabla_{Y^D}^D N_J)(Z^D, W^D)) \\
& +g(N_J(JZ^D, JW^D), (\nabla_{Y^D}^D J)X^D) + g(N_J((\nabla_{Y^D}^D J)Z^D, JW^D), JX^D) \\
& +g(N_J(JZ^D, (\nabla_{JY^D}^D J)W^D), JX^D) + g((\nabla_{JY^D}^D N_J)(JZ^D, JW^D), JX^D) \} \\
& +g(N_J((\nabla_{X^D}^D J)Y^D, JW^D), Z^D) + g(N_J(JY^D, (\nabla_{X^D}^D J)W^D), Z^D) \\
& +g((\nabla_{X^D}^D N_J)(JY^D, JW^D), Z^D) + g(N_J(Y^D, W^D), (\nabla_{JX^D}^D J)Z^D) \\
& +g((\nabla_{JX^D}^D N_J)(Y^D, W^D), JZ^D) + g(X^D, (\nabla_{Z^D}^D N_J)(Y^D, W^D)) \\
& +g(N_J(JY^D, JW^D), (\nabla_{JZ^D}^D J)X^D) + g(N_J((\nabla_{JZ^D}^D J)Y^D, JW^D), JX^D) \\
& +g(N_J(JY^D, (\nabla_{JZ^D}^D J)W^D), JX^D) + g((\nabla_{JZ^D}^D N_J)(JY^D, JW^D), JX^D) \\
& -g(N_J((\nabla_{X^D}^D J)Y^D, JZ^D), W^D) - g(N_J(JY^D, (\nabla_{X^D}^D J)Z^D), W^D) \\
& -g((\nabla_{X^D}^D N_J)(JY^D, JZ^D), W^D) - g(N_J(Y^D, Z^D), (\nabla_{JX^D}^D J)W^D) \\
& -g((\nabla_{JX^D}^D N_J)(Y^D, Z^D), JW^D) - g(X^D, (\nabla_{W^D}^D N_J)(Y^D, Z^D)) \\
& -g(N_J(JY^D, JZ^D), (\nabla_{JW^D}^D J)X^D) - g(N_J((\nabla_{JW^D}^D J)Y^D, JZ^D), JX^D) \\
& -g(N_J(JY^D, (\nabla_{JW^D}^D J)Z^D), JX^D) - g((\nabla_{JW^D}^D N_J)(JY^D, JZ^D), JX^D).
\end{aligned}$$

Since  $F_\xi$  is a Riemannian flow and  $g$  bundle-like, we can use the Gray and O'Neill tensors  $A$  and  $T$  (p.49 and p.50 in [4] about the properties of  $A$  and  $T$ ). Applying tensors  $A$  and  $T$  to a  $K$ -contact Riemannian manifold  $M$  and an almost Kähler manifold  $(M, F_\xi)$ , we have the following (see[4]).

$$(2.11) \quad A_\xi X^D = 0, \quad A_\xi \xi = 0, \quad A_{X^D} \xi = \pi(\nabla_{X^D}^M \xi),$$

$$A_{X^D} Y^D = \pi^\perp(\nabla_{X^D}^M Y^D), \quad A_{X^D} Y^D = -A_{Y^D} X^D$$

$$(2.12) \quad T_{X^D} \xi = 0, \quad T_{X^D} Y^D = 0, \quad T_\xi \xi = \pi(\nabla_\xi^M \xi) = 0, \quad T_\xi X^D = \pi^\perp(\nabla_\xi^M X^D),$$

and  $A_{X^D}$  is alternating, in particular  $g(A_{X^D} Y^D, \xi) = -g(Y^D, A_{X^D} \xi)$ , where  $\pi_*$  and  $\pi_*^\perp$  are the canonical projections  $\pi_* : TM \longrightarrow D$  and  $\pi_*^\perp : TM \longrightarrow \{\xi\}$  respectively. And we have (see [4] 5.32 Lemma)

$$(2.13) \quad (\nabla_{X^D}^M A)_{W^D} = -A_{A_{X^D} W^D}, \quad (\nabla_{X^D}^M T)_{Y^D} = -T_{A_{X^D} Y^D},$$

and

$$(\nabla_\xi^M T)_{Y^D} = -T_{T_\xi Y^D}.$$

Moreover  $(\nabla_X^M T)_Y$  and  $(\nabla_X^M A)_Y$  are also alternating. Then we have the following identities (see [4], p.51), that is,

$$(2.14) \quad \begin{aligned} g(R^M(X^D, \xi)Y^D, \xi) &= -g((\nabla_{X^D}^M T)_\xi \xi, Y^D) \\ &+ g(T_\xi X^D, T_\xi Y^D) - g((\nabla_\xi^M A)_{X^D} Y^D, \xi) - g(A_{X^D} \xi, A_{Y^D} \xi). \end{aligned}$$

$$(2.15) \quad \begin{aligned} g(R^M(X^D, Y^D)Z^D, \xi) &= -g((\nabla_{Z^D}^M A)_{X^D} Y^D, \xi) \\ &- g(A_{X^D} Y^D, T_\xi Z^D) + g(A_{Y^D} Z^D, T_\xi X^D) + g(A_{Z^D} X^D, T_\xi Y^D). \end{aligned}$$

$$(2.16) \quad \begin{aligned} g(R^M(X^D, Y^D)Z^D, W^D) &= g(R^D(X^D, Y^D)Z^D, W^D) \\ &+ 2g(A_{X^D} Y^D, A_{Z^D} W^D) \\ &- g(A_{Y^D} Z^D, A_{X^D} W^D) + g(A_{X^D} Z^D, A_{Y^D} W^D). \end{aligned}$$

For any vector field  $X$ ,  $\|X\| = 1$ ,  $X \perp \xi$ , we find that  $X = X^D$ . Then the relation between the  $\phi$ -sectional curvature  $H^M$  on  $M$  and the holomorphic sectional curvature  $H^D$  on  $(M, F_\xi)$  is given by

$$(2.17) \quad H^M(X, \phi X) = H^D(X^D, JX^D) - 3|A_{X^D}(JX^D)|.$$

Then we have the following Proposition.

**Proposition 2.2.** *Let  $M$  be a  $K$ -contact Riemannian manifold with a submersion of geodesic fibres  $\pi: M \rightarrow (M, F_\xi)$ . Then for the  $\phi$ -sectional curvature  $H^M$  on  $M$  and the holomorphic sectional curvature  $H^D$  on  $(M, F_\xi)$ , we have  $H^M = H^D - 3$ .*

*Proof.* From (2.17) we have

$$H^M(X, \phi X) = H^D(X^D, JX^D) - 3|A_{X^D}(JX^D)|.$$

for any vector field  $X$ ,  $\|X\| = 1$ ,  $X \perp \xi$ . Here, by (2.3), (2.4) and (2.11) we see that

$$\begin{aligned} A_{X^D}(JX^D) &= \pi_*^\perp(\nabla_{X^D}^M(\phi X^D)) = g(\nabla_{X-\eta(X)\xi}^M(\phi X), \xi)\xi \\ &= g((\nabla_X^M \phi)X, \xi)\xi = -g((\nabla_X^M \phi)\xi, X)\xi = \xi. \end{aligned}$$

Thus, we get  $H^M = H^D - 3$ .

**Lemma 2.1.** *For the Nijenhuis tensors  $\tilde{N}_\phi$  on  $M$  and  $N_J$  on  $(M, F_\xi)$  respectively, we have*

- (1)  $\tilde{N}_\phi(fX, Y) = f\tilde{N}_\phi(X, Y)$ , where  $f$  is a  $C^\infty$  function on  $M$ .
- (2)  $N_J(X^D, Y^D) = \tilde{N}_\phi(X, Y) = N_\phi(X, Y) + 2d\eta \otimes \xi$

$$\begin{aligned}
(3) \quad & \tilde{N}_\phi(\phi X, Y) = \tilde{N}_\phi(X, \phi Y) = -\phi \tilde{N}_\phi(X, Y), \text{ In particular } \phi \tilde{N}_\phi(\xi, Y) = 0 \text{ and} \\
& (\nabla_Y^M \tilde{N}_\phi)(\phi X, Z) + \tilde{N}_\phi((\nabla_Y^M \phi)X, Z) = (\nabla_Y^M \tilde{N}_\phi)(X, \phi Z) + \tilde{N}_\phi(X, (\nabla_Y^M \phi)Z) \\
& = -(\nabla_Y^M \phi) \tilde{N}_\phi(X, Z) - \phi(\nabla_Y^M \tilde{N}_\phi)(X, Z).
\end{aligned}$$

*Proof.* (1): we see that, by the definition of  $\tilde{N}_\phi$ ,

$$\begin{aligned}
& \tilde{N}_\phi(fX, Z) = N_\phi + d\eta(fX, Z)\xi \\
& = \phi^2[fX, Z] + [f\phi X, \phi Z] - \phi[f\phi X, Z] - \phi[fX, \phi Z] \\
& \quad + fX(\eta(Z)) - Z(\eta(fX)) - \eta([fX, Z]) \\
& = \phi^2(f[X, Z] - (Zf)X) + f[\phi X, \phi Z] - ((\phi Z)f)\phi X \\
& \quad - \phi(f[\phi X, Z] - (Zf)\phi X) - \phi(f[X, \phi Z] - ((\phi Z)f)X) + fd\eta(X, Z)\xi. \\
& = N_\phi(X, Z) + fd\eta(X, Z)\xi = f\tilde{N}_\phi(X, Z).
\end{aligned}$$

(2): First we prove the following:

$$\begin{aligned}
(2.18) \quad & N_J(X^D, Y^D) = [\phi X, \phi Y]^M - [X - \eta(X)\xi, Y - \eta(Y)\xi]^M \\
& \quad - \phi[\phi X, Y - \eta(Y)\xi]^M + \phi[\phi Y, X - \eta(X)\xi]^M.
\end{aligned}$$

From the definition of  $\nabla^D$  we have

$$\begin{aligned}
[X^D, Y^D]^D &= \nabla_{X^D}^D Y^D - \nabla_{Y^D}^D X^D = (\nabla_{X^D}^M Y^D)^h - (\nabla_{Y^D}^M X^D)^h \\
&= \nabla_{X^D}^M Y^D - \eta(\nabla_{X^D}^M Y^D)\xi - \nabla_{Y^D}^M X^D + \eta(\nabla_{Y^D}^M X^D)\xi.
\end{aligned}$$

Here, differentiating  $g(\xi, Z^D) = 0$  covariantly, we get

$$\eta(\nabla_{Y^D}^M X^D)\xi = g(\phi X, Y)\xi.$$

Thus we find

$$\begin{aligned}
(2.19) \quad & [X^D, Y^D]^D = [X^D, Y^D]^M - 2g(\phi X, Y)\xi \\
& = [X - \eta(X)\xi, Y - \eta(Y)\xi]^M - 2g(\phi X, Y)\xi.
\end{aligned}$$

Using (2.18), we have the following:

$$(2.20) \quad [JX^D, JY^D]^D = [\phi X, \phi Y]^M - 2g(\phi X, Y)\xi$$

$$\begin{aligned}
(2.21) \quad J[JX^D, Y^D]^D &= \phi[JX^D, Y^D]^D = \phi[(\phi X)^D, Y^D]^M \\
&= \phi[\phi X, Y - \eta(Y)\xi]^M.
\end{aligned}$$

$$(2.22) \quad J[X^D, JY^D]^D = \phi[X - \eta(X)\xi, \phi Y]^M.$$

Combining (2.19), (2.20), (2.21) and (2.22), we get (2.18). Thus we see that

$$\begin{aligned}
N_J(X^D, Y^D) &= [\phi X, \phi Y]^M - [X, Y]^M + \eta(X)[\xi, Y]^M \\
&\quad - Y(\eta(X))\xi + \eta(Y)[X, \xi]^M + X(\eta(Y))\xi - \{\eta(X)\xi(\eta(Y))\xi - \eta(Y)\xi(\eta(X))\xi\} \\
&\quad - \phi\{[\phi X, Y]^M - \eta(Y)[\phi X, \xi]^M - \phi X(\eta(Y))\xi\} \\
&\quad - \phi\{[X, \phi Y]^M - \eta(X)[\xi, \phi Y]^M + \phi Y(\eta(X))\xi\} \\
&= \phi^2[X, Y]^M + [\phi X, \phi Y]^M - \phi[\phi X, Y]^M - \phi[X, \phi Y]^M \\
&\quad - g(\nabla_X^M Y, \xi)\xi + g(\nabla_Y^M X, \xi)\xi + \eta(X)\nabla_\xi^M Y - \eta(X)\nabla_Y^M \xi \\
&\quad + \eta(Y)\nabla_X^M \xi - \eta(Y)\nabla_\xi^M X - \eta(X)(g(\nabla_\xi^M Y, \xi))\xi + \eta(Z)(g(\nabla_\xi^M Y, \xi))\xi \\
&\quad - \phi\{-\eta(Y)\nabla_{\phi X}^M \xi + \eta(Y)\phi(\nabla_\xi^M X)\} - \phi\{-\eta(X)\phi(\nabla_\xi^M Y) + \eta(X)\nabla_{\phi Y}^M \xi\} \\
&\quad - Y(\eta(X))\xi + X(\eta(Y))\xi \\
&= N_\phi(X, Y) - \eta(\nabla_X^M Y)\xi + \eta(\nabla_Y^M X)\xi + \eta(X)\nabla_\xi^M Y - \eta(X)\nabla_Y^M \xi \\
&\quad + \eta(Y)\nabla_X^M \xi - \eta(Y)\nabla_\xi^M X - \eta(X)\eta(\nabla_\xi^M Y)\xi + \eta(Y)\eta(\nabla_\xi^M X)\xi \\
&\quad + \eta(Y)\phi(\nabla_{\phi X}^M \xi) - \eta(Y)\phi^2\nabla_\xi^M X + \eta(X)\phi^2(\nabla_\xi^M Y) - \eta(X)\phi(\nabla_{\phi Y}^M \xi) \\
&\quad - Y(\eta(X))\xi + X(\eta(Y))\xi \\
&= N_\phi(X, Y) + 2d\eta \otimes \xi = \tilde{N}_\phi(X, Y),
\end{aligned}$$

where we have used (2.1), and  $-[X, Y]^M = \phi^2[X, Y]^M - g([X, Y]^M, \xi)\xi$ .

(3): from (2) we have, for  $X = X^D + \eta(X)\xi$  and  $Y = Y^D + \eta(Y)\xi$ ,

$$N_J(JX^D, Y^D) = \tilde{N}_\phi(\phi X, Y)$$

$$N_J(X^D, JY^D) = \tilde{N}_\phi(X, \phi Y)$$

$$-JN_J(X^D, Y^D) = -\phi N_J(X^D, Y^D) = -\phi\tilde{N}_\phi(X, Y).$$

Here, by (2.9), we get (3).

**Lemma 2.2** *On a  $K$ -contact Riemannian manifold  $M$  and an almost Kähler manifold  $(M, F_\xi)$ , the following identities are satisfied*

$$(1) \quad (\nabla_{X^D}^D J)Y^D = (\nabla_X^M \phi)Y + \eta(Y)X - g(X, Y)\xi$$

$$(2) \quad (\nabla_{X^D}^D N_J)(Z^D, W^D) = (\nabla_X^M \tilde{N}_\phi)(Z, W) - \eta(X)(\nabla_\xi^M \tilde{N}_\phi)(Z, W)$$

$$-\eta(Z)\tilde{N}_\phi(\phi X, W) + \eta(W)\tilde{N}_\phi(\phi X, Z) - g(\tilde{N}_\phi(Z, W), \phi X)\xi.$$



*Proof.* First we show (1). From (2.7) it is evident. Next we prove (2). Putting  $X = X^D + \eta(X)\xi$ ,  $Z = Z^D + \eta(Z)\xi$  and  $W = W^D + \eta(W)\xi$ , we have

$$N_J(Z^D, W^D) = \tilde{N}_\phi(Z, W),$$

from which

$$\nabla_X^M(N_J(Z^D, W^D)) = \nabla_X^M(\tilde{N}_\phi(Z, W)),$$

that is,

$$(2.23) \quad \nabla_{X^D}^M(N_J(Z^D, W^D)) + \eta(X)\nabla_\xi^M(N_J(Z^D, W^D))$$

$$= (\nabla_X^M \tilde{N}_\phi)(Z, W) + \tilde{N}_\phi(\nabla_X^M Z, W) + \tilde{N}_\phi(Z, \nabla_X^M W).$$

Here, for the left hand members we find, from the definition of  $\nabla^D$  and Lemma 2.1,

$$(2.24) \quad \nabla_{X^D}^M(N_J(Z^D, W^D)) = (\nabla_{X^D}^M(N_J(Z^D, W^D)))^h + \eta(\nabla_{X^D}^M(N_J(Z^D, W^D)))\xi$$

$$= \nabla_{X^D}^D(N_J(Z^D, W^D)) + \eta(\nabla_{X^D}^M(N_J(Z^D, W^D)))\xi.$$

$$= (\nabla_{X^D}^D N_J)(Z^D, W^D) + N_J(\nabla_{X^D}^D Z^D, W^D)$$

$$+ N_J(Z^D, \nabla_{X^D}^D W^D) + \eta(\nabla_{X^D}^M(N_J(Z^D, W^D)))\xi$$

$$\eta(X)\nabla_\xi^M(N_J(Z^D, W^D)) = \eta(X)\nabla_\xi^M(\tilde{N}_\phi(Z, W)) = \eta(X)(\nabla_\xi^M \tilde{N}_\phi)(Z, W)$$

$$+ \eta(X)\tilde{N}_\phi(\nabla_\xi^M Z, W) + \eta(X)\tilde{N}_\phi(Z, \nabla_\xi^M W).$$

Moreover, by means of  $g(N_J(Z^D, W^D), \xi) = 0$ , we see that

$$g(\nabla_{X^D}^M(N_J(Z^D, W^D)), \xi) + g(N_J(Z^D, W^D), \nabla_{X^D}^M \xi) = 0.$$

Substituting  $X^D = X - \eta(X)\xi$  in  $X^D$  of  $\nabla_{X^D}^M \xi$  and using (2.1) and Lemma 2.1, we get

$$(2.25) \quad \eta(\nabla_{X^D}^M(N_J(Z^D, W^D))) = g(\nabla_{X^D}^M(N_J(Z^D, W^D)), \xi) = g(\tilde{N}_\phi(Z, W), \phi X).$$

And, by Lemma 2.1, it follows that

$$(2.26) \quad \eta(X)\nabla_\xi^M(N_J(Z^D, W^D)) = \eta(X)\nabla_\xi^M(\tilde{N}_\phi(Z, W))$$

$$= \eta(X)(\nabla_\xi^M \tilde{N}_\phi)(Z, W) + \eta(X)\tilde{N}_\phi(\nabla_\xi^M Z, W) + \eta(X)\tilde{N}_\phi(Z, \nabla_\xi^M W).$$

From (2.24), (2.25) and (2.26), (2.23) becomes as follows:

$$(2.27) \quad (\nabla_{X^D}^D N_J)(Z^D, W^D) + N_J(\nabla_{X^D}^D Z^D, W^D)$$

$$\begin{aligned}
& +N_J(Z^D, \nabla_{X^D}^D W^D) + g(\tilde{N}_\phi(Z, W), \phi X)\xi \\
& +\eta(X)(\nabla_\xi^M \tilde{N}_\phi)(Z, W)) + \eta(X)\tilde{N}_\phi(\nabla_\xi^M Z, W) + \eta(X)\tilde{N}_\phi(Z, \nabla_\xi^M W) \\
& = (\nabla_X^M \tilde{N}_\phi)(Z, W)) + \tilde{N}_\phi(\nabla_X^M Z, W) + \tilde{N}_\phi(Z, \nabla_X^M W).
\end{aligned}$$

On the other hand, we have

$$\nabla_{X^D}^M Y^D = \nabla_{X-\eta(X)\xi}^M (Y - \eta(Y)\xi),$$

from which, by (2.1) and the definition of  $\nabla^D$

$$\begin{aligned}
\nabla_{X^D}^D Y^D + \eta(\nabla_{X^D}^M Y^D)\xi &= \nabla_X^M Y - \eta(X)\nabla_\xi^M Y - g(\nabla_X^M Y, \xi)\xi \\
&- g(Y, \nabla_X^M \xi)\xi + g(Y, \xi)\phi X + \eta(X)g(\nabla_\xi^M Y, \xi)\xi.
\end{aligned}$$

Here, taking the inner product of this equation and  $\xi$ , we get

$$\eta(\nabla_{X^D}^M Y^D)\xi = -g(Y, \nabla_X^M \xi)\xi.$$

From this, it follows that

$$\nabla_X^M Y - \eta(X)\nabla_\xi^M Y + \eta(Y)\phi X = \nabla_{X^D}^D Y^D + g(\nabla_X^M Y, \xi)\xi - \eta(X)g(\nabla_\xi^M Y, \xi)\xi.$$

Thus we get

$$\begin{aligned}
(2.28) \quad N_J(\nabla_{X^D}^D Y^D, W^D) &= \tilde{N}_\phi(\nabla_X^M Y - \eta(X)\nabla_\xi^M Y + \eta(Y)\phi X, W) \\
&= \tilde{N}_\phi(\nabla_X^M Y, W) - \eta(X)\tilde{N}_\phi(\nabla_\xi^M Y, W) + \eta(Y)\tilde{N}_\phi(\phi X, W).
\end{aligned}$$

Substituting (2.28) into (2.27), we obtain (2).

### 3 Results

**Theorem 3.1.** *Let  $M$  be a  $K$ -contact Riemannian manifold of pointwise constant  $\phi$ -sectional curvature  $H^M$  with a submersion of geodesic fibres  $\pi: M \rightarrow (M, F_\xi)$ .*

Then we have the following:

$$\begin{aligned}
& g(R^M(X, Y)Z, W) = \frac{H^M+3}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
& + \frac{H^M-1}{4} \{g(Y, \phi Z)g(X, \phi W) - g(X, \phi Z)g(Y, \phi W) - 2g(X, \phi Y)g(Z, \phi W)\} \\
& + \frac{H^M+7}{4} \{ \eta(X)\eta(Z)g(Y, W) + \eta(Y)\eta(W)g(X, Z) \\
& \quad - \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(W)g(Y, Z) \} \\
& + \eta(X)g((\nabla_Y^M \phi)Z, W) + \eta(Y)g((\nabla_X^M \phi)W, Z) \\
& + \eta(Z)g((\nabla_W^M \phi)X, Y) + \eta(W)g((\nabla_Z^M \phi)Y, X) \\
& + \frac{1}{16} \{ g(\widetilde{N}_\phi(X, Z), \widetilde{N}_\phi(Y, W)) - g(\widetilde{N}_\phi(X, W), \widetilde{N}_\phi(Y, Z)) \\
& \quad + 2g(\widetilde{N}_\phi(X, Y), \widetilde{N}_\phi(Z, W)) \} \\
& + \frac{11}{192} \left( \eta(W)g(\phi X, \widetilde{N}_\phi(Z, Y)) - \eta(Z)g(\phi X, \widetilde{N}_\phi(W, Y)) \right) \\
& + \frac{5}{192} \left( \eta(Z)g(\phi Y, \widetilde{N}_\phi(W, X)) - \eta(W)g(\phi Y, \widetilde{N}_\phi(Z, X)) \right) \\
& + \frac{5}{64} \eta(X) \left( g(\phi W, \widetilde{N}_\phi(Y, Z)) - g(\phi Z, \widetilde{N}_\phi(Y, W)) \right) \\
& + \frac{17}{192} \eta(Y) \left( g(\phi Z, \widetilde{N}_\phi(X, W)) - g(\phi W, \widetilde{N}_\phi(X, Z)) \right) \\
& + \frac{11}{96} \left( \eta(W)g(\phi Z, \widetilde{N}_\phi(X, Y)) - \eta(Z)g(\phi W, \widetilde{N}_\phi(X, Y)) \right) \\
& + \frac{11}{96} \eta(Y)g(\phi X, \widetilde{N}_\phi(Z, W)) - \frac{15}{96} \eta(X)g(\phi Y, \widetilde{N}_\phi(Z, W)) \\
& + \frac{5}{96} \left( g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi Y}^M \phi)X) - g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi X}^M \phi)Y) \right) \\
& + \frac{1}{64} \left( g(\widetilde{N}_\phi(W, X), (\nabla_{\phi Y}^M \phi)Z) - g(\widetilde{N}_\phi(W, X), (\nabla_{\phi Z}^M \phi)Y) \right. \\
& \quad \left. + g(\widetilde{N}_\phi(Z, X), (\nabla_{\phi W}^M \phi)Y) - g(\widetilde{N}_\phi(Z, X), (\nabla_{\phi Y}^M \phi)W) \right) \\
& + \frac{5}{192} \left( g(\widetilde{N}_\phi(W, Y), (\nabla_{\phi Z}^M \phi)X) - g(\widetilde{N}_\phi(W, Y), (\nabla_{\phi X}^M \phi)Z) \right. \\
& \quad \left. + g(\widetilde{N}_\phi(Z, Y), (\nabla_{\phi X}^M \phi)W) - g(\widetilde{N}_\phi(Z, Y), (\nabla_{\phi W}^M \phi)X) \right) \\
& + \frac{1}{32} \left( g(\widetilde{N}_\phi(Y, X), (\nabla_{\phi W}^M \phi)Z) - g(\widetilde{N}_\phi(Y, X), (\nabla_{\phi Z}^M \phi)W) \right) \\
& + \frac{11}{96} \left( g((\nabla_X^M \widetilde{N}_\phi)(Z, W), Y) - g((\nabla_Y^M \widetilde{N}_\phi)(Z, W), X) \right) \\
& + \frac{13}{96} \left( g((\nabla_Z^M \widetilde{N}_\phi)(X, Y), W) - g((\nabla_W^M \widetilde{N}_\phi)(X, Y), Z) \right) \\
& + \frac{11}{192} \left( g((\nabla_X^M \widetilde{N}_\phi)(Y, W), Z) - g((\nabla_Z^M \widetilde{N}_\phi)(Y, W), X) \right. \\
& \quad \left. + g((\nabla_W^M \widetilde{N}_\phi)(Y, Z), X) - g((\nabla_X^M \widetilde{N}_\phi)(Y, Z), W) \right) \\
& + \frac{13}{192} \left( g((\nabla_Y^M \widetilde{N}_\phi)(X, Z), W) - g((\nabla_W^M \widetilde{N}_\phi)(X, Z), Y) \right. \\
& \quad \left. + g((\nabla_Z^M \widetilde{N}_\phi)(X, W), Y) - g((\nabla_Y^M \widetilde{N}_\phi)(X, W), Z) \right) \\
& + \frac{1}{64} \left( g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Z, X), \phi Y) - g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(Z, X), \phi W) \right. \\
& \quad \left. + g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(W, X), \phi Z) - g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(W, X), \phi Y) \right) \\
& + \frac{5}{192} \left( g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(W, Y), \phi X) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Y), \phi Z) \right. \\
& \quad \left. + g((\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, Y), \phi W) - g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Z, Y), \phi X) \right) \\
& + \frac{5}{96} \left( g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(W, Z), \phi X) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Z), \phi Y) \right) \\
& + \frac{1}{32} \left( g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Y, X), \phi Z) - g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(Y, X), \phi W) \right) \\
& + \frac{11}{96} \left( \eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), X) - \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), Y) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{13}{96} \left( \eta(W)g((\nabla_\xi^M \widetilde{N}_\phi)(X, Y), Z) - \eta(Z)g((\nabla_\xi^M \widetilde{N}_\phi)(X, Y), W) \right) \\
& + \frac{11}{192} \left( \eta(Z)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, W), X) - \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, W), Z) \right. \\
& \quad \left. + \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, Z), W) - \eta(W)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, Z), X) \right) \\
& + \frac{13}{192} \left( \eta(Z)g((\nabla_\xi^M \widetilde{N}_\phi)(W, X), Y) - \eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(W, X), Z) \right. \\
& \quad \left. + \eta(W)g((\nabla_\xi^M \widetilde{N}_\phi)(X, Z), Y) - \eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(X, Z), W) \right) \\
& + \frac{11}{96} \left( 2g(\phi X, Y)\eta(\widetilde{N}_\phi(Z, W)) + g(\phi X, Z)\eta(\widetilde{N}_\phi(Y, W)) - g(\phi X, W)\eta(\widetilde{N}_\phi(Y, Z)) \right) \\
& + \frac{13}{96} \left( 2g(\phi Z, W)\eta(\widetilde{N}_\phi(X, Y)) + g(\phi Y, W)\eta(\widetilde{N}_\phi(X, Z)) - g(\phi Y, Z)\eta(\widetilde{N}_\phi(X, W)) \right) \\
& \quad + \frac{11}{192} \left( \eta(W)\eta((\nabla_X^M \widetilde{N}_\phi)(Y, Z)) - \eta(X)\eta((\nabla_W^M \widetilde{N}_\phi)(Y, Z)) \right. \\
& \quad \left. + \eta(X)\eta((\nabla_Z^M \widetilde{N}_\phi)(Y, W)) - \eta(Z)\eta((\nabla_X^M \widetilde{N}_\phi)(Y, W)) \right) \\
& \quad + \frac{13}{192} \left( \eta(Y)\eta((\nabla_Z^M \widetilde{N}_\phi)(W, X)) - \eta(Z)\eta((\nabla_Y^M \widetilde{N}_\phi)(W, X)) \right. \\
& \quad \left. + \eta(Y)\eta((\nabla_W^M \widetilde{N}_\phi)(X, Z)) - \eta(W)\eta((\nabla_Y^M \widetilde{N}_\phi)(X, Z)) \right) \\
& \quad + \frac{13}{96} \left( \eta(Z)\eta((\nabla_W^M \widetilde{N}_\phi)(X, Y)) - \eta(W)\eta((\nabla_Z^M \widetilde{N}_\phi)(X, Y)) \right) \\
& \quad + \frac{15}{96} \eta(X)\eta((\nabla_Y^M \widetilde{N}_\phi)(Z, W)) - \frac{11}{96} \eta(Y)\eta((\nabla_X^M \widetilde{N}_\phi)(Z, W)).
\end{aligned}$$

*Proof.* Putting  $X = X^D + \eta(X)\xi$ ,  $Y = Y^D + \eta(Y)\xi$ ,  $Z = Z^D + \eta(Z)\xi$  and  $W = W^D + \eta(W)\xi$ , we calculate  $g(R^M(X, Y)Z, W)$  :

$$\begin{aligned}
& g(R^M(X, Y)Z, W) \\
& = g(R^M(X^D + \eta(X)\xi, Y^D + \eta(Y)\xi)(Z^D + \eta(Z)\xi), W^D + \eta(W)\xi) \\
& = g(R^M(X^D, Y^D)Z^D, W^D) + \eta(X)g(R^M(\xi, Y^D)Z^D, W^D) \\
(3.1) \quad & + \eta(Y)g(R^M(X^D, \xi)Z^D, W^D) + \eta(Z)g(R^M(X^D, Y^D)\xi, W^D) \\
& + \eta(W)g(R^M(X^D, Y^D)Z^D, \xi) + \eta(X)\eta(Z)g(R^M(\xi, Y^D)\xi, W^D) \\
& + \eta(Y)\eta(Z)g(R^M(X^D, \xi)\xi, W^D) + \eta(X)\eta(W)g(R^M(\xi, Y^D)Z^D, \xi) \\
& + \eta(Y)\eta(W)g(R^M(X^D, \xi)Z^D, \xi).
\end{aligned}$$

First we calculate  $g(R^M(X^D, Y^D)Z^D, W^D)$ .

Since we have

$$A_{X^D}Y^D = \pi_*^\perp(\nabla_{X^D}^M Y^D) = g(\nabla_{X^D}^M Y^D, \xi)\xi = \eta(\nabla_{X^D}^M Y^D)\xi,$$

by means of (2.1) and (2.24), we find

$$A_{X^D}Y^D = g(\nabla_{X^D}^M Y^D, \xi)\xi = -g(Y, \nabla_X^M \xi)\xi = g(Y, \phi X)\xi.$$

From this result and (2.16), we see that

$$\begin{aligned}
(3.2) \quad g(R^M(X^D, Y^D)Z^D, W^D) & = g(R^D(X^D, Y^D)Z^D, W^D) + 2g(Y, \phi X)g(W, \phi Z) \\
& \quad - g(Z, \phi Y)g(W, \phi X) + g(Z, \phi X)g(W, \phi Y).
\end{aligned}$$

Here we rewrite the first term of the right hand members of (3.2), that is, each term of the right hand members in (2.10): The first term of (2.10):

$$g(X^D, W^D)g(Y^D, Z^D) = g(X - \eta(X)\xi, W - \eta(W)\xi)g(Y - \eta(Y)\xi, Z - \eta(Z)\xi)$$

$$= g(X, W)g(Y, Z) - \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(W)g(Y, Z) + \eta(X)\eta(Y)\eta(Z)\eta(W).$$

The third term of (2.10) is:

$$\begin{aligned} g(JX^D, W^D)g(JY^D, Z^D) &= g(\phi(X - \eta(X)\xi), W - \eta(W)\xi)g(\phi(Y - \eta(Y)\xi), Z - \eta(Z)\xi) \\ &= g(\phi X, W)g(\phi Y, Z). \end{aligned}$$

The sixth term of (2.10):

$$g(N(X^D, Z^D), N(Y^D, W^D)) = g(\widetilde{N}_\phi(X, Z), \widetilde{N}_\phi(Y, W)).$$

From now on we represent the  $n$  th term of  $\widetilde{Q}$  as  $\widetilde{Q}_n$ .

$$\widetilde{Q}_1 = g(N_J(JZ^D, W^D), (\nabla_{X^D}^D J)Y^D) = g(\widetilde{N}_\phi(\phi Z, W), (\nabla_X^M \phi)Y + \eta(Y)X - g(X, Y)\xi).$$

From Lemma 2.1 (3) and (2.4), we see that

$$\begin{aligned} \widetilde{Q}_1 &= g(\widetilde{N}_\phi(Z, W), \phi(\nabla_X^M \phi)Y) + \eta(Y)g(\widetilde{N}_\phi(\phi Z, W), X) \\ &= -g(\widetilde{N}_\phi(Z, W), (\nabla_X^M \phi)\phi Y) - g(\phi X, Y)g(\widetilde{N}_\phi(Z, W), \xi). \end{aligned}$$

By Lemma 2.2 (1), and Lemma 2.1, we have

$$\begin{aligned} \widetilde{Q}_2 &= g(JY^D, N_J((\nabla_{X^D}^D J)Z^D, W^D)) = g(\phi Y, \widetilde{N}_\phi((\nabla_X^M \phi)Z, W)) \\ &\quad + \eta(Z)g(\phi Y, \widetilde{N}_\phi(X, W)). \end{aligned}$$

From Lemma 2.2 (2) we find

$$\begin{aligned} \widetilde{Q}_3 &= g(JY^D, (\nabla_{X^D}^D N_J)(JZ^D, W^D)) = g(\phi Y, (\nabla_X^M \widetilde{N}_\phi)(\phi Z, W)) \\ &\quad - \eta(X)g(\phi Y, (\nabla_\xi^M \widetilde{N}_\phi)(\phi Z, W)) + \eta(W)g(\phi Y, \widetilde{N}_\phi(\phi X, \phi Z)). \end{aligned}$$

Using Lemma 2.1 (3), we have

$$\begin{aligned} \widetilde{Q}_3 &= -g(\phi Y, \widetilde{N}_\phi((\nabla_X^M \phi)Z, W)) - g(\phi Y, (\nabla_X^M \phi)\widetilde{N}_\phi(Z, W)) \\ &\quad - g(Y, (\nabla_X^M \widetilde{N}_\phi)(Z, W)) + \eta(Y)\eta((\nabla_X^M \widetilde{N}_\phi)(Z, W)) \\ &\quad + \eta(X)g(Y, (\nabla_\xi^M \widetilde{N}_\phi)(Z, W)) - \eta(X)\eta(Y)\eta((\nabla_\xi^M \widetilde{N}_\phi)(Z, W)) \\ &\quad - \eta(W)g(\phi Y, \widetilde{N}_\phi(X, Z)). \end{aligned}$$

$\tilde{Q}_4, \tilde{Q}_5$  and  $\tilde{Q}_6$  are obtained by exchanging  $X$  with  $Y$  in  $\tilde{Q}_1, \tilde{Q}_2$  and  $\tilde{Q}_3$  respectively and taking the minus of it. As well,  $\tilde{Q}_7, \tilde{Q}_8$  and  $\tilde{Q}_9$  are got by exchanging  $X$  with  $Z$  and  $Y$  with  $W$  in  $\tilde{Q}_1, \tilde{Q}_2$  and  $\tilde{Q}_3$  respectively. Moreover we get  $\tilde{Q}_{10}, \tilde{Q}_{11}$  and  $\tilde{Q}_{12}$  by exchanging  $Z$  with  $W$  in  $\tilde{Q}_7, \tilde{Q}_8$  and  $\tilde{Q}_9$  respectively and taking the minus of it. Adding from  $\tilde{Q}_1$  to  $\tilde{Q}_{12}$  in  $\tilde{Q}$ , we have

$$\begin{aligned}
(3.3) \quad & -\eta(Z)g(\phi Y, \tilde{N}_\phi(W, X)) + \eta(W)g(\phi Y, \tilde{N}_\phi(Z, X)) + \eta(Z)g(\phi X, \tilde{N}_\phi(W, Y)) \\
& -\eta(W)g(\phi X, \tilde{N}_\phi(Z, Y)) - \eta(X)g(\phi W, \tilde{N}_\phi(Y, Z)) + \eta(Y)g(\phi W, \tilde{N}_\phi(X, Z)) \\
& +\eta(X)g(\phi Z, \tilde{N}_\phi(Y, W)) - \eta(Y)g(\phi Z, \tilde{N}_\phi(X, W)) - g(Y, (\nabla_X^M \tilde{N}_\phi)(Z, W)) \\
& +g(X, (\nabla_Y^M \tilde{N}_\phi)(Z, W)) - g(W, (\nabla_Z^M \tilde{N}_\phi)(X, Y)) + g(Z, (\nabla_W^M \tilde{N}_\phi)(X, Y)) \\
& +\eta(X)g(Y, (\nabla_\xi^M \tilde{N}_\phi)(Z, W)) - \eta(Y)g(X, (\nabla_\xi^M \tilde{N}_\phi)(Z, W)) + \eta(Z)g(W, (\nabla_\xi^M \tilde{N}_\phi)(X, Y)) \\
& -\eta(W)g(Z, (\nabla_\xi^M \tilde{N}_\phi)(X, Y)) - 2g(\phi X, Y)\eta(\tilde{N}_\phi(Z, W)) - 2g(\phi Z, W)\eta(\tilde{N}_\phi(X, Y)) \\
& +\eta(Y)\eta((\nabla_X^M \tilde{N}_\phi)(Z, W)) - \eta(X)\eta((\nabla_Y^M \tilde{N}_\phi)(Z, W)) + \eta(W)\eta((\nabla_Z^M \tilde{N}_\phi)(X, Y)) \\
& -\eta(Z)\eta((\nabla_W^M \tilde{N}_\phi)(X, Y)).
\end{aligned}$$

From Lemma 2.2 (1) and Lemma 2.1 (1) (3), we have

$$\begin{aligned}
\tilde{Q}_{13} &= g(N_J(Z^D, (\nabla_{JX^D}^D J)W^D), Y^D) \\
&= g(\tilde{N}_\phi(Z, (\nabla_{\phi X}^M \phi)W), Y) + \eta(W)g(\tilde{N}_\phi(Z, \phi X), Y) \\
&\quad -\eta(Y)g(\tilde{N}_\phi(Z, (\nabla_{\phi X}^M \phi)W), \xi) - \eta(Y)\eta(W)g(\tilde{N}_\phi(Z, \phi X), \xi) \\
&= g(\tilde{N}_\phi(Z, (\nabla_{\phi X}^M \phi)W), Y) + \eta(W)g(\tilde{N}_\phi(Z, X), \phi Y) - \eta(Y)\eta(\tilde{N}_\phi(Z, (\nabla_{\phi X}^M \phi)W)).
\end{aligned}$$

By Lemma 2.2 (2) it follows that

$$\begin{aligned}
\tilde{Q}_{14} &= g((\nabla_{JX^D}^D N_J)(Z^D, JW^D), Y^D) = g((\nabla_{JX^D}^D N_J)(Z^D, JW^D), Y - \eta(Y)\xi) \\
&= g((\nabla_{\phi X}^M \tilde{N}_\phi)(Z, \phi W), Y) + \eta(Z)g(\tilde{N}_\phi(X, \phi W), Y) - \eta(Y)g((\nabla_{\phi X}^M \tilde{N}_\phi)(Z, \phi W), \xi).
\end{aligned}$$

From Lemma 2.1 (3)

$$\tilde{Q}_{14} = g(\tilde{N}_\phi((\nabla_{\phi X}^M \phi)W, Z), Y) + g((\nabla_{\phi X}^M \phi)\tilde{N}_\phi(W, Z), Y) + g(\phi(\nabla_{\phi X}^M \tilde{N}_\phi)(W, Z), Y)$$

$$\begin{aligned}
& +\eta(Z)g(\widetilde{N}_\phi(X, W), \phi Y) - \eta(Y)\eta(\widetilde{N}_\phi((\nabla_{\phi X}^M \phi)W, Z)) - \eta(Y)g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi X}^M \phi)\xi) \\
& = g(\widetilde{N}_\phi((\nabla_{\phi X}^M \phi)W, Z), Y) - g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi X}^M \phi)Y) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Z), \phi Y) \\
& +\eta(Z)g(\widetilde{N}_\phi(X, W), \phi Y) - \eta(Y)\eta(\widetilde{N}_\phi((\nabla_{\phi X}^M \phi)W, Z)) - \eta(Y)g(\widetilde{N}_\phi(W, Z), \phi X).
\end{aligned}$$

$\widetilde{Q}_{15}$  and  $\widetilde{Q}_{16}$  are obtained by exchanging  $X$  with  $Y$  in  $\widetilde{Q}_{13}$  and  $\widetilde{Q}_{14}$  and taking the minus of it. As well,  $\widetilde{Q}_{17}, \widetilde{Q}_{18}, \widetilde{Q}_{19}$  and  $\widetilde{Q}_{20}$  are got by exchanging  $X$  with  $Z$  and  $Y$  with  $W$  in  $\widetilde{Q}_{13}, \widetilde{Q}_{14}, \widetilde{Q}_{15}$  and  $\widetilde{Q}_{16}$  respectively. Adding from  $\widetilde{Q}_{13}$  to  $\widetilde{Q}_{20}$  in  $\widetilde{Q}$ , we get

$$\begin{aligned}
(3.4) \quad & \eta(W)g(\widetilde{N}_\phi(Z, X), \phi Y) + \eta(Z)g(\widetilde{N}_\phi(X, W), \phi Y) - \eta(Y)g(\widetilde{N}_\phi(W, Z), \phi X) \\
& -\eta(W)g(\widetilde{N}_\phi(Z, Y), \phi X) - \eta(Z)g(\widetilde{N}_\phi(Y, W), \phi X) + \eta(X)g(\widetilde{N}_\phi(W, Z), \phi Y) \\
& +\eta(Y)g(\widetilde{N}_\phi(X, Z), \phi W) - \eta(X)g(\widetilde{N}_\phi(Y, Z), \phi W) - \eta(W)g(\widetilde{N}_\phi(Y, X), \phi Z) \\
& -\eta(Y)g(\widetilde{N}_\phi(X, W), \phi Z) - \eta(X)g(\widetilde{N}_\phi(W, Y), \phi Z) - \eta(Z)g(\widetilde{N}_\phi(X, Y), \phi W) \\
& -g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi X}^M \phi)Y) + g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi Y}^M \phi)X) - g(\widetilde{N}_\phi(Y, X), (\nabla_{\phi Z}^M \phi)W) \\
& +g(\widetilde{N}_\phi(Y, X), (\nabla_{\phi W}^M \phi)Z) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Z), \phi Y) + g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(W, Z), \phi X) \\
& -g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(Y, X), \phi W) + g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Y, X), \phi Z).
\end{aligned}$$

Moreover, by exchanging  $Z$  and  $Y$  in the terms which are from  $\widetilde{Q}_1$  to  $\widetilde{Q}_{12}$ , we get the terms which are from  $\widetilde{Q}_{21}$  to  $\widetilde{Q}_{32}$ . And the terms which are from  $\widetilde{Q}_{33}$  to  $\widetilde{Q}_{44}$  are got by exchanging  $W$  with  $Z$  in the terms which are from  $\widetilde{Q}_{21}$  to  $\widetilde{Q}_{32}$  respectively and taking the minus of it. Summing up from  $\widetilde{Q}_{21}$  to  $\widetilde{Q}_{44}$ , we find

$$\begin{aligned}
(3.5) \quad & \eta(Y)g(\phi Z, \widetilde{N}_\phi(X, W)) - 2\eta(W)g(\phi Z, \widetilde{N}_\phi(X, Y)) - 2\eta(Y)g(\phi X, \widetilde{N}_\phi(Z, W)) \\
& +\eta(W)g(\phi X, \widetilde{N}_\phi(Z, Y)) + \eta(X)g(\phi W, \widetilde{N}_\phi(Y, Z)) + 2\eta(Z)g(\phi W, \widetilde{N}_\phi(X, Y)) \\
& +2\eta(X)g(\phi Y, \widetilde{N}_\phi(Z, W)) + \eta(Z)g(\phi Y, \widetilde{N}_\phi(W, X)) - \eta(Y)g(\phi W, \widetilde{N}_\phi(X, Z)) \\
& -\eta(Z)g(\phi X, \widetilde{N}_\phi(W, Y)) - \eta(X)g(\phi Z, \widetilde{N}_\phi(Y, W)) - \eta(W)g(\phi Y, \widetilde{N}_\phi(Z, X)) \\
& +g(Z, (\nabla_X^M \widetilde{N}_\phi)(W, Y)) + g(X, (\nabla_Z^M \widetilde{N}_\phi)(Y, W)) + g(W, (\nabla_Y^M \widetilde{N}_\phi)(Z, X))
\end{aligned}$$

$$\begin{aligned}
& +g(Y, (\nabla_W^M \widetilde{N}_\phi)(X, Z)) - g(W, (\nabla_X^M \widetilde{N}_\phi)(Z, Y)) - g(X, (\nabla_W^M \widetilde{N}_\phi)(Y, Z)) \\
& -g(Z, (\nabla_Y^M \widetilde{N}_\phi)(W, X)) - g(Y, (\nabla_Z^M \widetilde{N}_\phi)(X, W)) \\
& +\eta(Z)g(X, (\nabla_\xi^M \widetilde{N}_\phi)(W, Y)) + \eta(Y)g(W, (\nabla_\xi^M \widetilde{N}_\phi)(X, Z)) + \eta(W)g(Y, (\nabla_\xi^M \widetilde{N}_\phi)(Z, X)) \\
& -\eta(X)g(W, (\nabla_\xi^M \widetilde{N}_\phi)(Y, Z)) - \eta(W)g(X, (\nabla_\xi^M \widetilde{N}_\phi)(Z, Y)) - \eta(Y)g(Z, (\nabla_\xi^M \widetilde{N}_\phi)(X, W)) \\
& -\eta(Z)g(Y, (\nabla_\xi^M \widetilde{N}_\phi)(W, X)) + \eta(X)g(Z, (\nabla_\xi^M \widetilde{N}_\phi)(Y, W)) \\
& -2g(\phi X, Z)\eta(\widetilde{N}_\phi(Y, W)) - 2g(\phi Y, W)\eta(\widetilde{N}_\phi(X, Z)) + 2g(\phi X, W)\eta(\widetilde{N}_\phi(Y, Z)) \\
& +2g(\phi Y, Z)\eta(\widetilde{N}_\phi(X, W)) \\
& -\eta(Z)\eta((\nabla_X^M \widetilde{N}_\phi)(W, Y)) - \eta(X)\eta((\nabla_Z^M \widetilde{N}_\phi)(Y, W)) - \eta(W)\eta((\nabla_Y^M \widetilde{N}_\phi)(Z, X)) \\
& -\eta(Y)\eta((\nabla_W^M \widetilde{N}_\phi)(X, Z)) + \eta(W)\eta((\nabla_X^M \widetilde{N}_\phi)(Z, Y)) + \eta(X)\eta((\nabla_W^M \widetilde{N}_\phi)(Y, Z)) \\
& +\eta(Z)\eta((\nabla_Y^M \widetilde{N}_\phi)(W, X)) + \eta(Y)\eta((\nabla_Z^M \widetilde{N}_\phi)(X, W)).
\end{aligned}$$

From exchanging  $Y$  and  $Z$  in the terms which are from  $\widetilde{Q}_{13}$  to  $\widetilde{Q}_{20}$ , we have ones which are from  $\widetilde{Q}_{45}$  to  $\widetilde{Q}_{52}$  respectively. And  $\widetilde{Q}_{53} \sim \widetilde{Q}_{60}$  are the terms which are got by exchanging  $W$  with  $Z$  in  $\widetilde{Q}_{45} \sim \widetilde{Q}_{52}$  respectively and taking the minus of it. Summing up from  $\widetilde{Q}_{45}$  to  $\widetilde{Q}_{60}$ , we see that

$$\begin{aligned}
(3.6) \quad & 2\eta(W)g(\widetilde{N}_\phi(Y, X), \phi Z) + 2\eta(Y)g(\widetilde{N}_\phi(X, W), \phi Z) - 2\eta(Z)g(\widetilde{N}_\phi(W, Y), \phi X) \\
& -2\eta(W)g(\widetilde{N}_\phi(Y, Z), \phi X) - 2\eta(Y)g(\widetilde{N}_\phi(Z, W), \phi X) - 2\eta(X)g(\widetilde{N}_\phi(Y, W), \phi Z) \\
& +2\eta(Z)g(\widetilde{N}_\phi(X, Y), \phi W) + 2\eta(X)g(\widetilde{N}_\phi(Y, Z), \phi W) - 2\eta(W)g(\widetilde{N}_\phi(Z, X), \phi Y) \\
& -2\eta(Z)g(\widetilde{N}_\phi(X, W), \phi Y) - 2\eta(X)g(\widetilde{N}_\phi(W, Z), \phi Y) - 2\eta(Y)g(\widetilde{N}_\phi(X, Z), \phi W) \\
& -g(\widetilde{N}_\phi(W, Y), (\nabla_{\phi X}^M \phi)Z) + g(\widetilde{N}_\phi(W, Y), (\nabla_{\phi Z}^M \phi)X) - g(\widetilde{N}_\phi(Z, X), (\nabla_{\phi Y}^M \phi)W) \\
& +g(\widetilde{N}_\phi(Z, X), (\nabla_{\phi W}^M \phi)Y) + g(\widetilde{N}_\phi(Z, Y), (\nabla_{\phi X}^M \phi)W) - g(\widetilde{N}_\phi(Z, Y), (\nabla_{\phi W}^M \phi)X)
\end{aligned}$$



$$\begin{aligned}
& +g(\widetilde{N}_\phi(W, X), (\nabla_{\phi Y}^M \phi)Z) - g(\widetilde{N}_\phi(W, X), (\nabla_{\phi Z}^M \phi)Y) \\
& -g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Y), \phi Z) + g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(W, Y), \phi X) - g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(Z, X), \phi W) \\
& +g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Z, X), \phi Y) + g((\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, Y), \phi W) - g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Z, Y), \phi X) \\
& +g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(W, X), \phi Z) - g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(W, X), \phi Y).
\end{aligned}$$

Next we calculate  $\widetilde{Q}_{61} \sim \widetilde{Q}_{70}$ . From Lemma 2.2 (1) and Lemma 2.1 (3) we find that

$$\begin{aligned}
\widetilde{Q}_{61} &= g(N_J((\nabla_{X^D}^D J)Z^D, JW^D), Y^D) \\
&= g(\widetilde{N}_\phi((\nabla_X^M \phi)Z + \eta(Z)X, \phi W), Y - \eta(Y)\xi) \\
&= g(\widetilde{N}_\phi((\nabla_X^M \phi)Z, \phi W), Y) + \eta(Z)g(\widetilde{N}_\phi(X, \phi W), Y)
\end{aligned}$$

$\widetilde{Q}_{62}$  is got by exchanging  $Z$  with  $W$  of each term in  $\widetilde{Q}_{61}$  and taking the minus of it. By Lemma 2.2 (2), we have

$$\begin{aligned}
\widetilde{Q}_{63} &= g((\nabla_{X^D}^D N_J)(JZ^D, JW^D), Y^D) \\
&= g((\nabla_X^M \widetilde{N}_\phi)(\phi Z, \phi W), Y) - \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(\phi Z, \phi W), Y) - \eta(Y)g(\widetilde{N}_\phi(\phi Z, \phi W), \phi X) \\
&\quad - \eta(Y)g((\nabla_X^M \widetilde{N}_\phi)(\phi Z, \phi W), \xi) + \eta(X)\eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(\phi Z, \phi W), \xi) \\
&\quad + \eta(Y)g(\widetilde{N}_\phi(\phi Z, \phi W), \phi X).
\end{aligned}$$

Here, from Lemma 2.1 (3), we find

$$(\nabla_X^M \widetilde{N}_\phi)(\phi Z, \phi W) = -\widetilde{N}_\phi((\nabla_X^M \phi)Z, \phi W) - (\nabla_X^M \phi)\widetilde{N}_\phi(Z, \phi W) - \phi(\nabla_X^M \widetilde{N}_\phi)(Z, \phi W)$$

Putting  $X = \xi$  in this equation,  $Y = \xi$  in the equation of Lemma 2.1 (3) and using (2.3) we have

$$\begin{aligned}
(\nabla_\xi^M \widetilde{N}_\phi)(\phi Z, \phi W) &= -\phi(\nabla_\xi^M \widetilde{N}_\phi)(Z, \phi W) = \phi^2(\nabla_\xi^M \widetilde{N}_\phi)(Z, W) \\
&= -(\nabla_\xi^M \widetilde{N}_\phi)(Z, W) + \eta((\nabla_\xi^M \widetilde{N}_\phi)(Z, W))\xi.
\end{aligned}$$

Thus

$$\widetilde{Q}_{63} = -g(\widetilde{N}_\phi((\nabla_X^M \phi)Z, \phi W), Y)$$

$$\begin{aligned}
& -g((\nabla_X^M \phi) \widetilde{N}_\phi(Z, \phi W), Y) - g(\phi(\nabla_X^M \widetilde{N}_\phi)(Z, \phi W), Y) \\
& + \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), Y) + \eta(Y)\eta(\widetilde{N}_\phi((\nabla_X^M \phi)Z, \phi W)) \\
& + \eta(Y)g((\nabla_X^M \phi) \widetilde{N}_\phi(Z, \phi W), \xi) \\
& - \eta(X)\eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), \xi).
\end{aligned}$$

Moreover we find, by Lemma 2.1 (3),

$$\begin{aligned}
& -g(\phi(\nabla_X^M \widetilde{N}_\phi)(Z, \phi W), Y) = -g((\nabla_X^M \widetilde{N}_\phi)(\phi W, Z), \phi Y) \\
& = g(\widetilde{N}_\phi((\nabla_X^M \phi)W, Z), \phi Y) + g((\nabla_X^M \phi) \widetilde{N}_\phi(W, Z), \phi Y) + g(\phi(\nabla_X^M \widetilde{N}_\phi)(W, Z), \phi Y)
\end{aligned}$$

and

$$\eta(Y)\eta(\widetilde{N}_\phi((\nabla_X^M \phi)Z, \phi W)) = 0.$$

Therefore we have

$$\begin{aligned}
\widetilde{Q}_{63} &= -g(\widetilde{N}_\phi((\nabla_X^M \phi)Z, \phi W), Y) \\
& + g(\widetilde{N}_\phi(Z, W), \phi(\nabla_X^M \phi)Y) + g(\widetilde{N}_\phi((\nabla_X^M \phi)W, Z), \phi Y) \\
& + g(\widetilde{N}_\phi(Z, W), (\nabla_X^M \phi)\phi Y) + g(\phi(\nabla_X^M \widetilde{N}_\phi)(W, Z), \phi Y) \\
& + \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), Y) - g(\widetilde{N}_\phi(Z, \phi W), (\nabla_X^M \phi)\xi) \\
& - \eta(X)\eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), \xi).
\end{aligned}$$

On the other hand, using (2.4), we get

$$\begin{aligned}
\widetilde{Q}_{63} &= -g(\phi X, Y)\eta(\widetilde{N}_\phi(Z, W)) + g(\widetilde{N}_\phi((\nabla_X^M \phi)W, Z), \phi Y) + g((\nabla_X^M \widetilde{N}_\phi)(W, Z), Y) \\
& - \eta(Y)\eta((\nabla_X^M \widetilde{N}_\phi)(W, Z)) + \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), Y) - \eta(X)\eta(Y)\eta((\nabla_\xi^M \widetilde{N}_\phi)(Z, W)).
\end{aligned}$$

From Lemma 2.2 (1), we have

$$\begin{aligned}
\widetilde{Q}_{64} &= g(N_J(Z^D, W^D), (\nabla_{JX^D}^D J)Y^D) = g(\widetilde{N}_\phi(Z, W), (\nabla_{\phi X}^M \phi)Y) \\
& + \eta(Y)g(\widetilde{N}_\phi(Z, W), \phi X) - g(\phi X, Y)\eta(\widetilde{N}_\phi(Z, W)).
\end{aligned}$$

By Lemma 2.2 (2) we find

$$\begin{aligned} (\nabla_{JX^D}^D N_J)(Z^D, W^D) &= (\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, W) + \eta(Z) \widetilde{N}_\phi(X, W) - \eta(X) \eta(Z) \widetilde{N}_\phi(\xi, W) \\ &\quad - \eta(W) \widetilde{N}_\phi(X, Z) + \eta(X) \eta(W) \widetilde{N}_\phi(\xi, Z) + g(\widetilde{N}_\phi(Z, W), X) \xi - \eta(X) g(\widetilde{N}_\phi(Z, W), \xi) \xi, \end{aligned}$$

from which, from Lemma 2.1 (3), we get

$$\widetilde{Q}_{65} = g((\nabla_{JX^D}^D N_J)(Z^D, W^D), JY^D) = g((\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, W), \phi Y).$$

$$+ \eta(Z) g(\widetilde{N}_\phi(X, W), \phi Y) - \eta(W) g(\widetilde{N}_\phi(X, Z), \phi Y).$$

From Lemma 2.2 (2) and Lemma 2.1 (3) we have

$$\begin{aligned} \widetilde{Q}_{66} &= g(X^D, (\nabla_{Y^D}^D N_J)(Z^D, W^D)) \\ &= g(X, (\nabla_Y^M \widetilde{N}_\phi)(Z, W)) - \eta(Y) g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), X) \\ &\quad - \eta(Z) g(\widetilde{N}_\phi(Y, W), \phi X) + \eta(W) g(\widetilde{N}_\phi(Y, Z), \phi X) - \eta(X) g(\widetilde{N}_\phi(Z, W), \phi Y) \\ &\quad - \eta(X) \eta((\nabla_Y^M \widetilde{N}_\phi)(Z, W)) + \eta(X) \eta(Y) \eta((\nabla_\xi^M \widetilde{N}_\phi)(Z, W)) \\ &\quad + \eta(X) g(\widetilde{N}_\phi(Z, W), \phi Y). \end{aligned}$$

By Lemma 2.2 (1) and Lemma 2.1 (3), we see that

$$\begin{aligned} \widetilde{Q}_{67} &= g(N_J(JZ^D, JW^D), (\nabla_{JY^D}^D J)X^D) \\ &= g(\widetilde{N}_\phi(\phi Z, \phi W), (\nabla_{\phi Y}^M \phi)X) + \eta(X) g(\widetilde{N}_\phi(\phi Z, \phi W), \phi Y) \\ &= -g(\widetilde{N}_\phi(Z, W), (\nabla_{\phi Y}^M \phi)X) + \eta(\widetilde{N}_\phi(Z, W)) \eta((\nabla_{\phi Y}^M \phi)X) - \eta(X) g(\widetilde{N}_\phi(Z, W), \phi Y) \\ &= -g(\widetilde{N}_\phi(Z, W), (\nabla_{\phi Y}^M \phi)X) + \eta(\widetilde{N}_\phi(Z, W)) g(\phi Y, X) - \eta(X) g(\widetilde{N}_\phi(Z, W), \phi Y), \end{aligned}$$

where we have used  $(\nabla_{\phi Y}^M \phi)\xi = -\phi Y$  ( from (2.4) ).

From Lemma 2.2 (1) and Lemma 2.1 (3) we have

$$\begin{aligned} \widetilde{Q}_{68} &= g(N_J((\nabla_{JY^D}^D J)Z^D, JW^D), JX^D) \\ &= g(\widetilde{N}_\phi((\nabla_{\phi Y}^M \phi)Z, \phi W), \phi X) + \eta(Z) g(\widetilde{N}_\phi(\phi Y, \phi W), \phi X) \end{aligned}$$

$$= -g(\widetilde{N}_\phi((\nabla_{\phi Y}^M \phi)Z, W), X) + \eta(\widetilde{N}_\phi((\nabla_{\phi Y}^M \phi)Z, W))\eta(X) - \eta(Z)g(\widetilde{N}_\phi(Y, W), \phi X).$$

By exchanging  $W$  and  $Z$  in  $\widetilde{Q}_{68}$  and taking the minus of it, we get  $\widetilde{Q}_{69}$ .  $\widetilde{Q}_{70}$  is got by substituting  $JX^D = \phi X$ ,  $JY^D = \phi Y$ ,  $JZ^D = \phi Z$  and  $JW^D = \phi W$  for  $X^D, Y^D, Z^D$  and  $W^D$  of  $\widetilde{Q}_{66}$  respectively.  $\widetilde{Q}_{71} \sim \widetilde{Q}_{80}$  are the terms which are got by exchanging  $Y$  with  $Z$  in

$\widetilde{Q}_{61} \sim \widetilde{Q}_{70}$  respectively.  $\widetilde{Q}_{81} \sim \widetilde{Q}_{90}$  are by exchanging  $W$  and  $Z$  in  $\widetilde{Q}_{71} \sim \widetilde{Q}_{80}$  respectively and taking the minus of it. Adding from  $\widetilde{Q}_{61}$  and  $\widetilde{Q}_{70}$ , we have

$$(3.7) \quad 2\eta(Z)g(\widetilde{N}_\phi(X, W), \phi Y) - 2\eta(W)g(\widetilde{N}_\phi(X, Z), \phi Y) + \eta(Y)g(\widetilde{N}_\phi(Z, W), \phi X)$$

$$- 2\eta(Z)g(\widetilde{N}_\phi(Y, W), \phi X) + 2\eta(W)g(\widetilde{N}_\phi(Y, Z), \phi X) - \eta(X)g(\widetilde{N}_\phi(Z, W), \phi Y)$$

$$+ g(\widetilde{N}_\phi(Z, W), (\nabla_{\phi X}^M \phi)Y) - g(\widetilde{N}_\phi(Z, W), (\nabla_{\phi Y}^M \phi)X)$$

$$+ g((\nabla_X^M \widetilde{N}_\phi)(W, Z), Y) + g((\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, W), \phi Y) + g((\nabla_Y^M \widetilde{N}_\phi)(Z, W), X)$$

$$+ g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(Z, W), \phi X)$$

$$+ \eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), Y) - \eta(Y)g((\nabla_\xi^M \widetilde{N}_\phi)(Z, W), X)$$

$$- 2g(\phi X, Y)\eta(\widetilde{N}_\phi(Z, W)) - \eta(Y)\eta((\nabla_X^M \widetilde{N}_\phi)(W, Z)) + \eta(X)\eta((\nabla_Y^M \widetilde{N}_\phi)(Z, W)).$$

Summing up from  $\widetilde{Q}_{71}$  to  $\widetilde{Q}_{90}$ , we get

$$(3.8) \quad 2\eta(Y)g(\widetilde{N}_\phi(X, W), \phi Z) - 2\eta(W)g(\widetilde{N}_\phi(X, Y), \phi Z) + 3\eta(Z)g(\widetilde{N}_\phi(Y, W), \phi X)$$

$$+ 4\eta(Y)g(\widetilde{N}_\phi(W, Z), \phi X) - 3\eta(W)g(\widetilde{N}_\phi(Y, Z), \phi X) - \eta(X)g(\widetilde{N}_\phi(Y, W), \phi Z)$$

$$+ \eta(X)g(\widetilde{N}_\phi(Y, Z), \phi W) + 2\eta(Z)g(\widetilde{N}_\phi(X, Y), \phi W) - 2\eta(Y)g(\widetilde{N}_\phi(X, Z), \phi W)$$

$$- g(\widetilde{N}_\phi(Y, W), (\nabla_{\phi Z}^M \phi)X) - g(\widetilde{N}_\phi(Y, Z), (\nabla_{\phi X}^M \phi)W)$$

$$+ g(\widetilde{N}_\phi(Y, Z), (\nabla_{\phi W}^M \phi)X) + g(\widetilde{N}_\phi(Y, W), (\nabla_{\phi X}^M \phi)Z)$$

$$+ g((\nabla_X^M \widetilde{N}_\phi)(W, Y), Z) - g((\nabla_Z^M \widetilde{N}_\phi)(W, Y), X)$$

$$+ g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(W, Y), \phi X) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Y), \phi Z)$$

$$\begin{aligned}
& +g((\nabla_W^M \widetilde{N}_\phi)(Z, Y), X) - g((\nabla_X^M \widetilde{N}_\phi)(Z, Y), W) \\
& +g((\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, Y), \phi W) - g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Z, Y), \phi X) \\
& +\eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, W), Z) - \eta(Z)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, W), X) \\
& -\eta(X)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, Z), W) + \eta(W)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, Z), X) \\
& -2g(\phi X, Z)\eta(\widetilde{N}_\phi(Y, W)) + 2g(\phi X, W)\eta(\widetilde{N}_\phi(Y, Z)) \\
& -\eta(Z)\eta((\nabla_X^M \widetilde{N}_\phi)(W, Y)) - \eta(X)\eta((\nabla_Z^M \widetilde{N}_\phi)(Y, W)) \\
& +\eta(W)\eta((\nabla_X^M \widetilde{N}_\phi)(Z, Y)) + \eta(X)\eta((\nabla_W^M \widetilde{N}_\phi)(Y, Z)).
\end{aligned}$$

Next we calculate the second, third, fourth and fifth terms in the right members in (3.1). First we calculate  $g(R^M(X^D, Y^D)Z^D, \xi)$  generally.

From (2.15), we have

$$\begin{aligned}
(3.9) \quad g(R^M(X^D, Y^D)Z^D, \xi) &= -g((\nabla_{Z^D}^M A)_{X^D} Y^D, \xi) - g(A_{X^D} Y^D, T_\xi Z^D) \\
&+ g(A_{Y^D} Z^D, T_\xi X^D) + g(A_{Z^D} X^D, T_\xi Y^D).
\end{aligned}$$

Since  $(\nabla_{Z^D}^M A)_{X^D}$  is, by (2.13), alternating, we find

$$-g((\nabla_{Z^D}^M A)_{X^D} Y^D, \xi) = g(Y^D, (\nabla_{Z^D}^M A)_{X^D} \xi).$$

Moreover, we have

$$(3.10) \quad (\nabla_{Z^D}^M A)_{X^D} \xi = \nabla_{Z^D}^M (A_{X^D} \xi) - A_{\nabla_{Z^D}^M X^D} \xi - A_{X^D} (\nabla_{Z^D}^M \xi).$$

Then, by (2.11),  $g(\xi, \xi) = 1$  and (2.1), the first term of the right hand members in (3.10) implies

$$\begin{aligned}
\nabla_{Z^D}^M (A_{X^D} \xi) &= \nabla_{Z^D}^M (\pi_*(\nabla_{X^D}^M \xi)) = \nabla_{Z-\eta(Z)\xi}^M (\nabla_{X^D}^M \xi - g(\nabla_{X^D}^M \xi, \xi)\xi) \\
&= \nabla_{Z-\eta(Z)\xi}^M \nabla_{X-\eta(X)\xi}^M \xi = \nabla_Z^M \nabla_X^M \xi - \eta(Z) \nabla_\xi^M \nabla_X^M \xi.
\end{aligned}$$

Since  $\nabla_X^M \xi = -\phi X$ , it follows that

$$\nabla_Z^M \nabla_X^M \xi = \nabla_Z^M (-\phi X) = -(\nabla_Z^M \phi)X - \phi(\nabla_Z^M X).$$

Moreover

$$-\eta(Z) \nabla_\xi^M \nabla_X^M \xi = -\eta(Z) \nabla_\xi^M (-\phi X) = \eta(Z) \phi(\nabla_\xi^M X).$$

Thus we have

$$(3.11) \quad \nabla_{Z^D}^M(A_{X^D}\xi) = -(\nabla_Z^M\phi)X - \phi(\nabla_Z^MX) + \eta(Z)\phi(\nabla_\xi^MX).$$

Using (2.11) and (2.1), the second term of the right hand members in (3.10) implies

$$\begin{aligned} -A_{\nabla_{Z^D}^MX^D}\xi &= -A_{(\nabla_{Z^D}^MX^D)^h + \eta(\nabla_{Z^D}^MX^D)\xi}\xi \\ &= -A_{(\nabla_{Z^D}^MX^D)^h}\xi \\ &= -\pi_*(\nabla_{(\nabla_{Z^D}^MX^D)^h}^M\xi) = -\pi_*(\nabla_{\nabla_{Z^D}^MX^D - \eta(\nabla_{Z^D}^MX^D)\xi}^M\xi) \\ &= -\pi_*(\nabla_{\nabla_{Z^D}^MX^D}^M\xi). \end{aligned}$$

On the other hand, from (2.1), we have

$$\begin{aligned} \nabla_{Z^D}^MX^D &= \nabla_{Z - \eta(Z)\xi}^M(X - \eta(X)\xi) = \nabla_Z^MX - \eta(Z)\nabla_\xi^MX - g(\nabla_Z^MX, \xi)\xi \\ &\quad + \eta(Z)g(\nabla_\xi^MX, \xi)\xi - g(X, \nabla_Z^M\xi)\xi - \eta(X)\nabla_Z^M\xi. \end{aligned}$$

Thus we get, by means of (2.1),

$$(3.12) \quad -A_{\nabla_{Z^D}^MX^D}\xi = -\pi_*(\nabla_{\nabla_Z^MX - \eta(Z)\nabla_\xi^MX - \eta(X)\nabla_Z^M\xi}^M\xi - \eta(Z)\nabla_{\nabla_\xi^MX}^M\xi - \eta(X)\nabla_{\nabla_Z^M\xi}^M\xi).$$

The third term  $-A_{X^D}(\nabla_{Z^D}^M\xi)$  of the right hand members in (3.10) is as follows:

$$\begin{aligned} (3.13) \quad -A_{X^D}(\nabla_{Z^D}^M\xi) &= -A_{X^D}(\nabla_{Z^D}^M\xi)^h - A_{X^D}(\eta(\nabla_{Z^D}^M\xi)\xi) \\ &= -\pi_*^\perp(\nabla_{X^D}(\nabla_{Z^D}^M\xi)^h). \end{aligned}$$

$(\nabla_{Z^D}^MA)_{X^D}$  being alternating and using (3.10), (3.11), (3.12) and (3.13), the first term of the right hand members in (3.9) is the following relations:

$$\begin{aligned} &-g((\nabla_{Z^D}^MA)_{X^D}Y^D, \xi) = g(Y^D, (\nabla_{Z^D}^MA)_{X^D}\xi) \\ &= g(Y - \eta(Y)\xi, \nabla_Z^M\nabla_X^M\xi) - \eta(Z)g(Y - \eta(Y)\xi, \nabla_\xi^M\nabla_X^M\xi) \\ &\quad -g(\nabla_{\nabla_Z^MX}^M\xi - g(\nabla_{\nabla_Z^MX}^M\xi, \xi)\xi, Y - \eta(Y)\xi) \\ &\quad + \eta(Z)g(\nabla_{\nabla_\xi^MX}^M\xi - g(\nabla_{\nabla_\xi^MX}^M\xi, \xi)\xi, Y - \eta(Y)\xi) \\ &\quad + \eta(X)g(\nabla_{\nabla_Z^M\xi}^M\xi - g(\nabla_{\nabla_Z^M\xi}^M\xi, \xi)\xi, Y - \eta(Y)\xi). \end{aligned}$$

Here, differentiating  $g(\xi, \xi) = 1$  and  $g(\nabla_X^M \xi, \xi) = 1$  covariantly, we find  $g(\nabla_\xi^M \nabla_X^M \xi, \xi) = 0$  and  $g(\nabla_Z^M \nabla_X^M \xi, \xi) = -g(\nabla_X^M \xi, \nabla_Z^M \xi)$ . Therefore we get, by means of (2.1),

$$\begin{aligned}
 (3.14) \quad & -g((\nabla_{Z^D}^M A)_{X^D} Y^D, \xi) = -g((\nabla_Z^M \phi)X, Y) \\
 & + \eta(Y)g(X, Z) + \eta(Z)g(\phi(\nabla_\xi^M X), Y) \\
 & - \eta(Z)g(\phi(\nabla_\xi^M X), Y) - \eta(X)g(Z, Y) \\
 & = -g((\nabla_Z^M \phi)X, Y) + \eta(Y)g(X, Z) - \eta(X)g(Z, Y).
 \end{aligned}$$

Next we calculate after the second term of the right hand members in (3.9). From (2.11) and (2.12) we have

$$\begin{aligned}
 -g(A_{X^D} Y^D, T_\xi Z^D) &= -g(\pi_*^\perp(\nabla_{X^D}^M Y^D), \pi_*^\perp(\nabla_\xi^M Z^D)) \\
 &= -g(\eta(\nabla_{X^D}^M Y^D)\xi, \eta(\nabla_\xi^M Z^D)\xi).
 \end{aligned}$$

However,  $\eta(\nabla_\xi^M Z^D) = 0$  because of  $g(Z^D, \xi) = 0$ . Thus

$$(3.15) \quad -g(A_{X^D} Y^D, T_\xi Z^D) = 0.$$

Similary we find

$$(3.16) \quad g(A_{Y^D} Z^D, T_\xi Z^D) = g(A_{Z^D} X^D, T_\xi Y^D) = 0.$$

From (3.14), (3.15) and (3.16), (3.9) implies

$$g(R^M(X^D, Y^D)Z^D, \xi) = -g((\nabla_Z^M \phi)X, Y) + \eta(Y)g(X, Z) - \eta(X)g(Z, Y),$$

from which, together with the second, the third, the forth, the fifth terms of the right hand members in (3.1), by

$$\begin{aligned}
 & \eta(X)g(R^M(\xi, Y^D)Z^D, W^D) = \eta(X)g(R^M(Z^D, W^D)\xi, Y^D) \\
 & = -\eta(X)g(R^M(Z^D, W^D)Y^D, \xi) \\
 & = -\eta(X)(-g((\nabla_Y^M \phi)Z, W) + \eta(W)g(Z, Y) - \eta(Z)g(Y, W)) \\
 & = \eta(X)g((\nabla_Y^M \phi)Z, W) - \eta(X)\eta(W)g(Z, Y) + \eta(X)\eta(Z)g(Y, W), \\
 & \eta(Y)g(R^M(X^D, \xi)Z^D, W^D) = -\eta(Y)g(R^M(\xi, X^D)Z^D, W^D) \\
 & = -\eta(Y)g((\nabla_X^M \phi)Z, W) + \eta(Y)\eta(W)g(Z, X) - \eta(Y)\eta(Z)g(X, W),
 \end{aligned}$$

$$\begin{aligned}
& \eta(Z)g(R^M(X^D, Y^D)\xi, W^D) = \eta(Z)g((\nabla_W^M \phi)X, Y) \\
& -\eta(Z)\eta(Y)g(X, W) + \eta(Z)\eta(X)g(W, Y), \\
& \eta(W)g(R^M(X^D, Y^D)Z^D, \xi) = -\eta(W)g(R^M(X^D, Y^D)\xi, Z^D) \\
& -\eta(W)g((\nabla_Z^M \phi)X, Y) + \eta(W)\eta(Y)g(X, Z) - \eta(W)\eta(X)g(Z, Y),
\end{aligned}$$

we get

$$\begin{aligned}
(3.17) \quad & \eta(X)g(R^M(\xi, Y^D)Z^D, W^D) + \eta(Y)g(R^M(X^D, \xi)Z^D, W^D) \\
& + \eta(Z)g(R^M(X^D, Y^D)\xi, W^D) + \eta(W)g(R^M(X^D, Y^D)Z^D, \xi) = \\
& = \eta(X)g((\nabla_Y^M \phi)Z, W) + \eta(Y)g((\nabla_X^M \phi)W, Z) + \eta(W)g((\nabla_Z^M \phi)Y, X) \\
& + \eta(Z)g((\nabla_W^M \phi)X, Y) + 2\eta(X)\eta(Z)g(Y, W) - 2\eta(X)\eta(W)g(Z, Y) \\
& + 2\eta(Y)\eta(W)g(Z, X) - 2\eta(Y)\eta(Z)g(X, W).
\end{aligned}$$

Next we calculate the sixth, seventh, eight and ninth terms of the right hand memmbers in (3.1). First we calculate  $g(R^M(\xi, Y^D)\xi, W^D)$  by means of (2.14):

$$\begin{aligned}
(3.18) \quad & g(R^M(\xi, Y^D)\xi, W^D) = g(R^M(Y^D, \xi)W^D, \xi) \\
& = -g((\nabla_{Y^D}^M T)_\xi \xi, W^D) + g(T_\xi Y^D, T_\xi W^D) \\
& - g((\nabla_\xi^M A)_{Y^D} W^D, \xi) - g(A_{Y^D} \xi, A_{W^D} \xi).
\end{aligned}$$

Moreover we have

$$(3.19) \quad (\nabla_{Y^D}^M T)_\xi \xi = \nabla_{Y^D}^M (T_\xi \xi) - T_{\nabla_{Y^D}^M \xi} \xi - T_\xi (\nabla_{Y^D}^M \xi).$$

Here,  $T_\xi \xi = \pi_*(\nabla_\xi^M \xi) = 0$  because of (2.12) and (2.1).

$$(3.20) \quad T_{\nabla_{Y^D}^M \xi} \xi = T_{(\nabla_{Y^D}^M \xi)^h + \eta(\nabla_{Y^D}^M \xi)\xi} \xi = \eta(\nabla_{Y^D}^M \xi) T_\xi \xi = 0$$

by means of (2.12) and (2.1). The third term of the right hand members in (3.19) is, from (2.12) and (2.1),

$$\begin{aligned}
(3.21) \quad & T_\xi (\nabla_{Y^D}^M \xi) = T_\xi ((\nabla_{Y^D}^M \xi)^h + \eta(\nabla_{Y^D}^M \xi)\xi) \\
& = \pi_*^\perp (\nabla_\xi^M (\nabla_{Y^D}^M \xi)^h).
\end{aligned}$$



Thus, substituting (3.20) and (3.21) in (3.19), the first term of the right hand members in (3.18) implies that

$$(3.22) \quad -g((\nabla_{Y^D}^M T)_\xi \xi, W^D) = 0.$$

Here we calculate the second term of the right hand members in (3.18). Since we have

$$T_\xi Y^D = \pi_*^\perp(\nabla_\xi^M Y^D) = g(\nabla_\xi^M Y^D, \xi)\xi = 0$$

by means of differentiating  $g(Y^D, \xi) = 0$  covariantly and (2.1), we get

$$(3.23) \quad g(T_\xi Y^D, T_\xi W^D) = 0.$$

Next we calculate the third term of the right hand members in (3.18). Since  $(\nabla_\xi^M A)_{Y^D}$  is alternating, we have

$$(3.24) \quad -g((\nabla_\xi^M A)_{Y^D} W^D, \xi) = g((\nabla_\xi^M A)_{Y^D} \xi, W^D).$$

Here we have

$$(3.25) \quad \begin{aligned} (\nabla_\xi^M A)_{Y^D} \xi &= \nabla_\xi^M (A_{Y^D} \xi) - A_{\nabla_\xi^M Y^D} \xi - A_{Y^D} (\nabla_\xi^M \xi) \\ &= \nabla_\xi^M (A_{Y^D} \xi) - A_{\nabla_\xi^M Y^D} \xi, \end{aligned}$$

by means of (2.1). The first term of the right hand members in (3.25) implies, by (2.11),

$$\nabla_\xi^M (A_{Y^D} \xi) = \nabla_\xi^M (\pi_*(\nabla_{Y^D}^M \xi)) = \nabla_\xi^M (\nabla_{Y^D}^M \xi) - \nabla_\xi^M (\eta(\nabla_{Y^D}^M \xi)\xi).$$

However,  $\eta(\nabla_{Y^D}^M \xi) = 0$  because of differentiating  $g(\xi, \xi) = 1$  covariantly, it follows that

$$(3.26) \quad \nabla_\xi^M (A_{Y^D} \xi) = \nabla_\xi^M (\nabla_{Y^D}^M \xi) = \nabla_\xi^M (\nabla_{Y - \eta(Y)\xi}^M \xi) = \nabla_\xi^M \nabla_Y^M \xi = -\phi(\nabla_\xi^M Y).$$

Next the second term of the right hand members in (3.25) implies that, from (2.11) and (2.1),

$$(3.27) \quad A_{\nabla_\xi^M Y^D} \xi = A_{(\nabla_\xi^M Y^D)^h + \eta(\nabla_\xi^M Y^D)\xi} \xi = \pi_*(\nabla_{(\nabla_\xi^M Y^D)^h}^M \xi)$$

$$= \pi_*(\nabla_{\nabla_\xi^M Y^D - \eta(\nabla_\xi^M Y^D)\xi}^M \xi) = \pi_*(\nabla_{\nabla_\xi^M (Y - \eta(Y)\xi)}^M \xi) = \pi_*(\nabla_{\nabla_\xi^M Y}^M \xi),$$

by making use of differentiating  $g(Y^D, \xi) = 0$  covariantly and (2.1). Substituting (3.26) and (3.27) in (3.25), (3.24) becomes to

$$\begin{aligned} -g((\nabla_\xi^M A)_{Y^D} W^D, \xi) &= g(\nabla_\xi^M \nabla_Y^M \xi, W^D) - g(\pi_*(\nabla_{\nabla_\xi^M Y}^M \xi), W^D) \\ &= g(\nabla_\xi^M \nabla_Y^M \xi, W - \eta(W)\xi) - g(\nabla_{\nabla_\xi^M Y}^M \xi - \eta(\nabla_{\nabla_\xi^M Y}^M \xi)\xi, W - \eta(W)\xi). \end{aligned}$$

However, from  $g(\xi, \xi) = 1$ , we have  $g(\nabla_{\nabla_{\xi}^M Y}^M \xi, \xi) = 0$  and  $g(\nabla_{\xi}^M \nabla_Y^M \xi, \xi) = 0$ . Thus we get

$$(3.28) \quad \begin{aligned} -g((\nabla_{\xi}^M A)_{Y^D} W^D, \xi) &= g(\nabla_{\xi}^M \nabla_Y^M \xi, W) - g(\nabla_{\nabla_{\xi}^M Y}^M \xi, W) \\ &= -g(\phi(\nabla_{\xi}^M Y), W) + g(\phi(\nabla_{\xi}^M Y), W) = 0. \end{aligned}$$

Next we calculate, by means of (2.11), differentiating  $g(\xi, \xi) = 1$  covariantly and (2.1), the forth term of right hand members in (3.18).

$$(3.29) \quad \begin{aligned} -g(A_{Y^D} \xi, A_{W^D} \xi) &= -g((\nabla_{Y^D}^M \xi)^h, (\nabla_{W^D}^M \xi)^h) \\ &= -g(\nabla_{Y^D}^M \xi - \eta(\nabla_{Y^D}^M \xi) \xi, \nabla_{W^D}^M \xi - \eta(\nabla_{W^D}^M \xi) \xi) = -g(\nabla_{Y^D}^M \xi, \nabla_{W^D}^M \xi) \\ &= -g(\phi Y, \phi W) = -g(Y, W) + \eta(Y) \eta(W). \end{aligned}$$

Thus, substituting (3.22), (3.23), (3.28) and (3.29) into (3.18), we get

$$(3.30) \quad \begin{aligned} \eta(X) \eta(Z) g(R^M(\xi, Y^D) \xi, W^D) &= \eta(X) \eta(Z) (-g(Y, W) \\ &+ \eta(Y) \eta(W)) = -\eta(X) \eta(Z) g(Y, W) + \eta(X) \eta(Y) \eta(Z) \eta(W) \end{aligned}$$

Similary, we have

$$(3.31) \quad \begin{aligned} \eta(Y) \eta(Z) g(R^M(X^D, \xi) \xi, W^D) &= -\eta(Y) \eta(Z) g(R^M(\xi, X^D) \xi, W^D) \\ &= \eta(Y) \eta(Z) g(X, W) - \eta(X) \eta(Y) \eta(Z) \eta(W) \end{aligned}$$

$$(3.32) \quad \begin{aligned} \eta(X) \eta(W) g(R^M(\xi, Y^D) Z^D, \xi) &= -\eta(X) \eta(W) g(R^M(\xi, Y^D) \xi, Z^D) \\ &= \eta(X) \eta(W) g(Y, Z) - \eta(X) \eta(Y) \eta(Z) \eta(W) \end{aligned}$$

$$(3.33) \quad \begin{aligned} \eta(Y) \eta(W) g(R^M(X^D, \xi) Z^D, \xi) &= \eta(Y) \eta(W) g(R^M(\xi, X^D) \xi, Z^D) \\ &= -\eta(Y) \eta(W) g(X, Z) + \eta(X) \eta(Y) \eta(Z) \eta(W). \end{aligned}$$

Thus, adding (3.30), (3.31), (3.32) and (3.33), we get

$$(3.34) \quad \begin{aligned} \eta(X) \eta(Z) g(R^M(\xi, Y^D) \xi, W^D) &+ \eta(Y) \eta(Z) g(R^M(X^D, \xi) \xi, W^D) \\ &+ \eta(X) \eta(W) g(R^M(\xi, Y^D) Z^D, \xi) + \eta(Y) \eta(W) g(R^M(X^D, \xi) Z^D, \xi) \\ &= -\eta(X) \eta(Z) g(Y, W) + \eta(Y) \eta(Z) g(X, W) + \eta(X) \eta(W) g(Y, Z) - \eta(Y) \eta(W) g(X, Z). \end{aligned}$$

Proposition 2.2, (2.10), (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), (3.17) and (3.34) lead us to

$$\begin{aligned}
g(R^M(X, Y)Z, W) &= \frac{H^M+3}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \\
&\quad + g(Y, \phi Z)g(X, \phi W) - g(X, \phi Z)g(Y, \phi W) - 2g(X, \phi Y)g(Z, \phi W) \\
&\quad + \eta(X)\eta(Z)g(Y, W) + \eta(Y)\eta(W)g(X, Z) - \eta(Y)\eta(Z)g(X, W) - \eta(X)\eta(W)g(Y, Z)\} \\
&\quad + \frac{1}{16} \{g(\widetilde{N}_\phi(X, Z), \widetilde{N}_\phi(Y, W)) - g(\widetilde{N}_\phi(X, W), \widetilde{N}_\phi(Y, Z)) + 2g(\widetilde{N}_\phi(X, Y), \widetilde{N}_\phi(Z, W))\} \\
&\quad + \frac{1}{96} \left[ -13 \left\{ \eta(Z) \left( g(\phi X, \widetilde{N}_\phi(W, Y)) - g(\phi Y, \widetilde{N}_\phi(W, X)) \right) \right. \right. \\
&\quad \quad + \eta(W) \left( g(\phi Y, \widetilde{N}_\phi(Z, X)) - g(\phi X, \widetilde{N}_\phi(Z, Y)) \right) \\
&\quad \quad + \eta(X) \left( g(\phi Z, \widetilde{N}_\phi(Y, W)) - g(\phi W, \widetilde{N}_\phi(Y, Z)) \right) \\
&\quad \quad + \eta(Y) \left( g(\phi W, \widetilde{N}_\phi(X, Z)) - g(\phi Z, \widetilde{N}_\phi(X, W)) \right) \\
&\quad \quad + g(X, (\nabla_Y^M \widetilde{N}_\phi)(Z, W)) - g(Y, (\nabla_X^M \widetilde{N}_\phi)(Z, W)) \\
&\quad \quad + g(Z, (\nabla_W^M \widetilde{N}_\phi)(X, Y)) - g(W, (\nabla_Z^M \widetilde{N}_\phi)(X, Y)) \\
&\quad \quad + \eta(X)g(Y, (\nabla_\xi^M \widetilde{N}_\phi)(Z, W)) - \eta(Y)g(X, (\nabla_\xi^M \widetilde{N}_\phi)(Z, W)) \\
&\quad \quad + \eta(Z)g(W, (\nabla_\xi^M \widetilde{N}_\phi)(X, Y)) - \eta(W)g(Z, (\nabla_\xi^M \widetilde{N}_\phi)(X, Y)) \\
&\quad \quad - 2g(\phi X, Y)\eta(\widetilde{N}_\phi(Z, W)) - 2g(\phi Z, W)\eta(\widetilde{N}_\phi(X, Y)) \\
&\quad \quad + \eta(Y)\eta((\nabla_X^M \widetilde{N}_\phi)(Z, W)) - \eta(X)\eta((\nabla_Y^M \widetilde{N}_\phi)(Z, W)) \\
&\quad \quad \left. + \eta(W)\eta((\nabla_Z^M \widetilde{N}_\phi)(X, Y)) - \eta(Z)\eta((\nabla_W^M \widetilde{N}_\phi)(X, Y)) \right\} \\
&\quad + 3 \left\{ \eta(W) \left( g(\widetilde{N}_\phi(Z, X), \phi Y) - g(\widetilde{N}_\phi(Z, Y), \phi X) - g(\widetilde{N}_\phi(Y, X), \phi Z) \right) \right. \\
&\quad \quad + \eta(Z) \left( g(\widetilde{N}_\phi(X, W), \phi Y) - g(\widetilde{N}_\phi(Y, W), \phi X) - g(\widetilde{N}_\phi(X, Y), \phi W) \right) \\
&\quad \quad + \eta(Y) \left( g(\widetilde{N}_\phi(X, Z), \phi W) - g(\widetilde{N}_\phi(W, Z), \phi X) - g(\widetilde{N}_\phi(X, W), \phi Z) \right) \\
&\quad \quad \left. + \eta(X) \left( g(\widetilde{N}_\phi(W, Z), \phi Y) - g(\widetilde{N}_\phi(Y, Z), \phi W) - g(\widetilde{N}_\phi(W, Y), \phi Z) \right) \right. \\
&\quad \quad - g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi X}^M \phi)Y) + g(\widetilde{N}_\phi(W, Z), (\nabla_{\phi Y}^M \phi)X) - g(\widetilde{N}_\phi(Y, X), (\nabla_{\phi Z}^M \phi)W) \\
&\quad \quad + g(\widetilde{N}_\phi(Y, X), (\nabla_{\phi W}^M \phi)Z) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Z), \phi Y) + g((\nabla_{\phi Y}^M \widetilde{N}_\phi)(W, Z), \phi X) \\
&\quad \quad \left. - g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(Y, X), \phi W) + g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Y, X), \phi Z) \right\} \\
&\quad - \frac{13}{2} \left\{ \eta(X) \left( g(\phi W, \widetilde{N}_\phi(Y, Z)) - g(\phi Z, \widetilde{N}_\phi(Y, W)) \right) \right. \\
&\quad \quad + \eta(Y) \left( g(\phi Z, \widetilde{N}_\phi(X, W)) - g(\phi W, \widetilde{N}_\phi(X, Z)) \right) \\
&\quad \quad + \eta(Z) \left( g(\phi Y, \widetilde{N}_\phi(W, X)) - g(\phi X, \widetilde{N}_\phi(W, Y)) \right) \\
&\quad \quad \left. + \eta(W) \left( g(\phi X, \widetilde{N}_\phi(Z, Y)) - g(\phi Y, \widetilde{N}_\phi(Z, X)) \right) \right. \\
&\quad \quad + 2 \left( \eta(X)g(\phi Y, \widetilde{N}_\phi(Z, W)) - \eta(Y)g(\phi X, \widetilde{N}_\phi(Z, W)) \right. \\
&\quad \quad \left. + \eta(Z)g(\phi W, \widetilde{N}_\phi(X, Y)) - \eta(W)g(\phi Z, \widetilde{N}_\phi(X, Y)) \right) \\
&\quad \quad + g(X, (\nabla_Z^M \widetilde{N}_\phi)(Y, W)) - g(X, (\nabla_W^M \widetilde{N}_\phi)(Y, Z)) \\
&\quad \quad + g(Y, (\nabla_W^M \widetilde{N}_\phi)(X, Z)) - g(Y, (\nabla_Z^M \widetilde{N}_\phi)(X, W)) \\
&\quad \quad + g(Z, (\nabla_X^M \widetilde{N}_\phi)(W, Y)) - g(Z, (\nabla_Y^M \widetilde{N}_\phi)(W, X)) \\
&\quad \quad + g(W, (\nabla_Y^M \widetilde{N}_\phi)(Z, X)) - g(W, (\nabla_X^M \widetilde{N}_\phi)(Z, Y)) \\
&\quad \quad \left. + \eta(X) \left( g(Z, (\nabla_\xi^M \widetilde{N}_\phi)(Y, W)) - g(W, (\nabla_\xi^M \widetilde{N}_\phi)(Y, Z)) \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& +\eta(Y) \left( g(W, (\nabla_{\xi}^M \widetilde{N}_{\phi})(X, Z)) - g(Z, (\nabla_{\xi}^M \widetilde{N}_{\phi})(X, W)) \right) \\
& +\eta(Z) \left( g(X, (\nabla_{\xi}^M \widetilde{N}_{\phi})(W, Y)) - g(Y, (\nabla_{\xi}^M \widetilde{N}_{\phi})(W, X)) \right) \\
& +\eta(W) \left( g(Y, (\nabla_{\xi}^M \widetilde{N}_{\phi})(Z, X)) - g(X, (\nabla_{\xi}^M \widetilde{N}_{\phi})(Z, Y)) \right) \\
& +2 \left( g(\phi X, W) \eta(\widetilde{N}_{\phi}(Y, Z)) + g(\phi Y, Z) \eta(\widetilde{N}_{\phi}(X, W)) \right. \\
& \quad \left. - g(\phi X, Z) \eta(\widetilde{N}_{\phi}(Y, W)) - g(\phi Y, W) \eta(\widetilde{N}_{\phi}(X, Z)) \right) \\
& +\eta(X) \left( \eta((\nabla_W^M \widetilde{N}_{\phi})(Y, Z)) - \eta((\nabla_Z^M \widetilde{N}_{\phi})(Y, W)) \right) \\
& +\eta(Y) \left( \eta((\nabla_Z^M \widetilde{N}_{\phi})(X, W)) - \eta((\nabla_W^M \widetilde{N}_{\phi})(X, Z)) \right) \\
& +\eta(Z) \left( \eta((\nabla_Y^M \widetilde{N}_{\phi})(W, X)) - \eta((\nabla_X^M \widetilde{N}_{\phi})(W, Y)) \right) \\
& +\eta(W) \left( \eta((\nabla_X^M \widetilde{N}_{\phi})(Z, Y)) - \eta((\nabla_Y^M \widetilde{N}_{\phi})(Z, X)) \right) \Big\} \\
& +\frac{3}{2} \Big\{ 2\eta(X) \left( g(\widetilde{N}_{\phi}(Y, Z), \phi W) - g(\widetilde{N}_{\phi}(W, Z), \phi Y) - g(\widetilde{N}_{\phi}(Y, W), \phi Z) \right) \\
& +2\eta(Y) \left( g(\widetilde{N}_{\phi}(X, W), \phi Z) - g(\widetilde{N}_{\phi}(Z, W), \phi X) - g(\widetilde{N}_{\phi}(X, Z), \phi W) \right) \\
& +2\eta(Z) \left( g(\widetilde{N}_{\phi}(X, Y), \phi W) - g(\widetilde{N}_{\phi}(X, W), \phi Y) - g(\widetilde{N}_{\phi}(W, Y), \phi X) \right) \\
& +2\eta(W) \left( g(\widetilde{N}_{\phi}(Y, X), \phi Z) - g(\widetilde{N}_{\phi}(Y, Z), \phi X) - g(\widetilde{N}_{\phi}(Z, X), \phi Y) \right) \\
& +g(\widetilde{N}_{\phi}(W, Y), (\nabla_{\phi Z}^M \phi)X) - g(\widetilde{N}_{\phi}(W, Y), (\nabla_{\phi X}^M \phi)Z) + g(\widetilde{N}_{\phi}(Z, X), (\nabla_{\phi W}^M \phi)Y) \\
& -g(\widetilde{N}_{\phi}(Z, X), (\nabla_{\phi Y}^M \phi)W) + g(\widetilde{N}_{\phi}(Z, Y), (\nabla_{\phi X}^M \phi)W) - g(\widetilde{N}_{\phi}(Z, Y), (\nabla_{\phi W}^M \phi)X) \\
& +g(\widetilde{N}_{\phi}(W, X), (\nabla_{\phi Y}^M \phi)Z) - g(\widetilde{N}_{\phi}(W, X), (\nabla_{\phi Z}^M \phi)Y) \\
& +g((\nabla_{\phi Z}^M \widetilde{N}_{\phi})(W, Y), \phi X) - g((\nabla_{\phi X}^M \widetilde{N}_{\phi})(W, Y), \phi Z) \\
& +g((\nabla_{\phi W}^M \widetilde{N}_{\phi})(Z, X), \phi Y) - g((\nabla_{\phi Y}^M \widetilde{N}_{\phi})(Z, X), \phi W) \\
& +g((\nabla_{\phi X}^M \widetilde{N}_{\phi})(Z, Y), \phi W) - g((\nabla_{\phi W}^M \widetilde{N}_{\phi})(Z, Y), \phi X) \\
& +g((\nabla_{\phi Y}^M \widetilde{N}_{\phi})(W, X), \phi Z) - g((\nabla_{\phi Z}^M \widetilde{N}_{\phi})(W, X), \phi Y) \Big\} \\
& +2 \Big\{ -\eta(X)g(\widetilde{N}_{\phi}(Z, W), \phi Y) + \eta(Y)g(\widetilde{N}_{\phi}(Z, W), \phi X) \\
& +2\eta(Z) \left( g(\widetilde{N}_{\phi}(X, W), \phi Y) - g(\widetilde{N}_{\phi}(Y, W), \phi X) \right) \\
& +2\eta(W) \left( g(\widetilde{N}_{\phi}(Y, Z), \phi X) - g(\widetilde{N}_{\phi}(X, Z), \phi Y) \right) \\
& +g(\widetilde{N}_{\phi}(Z, W), (\nabla_{\phi X}^M \phi)Y) - g(\widetilde{N}_{\phi}(Z, W), (\nabla_{\phi Y}^M \phi)X) \\
& +g((\nabla_X^M \widetilde{N}_{\phi})(W, Z), Y) + g((\nabla_Y^M \widetilde{N}_{\phi})(Z, W), X) \\
& +g((\nabla_{\phi X}^M \widetilde{N}_{\phi})(Z, W), \phi Y) + g((\nabla_{\phi Y}^M \widetilde{N}_{\phi})(W, Z), \phi X) \\
& +\eta(X)g((\nabla_{\xi}^M \widetilde{N}_{\phi})(Z, W), Y) - \eta(Y)g((\nabla_{\xi}^M \widetilde{N}_{\phi})(Z, W), X) \\
& -2g(\phi X, Y)\eta(\widetilde{N}_{\phi}(Z, W)) + \eta(X)\eta((\nabla_Y^M \widetilde{N}_{\phi})(Z, W)) \\
& \quad -\eta(Y)\eta((\nabla_X^M \widetilde{N}_{\phi})(W, Z)) \\
& +\eta(X) \left( g(\widetilde{N}_{\phi}(Y, Z), \phi W) - g(\widetilde{N}_{\phi}(Y, W), \phi Z) \right) \\
& +2 \left( \eta(Y)g(\widetilde{N}_{\phi}(X, W), \phi Z) - \eta(W)g(\widetilde{N}_{\phi}(X, Y), \phi Z) \right) \\
& +2 \left( \eta(Z)g(\widetilde{N}_{\phi}(X, Y), \phi W) - \eta(Y)g(\widetilde{N}_{\phi}(X, Z), \phi W) \right) \\
& +3 \left( \eta(Z)g(\widetilde{N}_{\phi}(Y, W), \phi X) - \eta(W)g(\widetilde{N}_{\phi}(Y, Z), \phi X) \right) \\
& +4\eta(Y)g(\widetilde{N}_{\phi}(W, Z), \phi X) \\
& +g(\widetilde{N}_{\phi}(Y, W), (\nabla_{\phi X}^M \phi)Z) - g(\widetilde{N}_{\phi}(Y, W), (\nabla_{\phi Z}^M \phi)X)
\end{aligned}$$

$$\begin{aligned}
& +g(\widetilde{N}_\phi(Y, Z), (\nabla_{\phi W}^M \phi)X) - g(\widetilde{N}_\phi(Y, Z), (\nabla_{\phi X}^M \phi)W) \\
& +g((\nabla_X^M \widetilde{N}_\phi)(W, Y), Z) - g((\nabla_Z^M \widetilde{N}_\phi)(W, Y), X) \\
& +g((\nabla_W^M \widetilde{N}_\phi)(Z, Y), X) - g((\nabla_X^M \widetilde{N}_\phi)(Z, Y), W) \\
& +g((\nabla_{\phi X}^M \widetilde{N}_\phi)(Z, Y), \phi W) - g((\nabla_{\phi W}^M \widetilde{N}_\phi)(Z, Y), \phi X) \\
& +g((\nabla_{\phi Z}^M \widetilde{N}_\phi)(W, Y), \phi X) - g((\nabla_{\phi X}^M \widetilde{N}_\phi)(W, Y), \phi Z) \\
& +\eta(X) \left( g((\nabla_\xi^M \widetilde{N}_\phi)(Y, W), Z) - g((\nabla_\xi^M \widetilde{N}_\phi)(Y, Z), W) \right) \\
& +\eta(W)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, Z), X) - \eta(Z)g((\nabla_\xi^M \widetilde{N}_\phi)(Y, W), X) \\
& +2 \left( g(\phi X, W)\eta(\widetilde{N}_\phi(Y, Z)) - g(\phi X, Z)\eta(\widetilde{N}_\phi(Y, W)) \right) \\
& +\eta(X) \left( \eta((\nabla_W^M \widetilde{N}_\phi)(Y, Z)) - \eta((\nabla_Z^M \widetilde{N}_\phi)(Y, W)) \right) \\
& +\eta(W)\eta((\nabla_X^M \widetilde{N}_\phi)(Z, Y)) - \eta(Z)\eta((\nabla_X^M \widetilde{N}_\phi)(W, Y)) \Big] \\
& +2g(\phi X, Y)g(\phi Z, W) - g(\phi Y, Z)g(\phi X, W) + g(\phi X, Z)g(\phi Y, W) \\
& +\eta(X)g((\nabla_Y^M \phi)Z, W) + \eta(Y)g((\nabla_X^M \phi)W, Z) \\
& +\eta(Z)g((\nabla_W^M \phi)X, Y) + \eta(W)g((\nabla_Z^M \phi)Y, X) \\
& +2\eta(X) (\eta(Z)g(Y, W) - \eta(W)g(Y, Z)) \\
& +2\eta(Y) (\eta(W)g(X, Z) - \eta(Z)g(X, W)) \\
& -\eta(X) (\eta(Z)g(Y, W) - \eta(W)g(Y, Z)) \\
& -\eta(Y) (\eta(W)g(X, Z) - \eta(Z)g(X, W)),
\end{aligned}$$

from which, we obtain our result.

**Remark.** If  $M$  is Sasakian with a submersion of geodesic fibres  $\pi: M \rightarrow (M, F_\xi)$ , it follows that  $\widetilde{N}_\phi = 0$  and  $(\nabla_Y^M \phi)Z = g(Y, Z)\xi - \eta(Z)Y$ . Consequently the identity in Theorem 3.1 is rewritten as follows (see [4] p.280):

$$\begin{aligned}
& g(R^M(X, Y)Z, W) = \frac{H^M+3}{4} \{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\} \\
& + \frac{H^M-1}{4} \{ \eta(X)\eta(Z)g(Y, W) - \eta(Y)\eta(Z)g(X, W) \\
& + \eta(Y)\eta(W)g(X, Z) - \eta(X)\eta(W)g(Y, Z) \} \\
& + g(\phi Y, Z)g(\phi X, W) - g(\phi X, Z)g(\phi Y, W) + 2g(X, \phi Y)g(\phi Z, W) \}.
\end{aligned}$$

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