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Lucky ANETOR

A Survey on Minimal Surfaces

Embedded in Euclidean Space

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A Survey on Minimal Surfaces Embedded in Euclidean Space

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Abstract. The main objective of this work is to present a brief tutorial on minimal surfaces. Furthermore, a great deal of effort was made to make this presentation as elementary and as self-contained as possible. It is hoped that the materials presented in this work will give some motivations for beginners to go further in the study of the exciting field of minimal surfaces in three dimensional Euclidean space and in other ambient spaces. The insight into some physical systems have been greatly improved by modeling them with surfaces that locally minimize area. Such systems include soap films, black holes, compound polymers, protein folding, crystals, etc. The relevant mathematical field started in the 1740s but has recently become an area of intensive research. This is due to the wide spread availability of powerful and relatively inexpensive computers combined with the preponderance of suitable graphical application software packages. Furthermore, its anticipated use in the medical, industrial and scientific applications have provided further impetus for research efforts concerning minimal surfaces. It was shown that minimal surfaces do not always minimize area. Some advantages and shortcomings of the Weierstrass equations were highlighted. Finally, the proofs of Bernstein's and Osserman's theorems were sketched out and some other improved (sharp) versions of these theorems were presented followed by some remarks on surfaces of finite total curvature.

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Key words: minimal surfaces; mean curvature; Bernstein's and Osserman's theorems.

Contents

Chapter 1

Introduction

In 1744, Euler posed and solved the problem of finding the surfaces of revolution that minimize area. The only solution was the catenoid. After about eleven years, in a series of letters to Euler, Lagrange, at age 19, discussed the problem of finding a graph over a region in the plane, with prescribed boundary values, that was a critical point for the area function. He wrote down what would now be called the Euler-Lagrange equation for the solution (a second-order, nonlinear and elliptic partial differential equation), but he did not provide any new solutions. In 1776, Meusnier showed that the helicoid was also a solution. Equally important, he gave a geometric interpretation of the Euler-Lagrange equation in terms of the vanishing of the mean of the principal curvatures of the surface, a quantity now known, after a suggestion of Sophie Germain, as the mean curvature [7, 11, 28].

Minimal surfaces are of interest in various branches of mathematics, science and engineering. In calculus of variations they could appear as surfaces of least or maximum area, (the surface with the least area is called the *stable minimal surface*). Furthermore, in differential geometry, a surface whose mean curvature is zero everywhere is called minimal surface. In gas dynamics, the equation of minimal surface is usually interpreted as the potential model of a hypothetical gas, which yield flows closely approximating that of an adiabatic flow of low Mach number. This interpretation is due to Chaplygin and has been used extensively in aerodynamic studies [3]. In the general theory of partial differential equations, the minimal surface model is the simplest nonlinear equation of the elliptic type. It has the defining property that every sufficiently small piece of it (small enough, say, to be a graph over some plane) is the surface of least area among all surfaces with the same boundary. The simplest minimal surface is the flat plane, but other minimal surfaces are far from simple [24]. Minimal surfaces are realized in the physical world by soap films spanning closed curves, and they appear as interfaces where the pressure is the same on either side. Finding a surface of least area spanning a given contour is known as the Plateau problem, after the nineteenth-century Belgian physicist Felix Plateau.

Surfaces that locally minimize area have been extensively used to model physical phenomena, including soap films, black holes, crystalline and polymer compounds, protein folding, etc [6]. Although the mathematical field started in the 1740s, it has recently become an area of intensive mathematical and scientific study, specifically in the areas of molecular engineering, materials science, and nano-technology, because of their many anticipated applications. Some of the areas where these phenomena are likely to play prominent roles are; interfaces in polymers, physical assembly during chemical reactions, microcellular membrane structures, architectural designs, etc.

Closely related to minimal surfaces are surfaces of constant mean curvature. The definition means what it says, with the only exception that the constant is nonzero. Constant mean curvature surfaces minimize area subject to a volume constraint. The most famous example is the sphere, which minimizes surface area subject to the constraint of enclosing a fixed volume. The physical realization of these type of surfaces appear as soap bubbles and, more generally, as interfaces when there is a pressure difference from one side to the other.

The objective of this survey is to briefly describe some of the major developments in the field of minimal surfaces. It is pretty obvious that it will be impossible to give a complete account of all the works that has been done in this field. In view of this, a number of results which seem to be both interesting and representative, and whose proofs should provide a good picture of some of the methods were presented. Furthermore, a great deal of effort was made to make this survey as elementary and as self-contained as possible. It is my hope that the materials presented in this work will motivate beginners to go further in the study of the exciting field of minimal surfaces.

The objective of the present study was achieved by dividing it into eight Chapters. Section 1 gives a brief introduction into some of the history of the studies that have been carried out in the area of minimal surfaces. In Chapter 2, a literature survey of recent works on minimal surfaces are presented. Some of the basics of differential geometry of surfaces which are relevant to this work were discussed in Chapter 3. In Chapter 4, examples of the classical minimal surfaces, namely, the helicoid, catenoid and Scherk's surface were presented. These surfaces were derived by imposing different conditions (algebraic and geometric) on the minimal surface elliptic partial differential equation. Since the theory of complex functional analysis is intimately intertwined with the study of minimal surfaces, Chapter 5 was devoted to examining the relevant portions of it. The Weierstrass-Enneper representation of minimal surfaces was covered in Chapter 6. It was shown in Chapter 7 that minimal surfaces do not always minimize area. The proofs of Bernstein's and Osserman's theorems were sketched out in Chapter 8 and some other improved (sharp) versions of these theorems were presented followed by some remarks on surfaces of finite total curvature. Some concluding remarks were presented in Chapter 9.

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Chapter 2

Literature review

The bulk of the results of the research that have been done in this field may be found in the works of Douglas [12, 13] and in the books of Rado, Courant and Osserman, [36, 10, 31], respectively. Of all the research that has been carried out in the field of minimal surfaces, Bernstein's work offered a perspective which was different from those of Rado and Courant [10, 36] in that he considered minimal surfaces mainly from the point of view of partial differential equations. Recent efforts in the study of minimal surfaces have been in the areas of generalizations to higher dimensions, Riemannian spaces and wider classes of surfaces. Most recent results on minimal surfaces, focusing on the classification and structure of embedded minimal surfaces and the stable singularities were discussed in [8]. The survey of recent spectacular successes in classical minimal surface theory were presented by [26]. For more information on the theory of minimal surfaces, the interested reader should consult the survey papers [19, 23, 35, 37].

Chapter 3

The basics of differential geometry of surfaces

What is a surface? A precise answer cannot really be given without introducing the concept of a manifold. An informal answer is to say that a surface is a set of points in $\mathbb{R}^3$ such that, for every point p on the surface, there is a small (perhaps very small) neighborhood U of p that is continuously deformable into a little flat open disk. Thus, a surface should really have some topology. Also, locally, unless the point p is "singular", the surface looks like a plane. Properties of surfaces can be classified into local properties and global properties. In the older literature, the study of local properties was called geometry in the small, and the study of global properties was called geometry in the large. Local properties are the properties that hold in a small neighborhood of a point on a surface. Curvature is a local property. Local properties can be studied more conveniently by assuming that the surface is parameterized locally. Therefore, it is relevant and useful to study parameterized patches.

Another more subtle distinction should be made between intrinsic and extrinsic properties of a surface. Roughly speaking, intrinsic properties are properties of a surface that do not depend on the way the surface is immersed in the ambient space, whereas extrinsic properties depend on properties of the ambient space. For example, we will see that the Gaussian curvature is an intrinsic concept, whereas the normal to a surface at a point is an extrinsic concept.

In this chapter, we shall focus exclusively on the study of local properties.

1. By studying the properties of the curvature of curves on a surface, we will be led to the First and the Second Fundamental Form of a surface.

2. The study of the normal and tangential components of the curve will lead to both the normal and geodesic curvatures. But we shall not be treating geodesic curvature, since the objective of this report is to study minimal surfaces.

3. We will study the normal curvature, and this will lead us to principal curvatures, principal directions, the Gaussian curvature, and the mean curvature.

4. The study of the variation of the normal at a point will lead to the Gauss map and its derivative.

3.1 Parameterized surfaces

In this chapter, we consider exclusively surfaces immersed in the affine space $\mathbb{R}^3$. In order to be able to define the normal to a surface at a point, and the notion of curvature, we assume that some inner product is defined on $\mathbb{R}^3$.
Unless specified otherwise, we assume that this inner product is the standard one, that is,

$$\langle (x_1,\, x_2,\, x_3)\,,\, (y_1,\, y_2,\, y_3)\rangle = x_1y_1 + x_2y_2 + x_3y_3. \tag{3.1.1}$$

A surface is a map $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$, where $\mathcal{D}$ is some open subset of the plane $\mathbb{R}^2$, and where $\mathbf{X}$ is at least C^3-continuously differentiable. Actually, we will need to impose an extra condition on a surface $\mathbf{X}$ so that the tangent and the normal planes at any point p are defined. Again, this leads us to consider curves on $\mathbf{X}$.

Definition 3.1.1. A smooth curve lying on the surface $\mathbf{X}$ is a map $t \to (u(t),\, v(t))$ with derivatives of all orders such that $\mathbf{C}(t) = \mathbf{X}(u(t), v(t))$ is a parameterized curve in $\mathbb{R}^3$.

For example, the curves $v \to \mathbf{X}(u_0,\, v)$ for some constant u_0 are called u-curves, and the curves $u \to \mathbf{X}(u,\, v_0)$ for some constant v_0 are called v-curves. Such curves are also called the coordinate curves.

A parameterized curve implies $u(t)$, $v(t)$ have derivatives of all orders and $\mathbf{C}'(t) = \mathbf{X}_u u' + \mathbf{X}_v v' \neq 0$. The definition of a surface implies that $\mathbf{X}_u$, $\mathbf{X}_v$ are linearly independent, so this condition is equivalent to $(u',\, v') \neq 0$.

When the curve $\mathbf{C}$ is parameterized by arc length, s [2], we denote

$$\frac{d\mathbf{C}(s)}{ds},\ \frac{du(s)}{ds} \ \text{and}\ \frac{dv(s)}{ds},$$

by $\mathbf{C}'(s)$, $u'(s)$, and $v'(s)$, or even as $\mathbf{C}'$, u', and v'. Thus, we reserve the prime notation for the case where the parametrization of $\mathbf{C}$ is by arc length.

Note that it is the curve $\mathbf{C} : t \to \mathbf{X}(u(t),\, v(t))$ which is parameterized by arc length, not the curve $t \to (u(t),\, v(t))$.

Using these notations, $\dot{\mathbf{C}}(t)$ is expressed as follows:

$$\dot{\mathbf{C}}(t) = \mathbf{X}_u(t)\dot{u}(t) + \mathbf{X}_v(t)\dot{v}(t), \tag{3.1.2}$$

or simply as

$$\dot{\mathbf{C}} = \mathbf{X}_u\dot{u} + \mathbf{X}_v\dot{v}.$$

Now, if we want $\dot{\mathbf{C}} \neq 0$ for all regular curves $t \to (u(t),\, v(t))$, we must require that $\mathbf{X}_u$ and $\mathbf{X}_v$ be linearly independent and are in the tangent space of surface, $\mathbf{X}$.

Equivalently, we must require that the cross-product $\mathbf{X}_u \times \mathbf{X}_v \neq 0$, that is, be non-null.

Definition 3.1.2. A smooth surface in $\mathbb{R}^3$ is a subset $X \subseteq \mathbb{R}^3$ such that each point has a neighbourhood $U \subseteq X$ and a map $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$, from an open set $\mathcal{D} \subseteq \mathbb{R}^2$, such that:

- $\mathbf{X} : \mathcal{D} \to U$ is a homeomorphism,
- $\mathbf{X}(u,\, v) = (x(u,\, v),\, y(u,\, v),\, z(u,\, v))$ has derivatives of all orders, and,
- At each point $\mathbf{X}_u = \partial\mathbf{X}/\partial u$ and $\mathbf{X}_v = \partial\mathbf{X}/\partial v$ are linearly independent.

We say that the surface $\mathbf{X}$ is regular at $(u,\, v) \in \mathcal{D}$ if and only if $\mathbf{X}_u \times \mathbf{X}_v \neq 0$, and we also say that $p = \mathbf{X}(u, v)$ is a regular point of $\mathbf{X}$. If $\mathbf{X}_u \times \mathbf{X}_v = 0$, we say that $p = \mathbf{X}(u, v)$ is a singular point of $\mathbf{X}$.

The surface $\mathbf{X}$ is regular on $\mathcal{D}$ if and only if $\mathbf{X}_u \times \mathbf{X}_v \neq 0$, for all $(u,\, v) \in \mathcal{D}$. The subset $\mathbf{X}(\mathcal{D})$ of $\mathbb{R}^3$ is called the trace of the surface $\mathbf{X}$.

Remark: It is often desirable to define a (regular) surface patch $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$ where $\mathcal{D}$ is a closed subset of $\mathbb{R}^2$. If $\mathcal{D}$ is a closed set, we assume that there is some open subset U containing $\mathcal{D}$ and such that $\mathbf{X}$ can be extended to a (regular) surface over U (that is, that $\mathbf{X}$ is at least $\mathcal{C}^3$-continuously differentiable).

Given a regular point $p = \mathbf{X}(u,\, v)$, since the tangent vectors to all the curves passing through a given point are of the form

$$\mathbf{X}_u\, \dot{u} + \mathbf{X}_v\, \dot{v}. \tag{3.1.3}$$

it is obvious that they form a vector space of dimension two isomorphic to $\mathbb{R}^2$, called the tangent space at p and denoted as $T_p(\mathbf{X})$.

Note that $(\mathbf{X}_u,\, \mathbf{X}_v)$ is a basis of this vector space $T_p(\mathbf{X})$.

The set of tangent lines passing through p and having some tangent vector in $T_p(\mathbf{X})$ as direction is an affine plane called the affine tangent plane at p. This is the projective space $PT_p(X)$ associated to $T_p(X)$. [41, 1]

The unit vector

$$\mathbf{N}_p = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}. \tag{3.1.4}$$

is called the unit normal vector at p, and the line through p of direction $\mathbf{N}_p$ is the normal line to $\mathbf{X}$ at p. This time, we can use the notation N_p for the line, to distinguish it from the vector $\mathbf{N}_p$.

The fact that we are not requiring the map $\mathbf{X}$ defining a surface $\mathbf{X} : \mathcal{D} \to \mathbb{R}^3$ to be injective may cause problems (Local injectivity might help in handling self-intersecting surfaces). Indeed, if $\mathbf{X}$ is not injective, it may happen that $p = \mathbf{X}(u_0,\, v_0) = \mathbf{X}(u_1,\, v_1)$ for some $(u_0,\, v_0)$ and $(u_1,\, v_1)$ such that $(u_0,\, v_0) \neq (u_1,\, v_1)$. In this case, the tangent plane $T_p(\mathbf{X})$ at p is not well defined. Indeed, we really have two pairs of partial derivatives $(\mathbf{X}_u(u_0,\, v_0)\,, \mathbf{X}_v(u_0,\, v_0))$ and $(\mathbf{X}_u(u_1,\, v_1),\, \mathbf{X}_v(u_1,\, v_1))$, and the planes spanned by these pairs could be distinct.

In this case, there are at least two, for a multiple point tangent plane $T_{(u_0,\, v_0)}(\mathbf{X})$ and $T_{(u_1,\, v_1)}(\mathbf{X})$ at the point p where $\mathbf{X}$ has a self-intersection. Similarly, the normal $\mathbf{N}_p$ is not well defined, and we really have two normals $\mathbf{N}_{(u_0,\, v_0)}$ and $\mathbf{N}_{(u_1,\, v_1)}$ at p.

We could avoid this problem entirely by assuming that $\mathbf{X}$ is injective (homeomorphic). Furthermore, in order to rule out many surfaces (self-intersecting surfaces) which may cause problems in practice, we only need to restrict the domain $\mathcal{D}$ of $\mathbf{X}$.

If necessary, we use the notation $T_{(u,\, v)}(\mathbf{X})$ or $\mathbf{N}_{(u,\, v)}$ which removes possible ambiguities. However, it is a more cumbersome notation, and we will continue to write $T_p(\mathbf{X})$ and $\mathbf{N}_p$, being aware that this may be an ambiguous notation, and that some additional information is needed.

The tangent space may also be undefined when p is not a regular point. For example, consider the surface $\mathbf{X} = (x(u,\, v),\, y(u,\, v),\, z(u,\, v))$ defined such that

$$x = u(u^2 + v^2), \qquad y = v(u^2 + v^2) \qquad \text{and} \qquad z = u^2 v - v^3/3. \tag{3.1.5}$$

Note that all the partial derivatives at the origin $(0,\, 0)$ are zero. Thus, the origin is a singular point of the surface $\mathbf{X}$. Indeed, one can check that the tangent lines at the origin do not lie in a plane.

It is interesting to see how the unit normal vector $\mathbf{N}_p$ changes under a change of parameters.

Assume that $u = u(r,\, s)$ and $v = v(r,\, s)$, where $(r,\, s) \to (u,\, v)$ is a diffeomorphism. By the chain rule

$$\begin{aligned}
\mathbf{X}_r \times \mathbf{X}_s &= \left(\mathbf{X}_u \frac{\partial u}{\partial r} + \mathbf{X}_v \frac{\partial v}{\partial r}\right) \times \left(\mathbf{X}_u \frac{\partial u}{\partial s} + \mathbf{X}_v \frac{\partial v}{\partial s}\right) \\
&= \left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial s} - \frac{\partial u}{\partial s}\frac{\partial v}{\partial r}\right) \mathbf{X}_u \times \mathbf{X}_v \\
&= \det \begin{pmatrix} u_r & u_s \\ v_r & v_s \end{pmatrix} \mathbf{X}_u \times \mathbf{X}_v = \frac{\partial(u,\, v)}{\partial(r,\, s)} \mathbf{X}_u \times \mathbf{X}_v. \qquad (3.1.6)
\end{aligned}$$

where we denoted the Jacobian determinant of the map $(r,\, s) \to (u,\, v)$ by $\partial(u,\, v)/\partial(r,\, s)$. Then, the relationship between the unit vectors $\mathbf{N}_{(u,\, v)}$ and $\mathbf{N}_{(r,\, s)}$ is

$$\mathbf{N}_{(r,\, s)} = \mathbf{N}_{(u,\, v)} \operatorname{sign} \frac{\partial(u,\, v)}{\partial(r,\, s)}. \tag{3.1.7}$$

We will therefore restrict our attention to changes of variables such that the Jacobian determinant $\frac{\partial(u,\, v)}{\partial(r,\, s)}$ is positive.

Notice also that the condition $\mathbf{X}_u \times \mathbf{X}_v \neq 0$ is equivalent to the fact that the Jacobian matrix of the derivative of the map $\mathbf{X} : D \to \mathbb{R}^3$ has rank 2, that is, the derivative $D\mathbf{X}(u, v)$ of $\mathbf{X}$ at $(u,\, v)$ is injective.

Indeed, the Jacobian matrix of the derivative of the map:

$$(u,\, v) \to \mathbf{X}(u,\, v) = (x(u,\, v),\, y(u,\, v),\, z(u,\, v)) \text{ is}$$

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ v_u & z_v \end{pmatrix}.$$

and $\mathbf{X}_u \times \mathbf{X}_v \neq 0$ is equivalent to saying that one of the minors of order 2 is invertible.

That is,

(3.1.8) $\quad x_u y_v - x_v y_u \neq 0, \qquad x_u z_v - x_v v_u \neq 0 \qquad \text{and} \qquad y_u z_v - y_v v_u \neq 0.$

Thus, a regular surface is an immersion of an open set of $\mathbb{R}^2$ into $\mathbb{R}^3$. It is worth mentioning that an embedding or a smooth embedding is an injective immersion which is an embedding in the topological sense, that is, homeomorphism onto its image.

Definition 3.1.3. Let S be an abstract surface. A differentiable map $\varphi : S \to \mathbb{R}^n$ is an embedding if φ is an immersion and a homeomorphism onto its image. For instance, a regular surface in $\mathbb{R}^3$ can be characterized as the image of a surface S by an embedding $\varphi : S \to \mathbb{R}^3$. That means only those abstract surfaces which can be embedded in $\mathbb{R}^3$ could be detected in our study of regular surfaces in $\mathbb{R}^3$.

To a great extent, the properties of a surface can be investigated by studying the properties of curves on the surface. One of the most important properties of a surface is its curvature. A gentle way to introduce the curvature of a surface is to study the curvature of a curve on a surface. For this, we will need to compute the norm of the tangent vector to a curve on a surface. This will lead us to the first fundamental form.

3.2 The First fundamental form (Riemannian metric)

Given a curve $\mathbf{C}$ on a surface $\mathbf{X}$, we first compute the element of arc length of the curve $\mathbf{C}$. For this, we need to compute the square norm of the tangent vector $\dot{\mathbf{C}}(t)$.

The square norm of the tangent vector $\dot{\mathbf{C}}(t)$ to the curve $\mathbf{C}$ at p is

(3.2.1) $$|\dot{\mathbf{C}}|^2 = \langle (\mathbf{X}_u \dot{u} + \mathbf{X}_v \dot{v})\,,\, (\mathbf{X}_u \dot{u} + \mathbf{X}_v \dot{v}) \rangle,$$

where $\langle \star\,, \star \rangle$ is the inner product in $\mathbb{R}^3$, and thus

(3.2.2) $$|\dot{\mathbf{C}}|^2 = \langle \mathbf{X}_u\,,\, \mathbf{X}_u \rangle\, \dot{u}^2 + 2\,(\langle \mathbf{X}_u\,,\, \mathbf{X}_v \rangle) \dot{u}\dot{v} + \langle \mathbf{X}_v\,,\, \mathbf{X}_v \rangle\, \dot{v}^2.$$

Following the common usage, we let

$$E = \langle \mathbf{X}_u\,,\, \mathbf{X}_u \rangle, \qquad F = \langle \mathbf{X}_u\,,\, \mathbf{X}_v \rangle \qquad \text{and} \qquad G = \langle \mathbf{X}_v\,,\, \mathbf{X}_v \rangle.$$

therefore

$$|\dot{\mathbf{C}}|^2 = E\,\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2. \tag{3.2.3}$$

Euler obtained this formula in 1760. Thus, the map

$$(x,\, y) \to Ex^2 + 2Fxy + Gy^2.$$

is a quadratic form on $\mathbb{R}^2$, and since it is equal to $|\dot{\mathbf{C}}|^2$, it is positive definite. This quadratic form plays a major role in the theory of surfaces, and deserves a formal definition.

Definition 3.2.1. Given a surface $\mathbf{X}$, for any point $p = \mathbf{X}(u,\, v)$ on $\mathbf{X}$, and letting

$$E = \langle \mathbf{X}_u\,,\, \mathbf{X}_u \rangle, \qquad F = \langle \mathbf{X}_u\,,\, \mathbf{X}_v \rangle \qquad \text{and} \qquad G = \langle \mathbf{X}_v\,,\, \mathbf{X}_v \rangle.$$

the positive definite quadratic form $(x,\, y) \to Ex^2 + 2Fxy + Gy^2$ is called the **First fundamental form** of $\mathbf{X}$ at p. It is often denoted as I_p and in matrix form, we have

$$I_p\,(x,\, y) = (x,\, y) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3.2.4}$$

Since the map $(x,\, y) \to Ex^2 + 2Fxy + Gy^2$ is a positive definite quadratic form, we must have $E \neq 0$ and $G \neq 0$.

Then, we can write

$$Ex^2 + 2Fxy + Gy^2 = E\left(x + \frac{F}{E}\,y\right)^2 + \frac{EG - F^2}{E}\,y^2.$$

Since this quantity must be positive, we must have $E > 0$, $G > 0$, and also $EG - F^2 > 0$.

The symmetric bilinear form ϕ_I associated with I is an inner product on the tangent space at p, such that

$$\phi_I((x_1,\, y_1)\,,\, (x_2,\, y_2)) = (x_1,\, y_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}. \tag{3.2.5}$$

This inner product is also denoted as $\langle (x_1,\, y_1)\,,\, (x_2,\, y_2) \rangle_p$.

The inner product ϕ_I can be used to determine the angle of two curves passing through p, that is, the angle θ of the tangent vectors to these two curves at p. We have

$$\cos\,\theta = \frac{\langle (\dot{u}_1,\, \dot{v}_1), (\dot{u}_2, \dot{v}_2) \rangle}{\sqrt{I(\dot{u}_1,\, \dot{v}_1)}\sqrt{I(\dot{u}_2, \dot{v}_2)}}. \tag{3.2.6}$$

For example, the angle θ between the u-curve and the v-curve passing through p (where u or v is constant) is given by

$$\cos\,\theta = \frac{F}{\sqrt{EG}}.$$

Thus, the u-curves and the v-curves are orthogonal if and only if $F(u,\, v) = 0$ on $\mathcal{D}$.

It is pertinent to mention that the coefficients, $E = g_{uu}$, $F = g_{uv}$ and $G = g_{vv}$ of the First Fundamental Form play important roles in many intrinsic properties of a surface.

In curvilinear coordinates (where $F = 0$), the quantities $h_u = \sqrt{g_{uu}} = \sqrt{E}$ and $h_v = \sqrt{g_{vv}} = \sqrt{G}$ are called scale factors. They show "how far" X_u and X_v are from being unit vectors. Furthermore, $I(\dot{u}\,,\dot{v})$ shows how far the speed on the curve is from being unit-speed.

Remarks:

(1) Since

$$\left(\frac{ds}{dt}\right)^2 = |\dot{\mathcal{C}}|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2,$$

represents the square of the "element of arc length" of the curve $\mathbf{C}$ on $\mathbf{X}$, and since $du = \dot{u}dt$ and $dv = \dot{v}dt$, one often writes the first fundamental form as

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2. \tag{3.2.7}$$

Thus, the length $l(pq)$ of an arc of curve on the surface joining $p = \mathbf{X}(u(t_0),\, v(t_0))$ and $q = \mathbf{X}(u(t_1),\, v(t_1))$, is

$$l(p,\, q) = \int_{t_0}^{t_1} \sqrt{E\,\dot{u}^2 + 2F\,\dot{u}\,\dot{v} + G\,\dot{v}^2}\; dt. \tag{3.2.8}$$

One also refers to $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ as a <u>Riemannian metric</u>. The symmetric matrix associated with the first fundamental form is also denoted as

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \text{where} \quad g_{12} = g_{21}.$$

(2) As in the previous chapter, if $\mathbf{X}$ is not injective, the first fundamental form I_p will not be well defined. What is well defined is $I_{(u,\,v)}$. In some sense, this is even worse, since one of the main themes of differential geometry is that the metric properties of a surface (or of a manifold) are captured by a Riemannian metric. We will not be too bordered about this, since we can always assume that $\mathbf{X}$ injective by reducing the study to a subdomain of $\mathcal{D}$ where injectivity holds.

(3) Using the Lagrange identity, $|\mathbf{X}_u|^2\ |\mathbf{X}_v|^2 = (\langle \mathbf{X}_u\,,\, \mathbf{X}_v\rangle)^2 + |\mathbf{X}_u \times \mathbf{X}_v|^2$ for any two vectors $\mathbf{X}_u,\, \mathbf{X}_v \in \mathbb{R}^3$, it can be shown that the element of area dA on a surface $\mathbf{X}$ is given by

$$dA = |\mathbf{X}_u \times \mathbf{X}_v|\, du\, dv = \sqrt{EG - F^2}\, du\, dv. \tag{3.2.9}$$

We just discovered that, contrary to a flat surface where the inner product is the same at every point, on a curved surface, the inner product induced by the Riemannian metric on the tangent space at every point changes as the point moves on the surface.

This fundamental idea is at the heart of the definition of an abstract Riemannian manifold. It is also important to observe that the first fundamental form of a surface does not characterize the surface.

For example, it is easy to see that the first fundamental form of a plane and the first fundamental form of a cylinder of revolution defined by

$$\mathbf{X}(u,\,v) = (\cos\,u,\,\sin\,u,\,v), \tag{3.2.10}$$

are identical: $(E,\,F,\,G) = (1,\,0,\,1)$.

Thus $ds^2 = du^2 + dv^2$, which is not surprising. A more striking example is that of the helicoid and of the catenoid.

The helicoid is the surface defined over $\mathbb{R} \times \mathbb{R}$ such that

$$\begin{aligned} x &= u_1\,\cos\,v_1, \\ y &= u_1\,\sin\,v_1, \\ z &= v_1. \end{aligned} \tag{3.2.11}$$

This is the surface generated by a line parallel to the xOy plane, touching the z axis, and also touching an helix of axis Oz. It is easily verified that $(E,\,F,\,G) = (1,\,0,\,u_1^2 + 1)$. Figures 9.1a and 9.1b show a portion of helicoid corresponding to $(0 \le v_1 \le 2\pi)$ and $(-2 \le u_1 \le 2)$.

The catenoid is the surface of revolution defined over $\mathbb{R} \times \mathbb{R}$ such that

$$\begin{aligned} x &= \cosh u_2\,\cos\,v_2, \\ y &= \cosh u_2\,\sin\,v_2, \\ z &= u_2. \end{aligned} \tag{3.2.12}$$

It is the surface obtained by rotating a catenary around the z-axis. It is easily verified that $(E,\,F,\,G) = (\cosh^2 u_2,\,0,\,\cosh^2 u_2)$. Figures 9.1c and 9.1d show a catenoid corresponding to $(0 \le v_2 \le 2\pi)$ and $(-2 \le u_2 \le 2)$.

We can make the change of variables $u_1 = \sinh u_3$, $v_1 = v_3$, which is bijective and whose Jacobian determinant is $\cosh u_3$, which is always positive, obtaining the following parametrization of the helicoid:

$$\begin{aligned} x &= \sinh u_3\,\cos\,v_3, \\ y &= \sinh u_3\,\sin\,v_3, \\ z &= v_3. \end{aligned} \tag{3.2.13}$$

It is easily verified that $(E,\,F,\,G) = (\cosh^2 u_3,\,0,\,\cosh^2 u_3)$, showing that the helicoid and the catenoid have the same first fundamental form. What is happening is that the two surfaces are locally isometric (roughly, this means that there is a smooth map between the two surfaces that preserves distances locally). Indeed, if we consider the portions of the two surfaces corresponding to the domain $\mathbb{R} \times (0,\,2\pi)$, it is possible to deform isometrically the portion of helicoid into the portion of catenoid (note that by excluding 0 and 2π, we made a "slit" in the catenoid (a portion of meridian), and thus we can open up the catenoid and deform it into the helicoid). This is illustrated in Figure 9.2.

An alternative method of deriving the First Fundamental Form is given below

The First fundamental form describes how the surface distorts length from their usual measurements in $\mathbb{R}^3$. If α is a unit speed curve with tangent vector α', then $|\alpha'|^2 = 1 = \langle \alpha' , \alpha' \rangle$ and we have

$$\begin{aligned} 1 &= \langle \alpha' , \alpha' \rangle, \qquad \langle (\mathbf{X}_u u' + \mathbf{X}_v v') , (\mathbf{X}_u u' + \mathbf{X}_v v') \rangle \\ &= \langle \mathbf{X}_u , \mathbf{X}_u \rangle {u'}^2 + (\langle \mathbf{X}_v , \mathbf{X}_u \rangle + \langle \mathbf{X}_v , \mathbf{X}_u \rangle) u'v' + \langle \mathbf{X}_v , \mathbf{X}_v \rangle {v'}^2 \\ &= E{u'}^2 + 2Fu'v' + G{v'}^2. \end{aligned} \tag{3.2.14}$$

where the coefficients E, F and G are called the coefficients of the First fundamental form and are given by

$$E = \langle \mathbf{X}_u , \mathbf{X}_u \rangle, \qquad F = \langle \mathbf{X}_u , \mathbf{X}_v \rangle \quad \text{and} \quad G = \langle \mathbf{X}_v , \mathbf{X}_v \rangle.$$

We will now show how the first fundamental form relates to the curvature of curves on a surface.

3.3 Normal curvature and the second fundamental form

Since the cross product of two vectors is always perpendicular to each of the vectors themselves, it follows that we can always find a unit normal vector $\mathbf{N}$ to the tangent plane of a surface M parameterized by $\mathbf{X}(u, v)$, such that

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}. \tag{3.3.1}$$

The normal curvature in some tangent direction $\mathbf{w}$ is defined as follows:

Take the plane determined by the chosen unit direction vector $\mathbf{w}$ and the unit normal $\mathbf{N}$, denoted by $P = \text{plane}(\mathbf{w}, \mathbf{N})$, and intersect this plane with surface M. The intersection is a curve $\alpha(s)$ (which we can assume is a unit speed curve). For unit speed curves, the curvature, $\kappa(s)$ is $|\alpha''(s)|$. Therefore the curvature in the normal direction should just be the projection of the acceleration onto the normal direction. In order words

Definition 3.3.1. For a unit tangent vector $\mathbf{w}$, the normal curvature in the $\mathbf{w}$-direction is defined as

$$k(\mathbf{w}) = \langle \alpha''(s) , \mathbf{N} \rangle,$$

where the derivatives are taken along the curve with respect to s.

Lemma 3.3.1. *If α is a curve in M, then $\langle \alpha''(s) , \mathbf{N} \rangle = - \langle \alpha' , \mathbf{N}' \rangle$.*

Proof. We know that α' is a tangent vector and $\mathbf{N}$ is normal to the tangent plane, so $\langle \alpha' \,, \mathbf{N} \rangle = 0$. On differentiating both sides of this equation, we have

$$(\langle \alpha' \,, \mathbf{N} \rangle)' = 0 \implies \langle \alpha'' \,, \mathbf{N} \rangle + \langle \alpha' \,, \mathbf{N}' \rangle = 0 \implies \langle \alpha'' \,, \mathbf{N} \rangle = -\langle \alpha' \,, \mathbf{N}' \rangle.$$

□

Remark

Again, the derivatives were taken along the curve with respect to s. To in order to obtain $\mathbf{N}'$, we again use the chain rule as follows:

$$\mathbf{N}' = \mathbf{N}_u u' + \mathbf{N}_v v' \quad \text{with} \quad u' = \frac{du}{ds}, \text{ and } v' = \frac{dv}{ds}.$$

We interpret $\langle \alpha'' \,, \mathbf{N} \rangle$ as the component of acceleration due to the bending of M. Recall that we assumed that α has unit speed so that the magnitude of α' does not affect our measurement. By Lemma 3.3.1 and Remark 1, we have

$$\begin{aligned} k(\mathbf{w}) &= -\langle \alpha' \,, \mathbf{N}' \rangle, \\ &= -\langle (\mathbf{X}_u u' + \mathbf{X}_v v') \,, (\mathbf{N}_u u' + \mathbf{N}_v v') \rangle, \\ &= -\langle \mathbf{X}_u \,, \mathbf{N}_u \rangle {u'}^2 - (\langle \mathbf{X}_v \,, \mathbf{N}_u \rangle + \langle \mathbf{X}_u \,, \mathbf{N}_v \rangle) u'v' - \langle \mathbf{X}_v \,, \mathbf{N}_v \rangle {v'}^2, \\ &= e\,{u'}^2 + 2f u'v' + g\,{v'}^2. \end{aligned} \tag{3.3.2}$$

where the coefficients e, f and g are given by

$$\begin{aligned} e &= -\langle \mathbf{X}_u \,, \mathbf{N}_u \rangle, \\ 2f &= -(\langle \mathbf{X}_v \,, \mathbf{N}_u \rangle + \langle \mathbf{X}_u \,, \mathbf{N}_v \rangle), \\ g &= -\langle \mathbf{X}_v \,, \mathbf{N}_v \rangle. \end{aligned} \tag{3.3.3}$$

These are the coefficients of the **Second fundamental form**.

Recalling that

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|}.$$

and using the Lagrange identity $(\langle \mathbf{X}_u \,, \mathbf{X}_v \rangle)^2 + |\mathbf{X}_u \times \mathbf{X}_v|^2 = |\mathbf{X}_u|^2 \, |\mathbf{X}_v|^2$,

we see that $|\mathbf{X}_u \times \mathbf{X}_v| = \sqrt{EG - F^2}$ and $e = \langle \mathbf{N} \,, \mathbf{X}_{uu} \rangle$ can be written as:

$$e = \frac{\langle \mathbf{X}_u \times \mathbf{X}_v, \, \mathbf{X}_{uu} \rangle}{\sqrt{EG - F^2}} = \frac{(\mathbf{X}_u, \, \mathbf{X}_v, \, \mathbf{X}_{uu})}{\sqrt{EG - F^2}}, \tag{3.3.4}$$

where $(\mathbf{X}_u, \, \mathbf{X}_v, \, \mathbf{X}_{uu})$ is the determinant of the three vectors. Some authors use the notation

$$D = (\mathbf{X}_u, \, \mathbf{X}_v, \, \mathbf{X}_{uu}), \qquad D' = (\mathbf{X}_u, \, \mathbf{X}_v, \, \mathbf{X}_{uv}), \qquad D'' = (\mathbf{X}_u, \, \mathbf{X}_v, \, \mathbf{X}_{vv}).$$

with these notations, we have:

$$e = \frac{D}{\sqrt{EG - F^2}}, \qquad f = \frac{D'}{\sqrt{EG - F^2}}, \qquad g = \frac{D''}{\sqrt{EG - F^2}}.$$

These expressions were used by Gauss to prove his famous **Theorema egregium**.

Since the quadratic form $(x, y) \to e(x^2) + 2fxy + g(y^2)$ plays a very important role in the theory of surfaces, we introduce the following definition

Definition 3.3.2. Given a surface $\mathbf{X}$, for any point $p = \mathbf{X}(u, v)$ on $\mathbf{X}$, the quadratic form $(x, y) \to e(x)^2 + 2fxy + g(y)^2$ is called the Second Fundamental Form of $\mathbf{X}$ at p.

It is often denoted as II_p. For a curve $\mathbf{C}$ on the surface $\mathbf{X}$ (parameterized by arc length), the quantity, $k(\mathbf{w})$ given by the formula

$$k(\mathbf{w}) = e(u')^2 + 2fu'\,v' + g(v')^2,$$

is called the normal curvature of $\mathbf{C}$ at p. Unlike the First Fundamental Form, the Second Fundamental Form is neither necessarily positive nor definite.

It is worth mentioning that the normal curvature at a point p on $\mathbf{X}$ can be shown to be equal to

$$k(\mathbf{w}) = \frac{e(\dot{u})^2 + 2f\dot{u}\,\dot{v} + g(\dot{v})^2}{E(\dot{u})^2 + 2F\dot{u}\,\dot{v} + 2G(\dot{v})^2}. \tag{3.3.5}$$

Properties expressible in terms of the First Fundamental Form are called intrinsic properties of the surface $\mathbf{X}$. Whereas, properties expressible in terms of the Second Fundamental Form are called extrinsic properties of the surface $\mathbf{X}$. They have to do with the way the surface is immersed in $\mathbb{R}^3$. It is pertinent to mention that certain notions that appear to be extrinsic turn out to be intrinsic, such as the geodesic and Gaussian curvatures.

3.4 Principal, Gaussian and Mean curvatures

The results in subsection 3.3 will enable us define various curvatures which are relevant to the study of minimal surfaces.

We will now study how the normal curvature at a point varies when a unit tangent vector varies. In general, we will see that the normal curvature has a maximum value κ_1 and a minimum value κ_2, and that the corresponding directions are orthogonal. This was shown by Euler in 1760.

The quantity $K = \kappa_1 \cdot \kappa_2$ is called the Gaussian curvature and the quantity $H = (\kappa_1 + \kappa_2)/2$ is called the mean curvature. These quantities play a very important role in the theory of surfaces.

We will compute H and K in terms of the first and the second fundamental form. We will also classify points on a surface according to the value and sign of the Gaussian curvature.

Recall that given a surface $\mathbf{X}$ and some point p on $\mathbf{X}$, the vectors $\mathbf{X}_u$, $\mathbf{X}_v$ form a basis of the tangent space $T_p(\mathbf{X})$.

Given a unit vector $\vec{t} = \mathbf{X}_u x + \mathbf{X}_v y$, the normal curvature is given by

$$\kappa_N(\vec{t}) = ex^2 + 2fx\,y + gy^2, \tag{3.4.1}$$

since $Ex^2 + 2Fx\,y + Gy^2 = 1$.

Usually, $(\mathbf{X}_u,\, \mathbf{X}_v)$ is not an orthonormal frame, and it is useful to replace the frame $(\mathbf{X}_u,\, \mathbf{X}_v)$ with an orthonormal frame. It is easy to verify that the frame $(\vec{e_1},\, \vec{e_2})$ defined such that

$$\vec{e_1} = \frac{\mathbf{X}_u}{\sqrt{E}} \qquad \text{and} \qquad \vec{e_2} = \frac{E\,\mathbf{X}_v - F\,\mathbf{X}_u}{\sqrt{E(EG - F^2)}}, \tag{3.4.2}$$

is indeed an orthonormal frame. Since $\langle \vec{e_1},\, \vec{e_2} \rangle = 0$.

With respect to this frame, every unit vector can be written as $\vec{t} = \cos\,\theta\,\vec{e_1} + \sin\,\theta\,\vec{e_2}$, and expressing $(\vec{e_1},\, \vec{e_2})$ in terms of $\mathbf{X}_u$ and $\mathbf{X}_v$, we have

$$\vec{t} = \left(\frac{w\,\cos\,\theta - F\,\sin\,\theta}{w\,\sqrt{E}}\right)\mathbf{X}_u + \frac{\sqrt{E}\,\sin\,\theta}{w}\,\mathbf{X}_v, \qquad \text{where} \qquad w = \sqrt{EG - F^2}.$$

We can now compute $\kappa_N(\vec{t})$, to get
(3.4.3)

$$\kappa_N(\vec{t}) = e\left(\frac{w\,\cos\,\theta - F\,\sin\,\theta}{w\,\sqrt{E}}\right)^2 + 2f\left(\frac{(w\,\cos\,\theta - F\,\sin\,\theta)\,\sin\,\theta}{w^2}\right) + g\,\frac{E\,\sin^2\,\theta}{w^2}.$$

After a lot of trigonometric and algebraic simplifications, Equation (3.4.3) can be written as

$$\kappa_N(\vec{t}) = H + A\,\cos\,2\theta + B\,\sin\,2\theta, \tag{3.4.4}$$

where

$$H = \frac{Ge - 2Ff + Eg}{2(EG - F^2)}, \quad A = \frac{e(EG - 2F^2) + 2EFf - E^2g}{2E(EG - F^2)} \quad \text{and} \quad B = \frac{Ef - Fe}{E\sqrt{EG - F^2}}.$$

If we let $C = \sqrt{A^2 + B^2}$, unless $A = B = 0$, the function

$$f(\theta) = H + A\,\cos\,2\theta + B\,\sin\,2\theta.$$

has a maximum $\kappa_1 = H + C$ for the angles θ_0 and $\theta_0 + \pi$, and a minimum $\kappa_2 = H - C$ for angles $\theta_0 + \pi/2$ and $\theta_0 + 3\pi/2$, where $\cos\,2\theta_0 = A/C$ and $\sin\,2\theta_0 = B/C$. The curvatures κ_1 and κ_2 play a major role in surface theory.

Definition 3.4.1. Given a surface $\mathbf{X}$, for any point p on $\mathbf{X}$, letting A, B, and H be as defined above, and $C = \sqrt{A^2 + B^2}$, unless $A = B = 0$, the normal curvature κ_N at p takes a maximum value κ_1 and a minimum value κ_2 called principal curvatures at p, where $\kappa_1 = H + C$ and $\kappa_2 = H - C$. The directions of the corresponding unit vectors are called the principal directions at p.

The average $H = (\kappa_1 + \kappa_2)/2$ of the principal curvatures is called the mean curvature, and the product $K = \kappa_1 \kappa_2$ of the principal curvatures is called the total curvature, or Gaussian curvature.

Notice that the principal directions θ_0 and $\theta_0 + \pi/2$ corresponding to κ_1 and κ_2 are orthogonal. Note that

$$K = \kappa_1 \kappa_2 = (H - C)(H + C) = H^2 - C^2 = H^2 - (A^2 + B^2).$$

After some laborious calculations, we get the following formulas for the mean curvature, H and the Gaussian curvature, K:

$$H = \frac{Ge - 2Ff + Eg}{2(EG - F^2)}, \tag{3.4.5}$$

and

$$K = \frac{e\,g - f^2}{EG - F^2}. \tag{3.4.6}$$

We have shown that the normal curvature κ_N can be expressed as

$$\kappa_N(\theta) = H + A \cos 2\theta + B \sin 2\theta.$$

over the orthonormal frame $(\vec{e_1}, \vec{e_2})$. We have also shown that the angle θ_0 such that $\cos 2\theta_0 = A/C$ and $\sin 2\theta_0 = B/C$, plays a special role. Indeed, it determines one of the principal directions.

If we rotate the basis $(\vec{e_1}, \vec{e_2})$ and pick a frame $(\vec{f_1}, \vec{f_2})$ corresponding to the principal directions, we obtain a particularly nice formula for κ_N. Since $A = C \cos 2\theta_0$ and $B = C \sin 2\theta_0$, if we let $\varphi = \theta - \theta_0$, we get

$$\kappa_N(\theta) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi.$$

Thus, for any unit vector $\vec{t}$ expressed as

$$\vec{t} = \cos \varphi \, \vec{f_1} + \sin \varphi \, \vec{f_2},$$

with respect to an orthonormal frame corresponding to the principal directions, the normal curvature $\kappa_N(\varphi)$ is given by Euler's formula (1760):

$$\kappa_N(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi. \tag{3.4.7}$$

3.4.1 Classification of the points on the surface

Recalling that $EG - F^2$ is always strictly positive, we can classify the points on the surface depending on the value of the Gaussian curvature K, and on the values of the principal curvatures κ_1 and κ_2 (or H).

Definition 3.4.2. Given a surface $\mathbf{X}$, a point p on $\mathbf{X}$ belongs to one of the following categories:

1. *Elliptic* if $e\,g - f^2 > 0$ or, equivalently, $K > 0$.
2. *Hyperbolic* if $e\,g - f^2 < 0$, or, equivalently, $K < 0$.
3. *Parabolic* if $e\,g - f^2 = 0$ and $e^2 + f^2 + g^2 > 0$, or, equivalently, $K = \kappa_1\kappa_2 = 0$ but either $\kappa_1 \neq 0$, or $\kappa_2 \neq 0$.
4. *Planar* if $e = f = g = 0$, or, equivalently, $\kappa_1 = \kappa_2 = 0$.
5. A point p is an *umbilical point (or umbilic)* if $K > 0$ and $\kappa_1 = \kappa_2$. Note that some authors allow a planar point to be an umbilical point, which we shall not - in this report.

Comments

- At an elliptic point, both principal curvatures are non-null and have the same sign. For example, most points on an ellipsoid are elliptic.
- At a hyperbolic point, the principal curvatures have opposite signs. For example, all points on the catenoid are hyperbolic.
- At a parabolic point, one of the two principal curvatures is zero, but not both. This is equivalent to $K = 0$ and $H \neq 0$. Points on a cylinder are parabolic.
- At a planar point, $\kappa_1 = \kappa_2 = 0$. This is equivalent to $K = H = 0$. Points on a plane are all planar points. On a monkey saddle, there is a planar point. The principal directions at that point are undefined.
- For an umbilical point, we have $\kappa_1 = \kappa_2 \neq 0$. This can only happen when $H - C = H + C$, which implies that $C = 0$, and since $C = \sqrt{A^2 + B^2}$, we have $A = B = 0$. Thus, for an umbilical point, $K = H^2$. In this case, the function κ_N is constant, and the principal directions are undefined. All points on a sphere are umbilics. A general ellipsoid (a, b, c pairwise distinct) has four umbilics. It can be shown that a connected surface consisting only of umbilical points is contained in a sphere.
- It can also be shown that a connected surface consisting only of planar points is contained in a plane.
- A surface can contain at the same time elliptic points, parabolic points, and hyperbolic points. This is the case of a torus. The parabolic points are on two circles also contained in two tangent planes to the torus (the two horizontal planes touching the top and the bottom of the torus on the following picture).

The elliptic points are on the outside part of the torus (with normal facing outward), delimited by the two parabolic circles. The hyperbolic points are on the inside part of the torus (with normal facing inward).

The normal curvature

$$\kappa_N(\mathbf{X}_u x + \mathbf{X}_v y) = ex^2 + 2fxy + gy^2,$$

will vanish for some tangent vector $(x,\, y) \neq (0, 0)$ if and only if $f^2 - e\,g \geq 0$.

Since

$$K = \frac{e\,g - f^2}{EG - F^2},$$

this can only happen if $(K \leq 0)$.

If $e = g = 0$, then there are two directions corresponding to $\mathbf{X}_u$ and $\mathbf{X}_v$ for which the normal curvature is zero.

If $e \neq 0$ or $g \neq 0$, say $e \neq 0$ (the other case being similar), then the equation

$$e\left(\frac{x}{y}\right)^2 + 2f\left(\frac{x}{y}\right) + g = 0,$$

has two distinct roots if and only if $K < 0$. The directions corresponding to the vectors $\mathbf{X}_u x + \mathbf{X}_v y$ associated with these roots are called the asymptotic directions at p. These are the directions for which the normal curvature is null at p.

3.5 Surfaces of Constant Gauss Curvature

1. There are surfaces of constant Gaussian curvature. For example, a cylinder or a cone is a surface of Gaussian curvature $K = 0$.

2. A sphere of radius R has positive constant Gaussian curvature $K = 1/R^2$.

3. There are surfaces of constant negative curvature, say $K = -1$. A famous one is the pseudosphere, also known as Beltrami's pseudosphere. This is the surface of revolution obtained by rotating a curve known as a tractrix around its asymptote. One possible parametrization is given by:

$$x = \frac{2\,\cos\, v}{e^u + e^{-u}}, \qquad y = \frac{2\,\sin\, v}{e^u + e^{-u}}, \qquad z = u - \frac{e^u - e^{-u}}{e^u + e^{-u}} \quad \text{over} \quad (0,\, 2\pi) \times \mathbb{R}.$$

The pseudosphere has a circle of singular points (for $u = 0$).

The Gaussian curvature at a point $(x,\, y,\, z)$ of an ellipsoid of equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

has the Gaussian curvature given by

$$K = \frac{p^4}{a^2\, b^2\, c^2},$$

where p is the distance from the origin $(0,\,0,\,0)$ to the tangent plane at the point $(x,\,y,\,z)$.

There are also surfaces for which $H = (\kappa_1 + \kappa_2)/2 = 0$. Such surfaces are called **minimal surfaces**, and they show up in nature, engineering and physics. It can be verified that the helicoid, catenoid and Enneper surfaces are minimal surfaces. The study of these type of surfaces is the main objective of this report.

Due to its importance to the study of surfaces, we conclude this chapter of the report by stating the **Theorema Egregium** of Gauss without proof.

Theorem 3.5.1. *Given a surface* $\mathbf{X}$ *and a point* $p = \mathbf{X}(u,\,v)$ *on* $\mathbf{X}$*, the Gaussian curvature,* K *at* $(u,\,v)$ *can be expressed as a function of* E*,* F*,* G *and their partial derivatives. In fact*

$$(3.5.1)\, K = \frac{1}{EG - F^2} \begin{vmatrix} C & F_v - \frac{1}{2}\, G_u & \frac{1}{2}\, G_v \\ \frac{1}{2}\, E_u & E & F \\ F_u - \frac{1}{2}\, E_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}\, E_v & \frac{1}{2}\, G_u \\ \frac{1}{2}\, E_v & E & F \\ \frac{1}{2}\, G_u & F & G \end{vmatrix},$$

where $C = \frac{1}{2}(-E_{vv} + 2F_{uv} - G_{uu})$.

When the surface is isothermally parameterized, $E = G$ and $F = 0$, then the Gaussian curvature reduces, as follows

Theorem 3.5.2. *The Gauss curvature depends only on the metric* E, $\quad F = 0 \quad$ *and* $\quad G$, *and has the explicit form:*

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) \right),$$

where

$$E_v = \frac{\partial E}{\partial v} = \frac{\partial}{\partial v}\,(\langle \mathbf{X}_u\,,\, \mathbf{X}_u \rangle) \qquad \text{and} \qquad G_u = \frac{\partial G}{\partial u} = \frac{\partial}{\partial u}\,(\langle \mathbf{X}_v\,,\, \mathbf{X}_v \rangle).$$

Chapter 4

Minimal surface equation

There is a general consensus that investigations concerning minimal surfaces started with Lagrange [25] in 1760. Lagrange considered surfaces immersed in $\mathbb{R}^3$ that were C^2-differentiable functions, $z = f(x, y)$. The Monge parametrization for its graph: $\mathbf{X}(u, v) = (u, v, f(u, v))$. For these type of surfaces, with its Monge parametrization, the area element is given by:

$$dA \quad = \quad |\mathbf{X}_u \times \mathbf{X}_v|\, du\, dv = \sqrt{1 + f_u^2 + f_v^2}\, du\, dv,$$

Hence

$$A = \iint\limits_{(u,\, v) \,\in\, \mathbb{R}^2} \sqrt{1 + f_u^2 + f_v^2}\ du\, dv. \tag{4.0.1}$$

He studied the problem of determining a surface of this kind with the minimum area among all surfaces that assume given values on the boundary of an open set $\mathcal{D}$ of $\mathbb{R}^2$ (with compact closure and smooth boundary).

We can use Green's theorem to calculate the area enclosed by closed curves in the plane. Suppose P and Q are two real valued (smooth) functions of two variables x and y defined on a simply connected region of the plane. Then Green's theorem says that;

$$\iint\limits_{(x,\, y) \,\in\, \mathbb{R}^2} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \int\limits_{\overline{\mathcal{D}}} (-\, Q\, dx + P\, dy),$$

where the right-hand side is the line integral around the boundary $\mathcal{D}$ of the region enclosed by $\mathcal{D}$. Since all the integrals are pulled back to the plane for computation, we will find Green's theorem particularly useful.

Now suppose M is the graph of a function of two variables $z = f(x, y)$ and is a surface of least area with boundary $\mathcal{D}$. Consider the nearby surfaces, which look like slightly deformed versions of M:

$$M^t : z^t(x, y) = f(x, y) + t\eta(x, y),$$

where η is a C^2-function that vanishes on the boundary of $\mathcal{D}$. Put differently, η is a function on the domain of f with $\eta|_{\overline{\mathcal{D}}} = 0$, where $\overline{\mathcal{D}}$ is the boundary of the domain of f and $f(\overline{\mathcal{D}}) = \mathcal{D}$. The perturbation $t\,\eta(x,\, y)$ then has the effect of moving points on M a small bit and leaving $\mathcal{D}$ constant. A Monge parametrization for M^t is given by

$$\mathbf{X}^t(u,\, v) = (u,\, v,\, f(u,\, v) + t\eta(u,\, v)).$$

Therefore, the area of M is given by

$$\begin{aligned}
dA(t) &= |\mathbf{X}^t_u \times \mathbf{X}^t_v|\, du\, dv, \\
&= \sqrt{1 + f_u^2 + f_v^2 + 2t(f_u\eta_u + f_v\eta_v) + t^2(\eta_u^2 + \eta_v^2)}\, du\, dv, \\
(4.0.2)\; A(t) &= \iint\limits_{(u,\, v)\, \in\, \mathbb{R}^2} \sqrt{1 + f_u^2 + f_v^2 + 2t(f_u\eta_u + f_v\eta_v) + t^2(\eta_u^2 + \eta_v^2)}\, du\, dv.
\end{aligned}$$

Recall that M is assumed to have least area, so the area function $A(t)$ must have a minimum at $t = 0$ (since $t = 0$ gives us M). We can take the derivative of $A(t)$ and impose the condition that $A'(0) = 0$. We were able to take the derivative with respect to t inside the integral because t has nothing to do with the parameters u and v, and using chain rule we get

$$(4.0.3)\; A'(t) = \iint\limits_{(u,\, v)\, \in\, \mathbb{R}^2} \frac{f_u\eta_u + f_v\eta_v + t(\eta_u^2 + \eta_v^2)}{\sqrt{1 + f_u^2 + f_v^2 + 2t(f_u\eta_u + f_v\eta_v) + t^2(\eta_u^2 + \eta_v^2)}}\, du\, dv.$$

Since we assumed that $z = z_0$ was a minimum, then $A'(0) = 0$. Therefore, setting $t = 0$ in Equation (4.0.3), we get

$$(4.0.4) \qquad \iint\limits_{(u,\, v)\, \in\, \mathbb{R}^2} \frac{f_u\eta_u + f_v\eta_v}{\sqrt{1 + f_u^2 + f_v^2}}\, du\, dv = 0.$$

If we let

$$P = \frac{f_u\eta}{\sqrt{1 + f_u^2 + f_v^2}} \qquad \text{and} \qquad Q = \frac{f_v\eta}{\sqrt{1 + f_u^2 + f_v^2}},$$

then compute $\partial P/\partial u,\quad \partial Q/\partial v$ and apply Green's theorem, we get:

$$\iint\limits_{(u,\, v)\, \in\, \mathbb{R}^2} \frac{f_u\eta_u + f_v\eta_v}{\sqrt{1 + f_u^2 + f_v^2}}\, du\, dv \;+ \iint\limits_{(u,\, v)\, \in\, \mathbb{R}^2} \frac{\eta[f_{uu}(1 + f_v^2) + f_{vv}(1 + f_u^2) - 2f_uf_vf_{uv}]}{\sqrt{(1 + f_u^2 + f_v^2)^{\frac{3}{2}}}}\, du\, dv,$$

$$(4.0.5) \qquad = \int\limits_{\overline{\mathcal{D}}} \left[\frac{f_u\eta\, dv}{\sqrt{1 + f_u^2 + f_v^2}} - \frac{f_v\eta\, du}{\sqrt{1 + f_u^2 + f_v^2}}\right] = 0.$$

Since $\eta|_{\overline{\mathcal{D}}} = 0$, the first integral is zero as well, so we end up with

$$\iint_{(u,v)\in\mathbb{R}^2} \frac{\eta[f_{uu}(1+f_v^2)+f_{vv}(1+f_u^2)-2f_uf_vf_{uv}]}{\sqrt{(1+f_u^2+f_v^2)^{\frac{3}{2}}}}\,du\,dv = 0.$$

Since this is true for all such η, we have,

$$f_{uu}(1+f_v^2)+f_{vv}(1+f_u^2)-2f_uf_vf_{uv} = 0. \tag{4.0.6}$$

This is the **minimal surface equation** and they are given by real analytic functions. Lagrange observed that a linear function (whose surface is a plane) is clearly a solution for Equation (4.0.6) and conjectured the existence of solutions containing any given curve given as a graphic along the boundary of $\mathcal{D}$.

It was in 1776 that Meusnier [25] gave a geometrical interpretation to Equation (4.0.6) as denoting the mean curvature, $H = (\kappa_1 + \kappa_2)/2$ of a surface, where κ_1 and κ_2 are the principal curvatures introduced earlier by Euler. Meusnier also did more work trying to find solutions to Equation (4.0.6) which are endowed with special properties. For example, he solved for level curves whose solutions were straight lines as follows

First he observed that when a curve is given implicitly by the equation $f(x, y) = c$, its curvature can be computed as

$$k = \frac{(-f_{xx}f_y^2 + 2f_xf_yf_{xy} - f_{yy}f_x^2)}{|\triangle f|^3}. \tag{4.0.7}$$

Equation (4.0.6) may be rewritten as,

$$k\,|\triangle f|^3 = f_{xx} + f_{yy}. \tag{4.0.8}$$

If the level curves of f are straight lines, then $k \equiv 0$, and f is a harmonic function; that is, f satisfies the equation

$$\triangle f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \tag{4.0.9}$$

The only solutions for Equation (4.0.9) whose level curves are straight lines are given by

$$f(x, y) = A\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) + B, \tag{4.0.10}$$

where A, B, x_0 and y_0 are constants. It is easy to check that the graphs of such functions is either a plane or a piece of helicoid whose equation is given by

$$\begin{aligned} x - x_0 &= u\cos v,\\ y - y_0 &= u\sin v,\\ z - B &= A\,v. \end{aligned} \tag{4.0.11}$$

4.1 The Helicoid

In 1842, Catalan proved that the helicoid is the only ruled minimal surface in $\mathbb{R}^3$. From Equation (4.0.11), we see that the helicoid can be described by the mapping $\mathbf{X}(u,\, v) : \mathbb{R}^2 \to \mathbb{R}^3$ given by:

$$\mathbf{X}(u,\, v) = (u \cos av,\, u \sin av,\, bv), \tag{4.1.1}$$

where a and b are nonzero constants. Geometrically, the helicoid is generated by a helicoidal motion of $\mathbb{R}^3$ acting on a straight line parallel to the rotation plane of the motion.

The helicoid is a complete minimal surface. Its Gaussian curvature, K is $K = -\,b^2/(b^2 + a^2\,u^2)^2$ and its total curvature, $\int_A K\, dA = \infty$, that is, the total curvature is infinite.

The helicoid is also an example of a ruled surface; that is, a surface described geometrically by a straight line sliding smoothly along a curve. A more precise definition can be found in [2].

Theorem 4.1.1. *Any ruled minimal surface of $\mathbb{R}^3$ is up to a rigid motion either part of a helicoid or part of a plane.*

Proof. If $M \subset \mathbb{R}^3$ is a ruled surface, then M can be parameterized locally by:

$$\mathbf{X}(s,\, t) = \alpha(s) + t\beta(s), \tag{4.1.2}$$

where $\alpha(s)$ is a curve perpendicular to the straight lines of M and $\beta(s)$ describes a unit length vector field along α pointing in the direction of the straight line through $\alpha(s)$. We may assume that s represents the arc-length of α and that α and β are analytic curves. A unit length normal vector field to $\mathbf{X}(s,\, t)$ is given by:

$$\mathbf{N} = \frac{\mathbf{X}_s \times \mathbf{X}_t}{|\mathbf{X}_s \times \mathbf{X}_t|} = \frac{(\alpha' \times \beta + t\beta' \times \beta)}{\sqrt{1 + 2t\langle \alpha,' \beta' \rangle + t^2|\beta'|^2}}.$$

We can easily verify that M is a minimal surface if and only if,

$$\langle \alpha' \times \beta, \alpha'' \rangle + t\langle \beta' \times \beta, \alpha'' \rangle + t\langle \alpha' \times \beta, \beta'' \rangle + t^2\langle \beta' \times \beta, \beta'' \rangle = 0. \tag{4.1.3}$$

Observe that the left hand side of Equation (4.1.3) is a polynomial in variable t, therefore,

$$\langle \alpha' \times \beta, \alpha'' \rangle = 0, \tag{4.1.4}$$

$$\langle \beta' \times \beta, \alpha'' \rangle + \langle \alpha' \times \beta,\, \beta'' \rangle = 0, \tag{4.1.5}$$

$$\text{and} \quad \langle \beta' \times \beta,\, \beta'' \rangle = 0. \tag{4.1.6}$$

From Equation (4.1.4), it follows that α'' must belong to the plane generated by α' and β. But, since α is parameterized by arc-length, then α and α' are perpendicular. Hence,

$$\alpha'' \quad \text{is parallel to} \quad \beta. \tag{4.1.7}$$

It follows that $\langle \beta' \times \beta, \alpha'' \rangle = 0$ and so, Equation (4.1.5) becomes,

$$\langle \alpha' \times \beta, \beta'' \rangle = 0. \tag{4.1.8}$$

From Equations (4.1.8) and (4.1.6), we can conclude that:

β'' belongs, simultaneously, to the planes generated by α' and β, and by β' and β.

The intersection of these two planes contains, at least, the subspace generated by the vector β. If there exists a point where β'' is not parallel to β, then in the neighborhood of this point, these two planes coincide and α' is parallel to β'. Since α and β are real analytic functions, this occurs everywhere. Hence $(\beta \times \alpha')' = \beta' \times \alpha' + \beta \times \alpha' = 0$. Thus, the plane generated by β and α' is constant. Therefore, α is a plane curve and the surface described by x is a plane.

On the other hand, if β'' is parallel to β everywhere and α' and β' are not parallel at one point, then this occurs in a neighborhood of this point. In this case, we claim that

(4.1.9) the curvature and the torsion of α are constant.

In fact, since $k = \langle \alpha'', \beta \rangle$, we have

$$\pm \frac{dk}{ds} = \langle \alpha'', \beta \rangle' = -\langle \alpha', \beta' \rangle' = -\langle \alpha'', \beta' \rangle - \langle \alpha', \beta'' \rangle = 0.$$

It is easy to see that $\pm\tau = \langle \alpha' \times \beta', \beta \rangle$ and that

$$\pm \frac{d\tau}{ds} = \langle \alpha' \times \beta', \beta \rangle' = \langle \alpha'' \times \beta', \beta \rangle + \langle \alpha' \times \beta'', \beta \rangle + \langle \alpha' \times \beta', \beta' \rangle = 0.$$

Hence, k and τ are constants. It follows that, up to a rigid motion of $\mathbb{R}^3$, α can be parameterized by

$$\alpha(s) = (A \cos as, A \sin as, bs), \tag{4.1.10}$$

where $A^2 a^2 + b^2 = 1$. Since β is parallel to α'', $\beta(s) = \pm(\cos as, \sin as, 0)$. If we take $u = A \pm t$ and $v = s$, then Equation (4.1.2) becomes

$$\mathbf{X}(u, v) = (u \cos as, u \sin as, bv). \tag{4.1.11}$$

Therefore, M is a piece of helicoid. Hence the theorem is proved. The helicoid is shown in Figures 9.1a and 9.1b. □

4.2 The Catenoid

Meusnier discovered that the catenoid is the only minimal surface of revolution in $\mathbb{R}^3$. The catenoid is a surface of revolution M in $\mathbb{R}^3$ obtained by rotating the curve,

$$\alpha(x) = \left(x, a \cosh\left(\frac{x}{a} + b\right)\right), \tag{4.2.1}$$

$x \in \mathbb{R}$ around the x-axis. This is an imposition of geometric condition on Equation (4.0.6). Such a surface is minimal and complete. Its Gaussian curvature, K is

$$K = -\frac{1}{a^2}\cosh^2\left(\frac{x}{a}+b\right) \text{ and its total curvature, } K_M = \int_M K\,dM = -4\pi.$$

Theorem 4.2.1. *Any minimal surface of revolution in $\mathbb{R}^3$ is up to a rigid motion, part of a catenoid or part of a plane.*

Proof. By a rigid motion we may assume that the surface in $\mathbb{R}^3$ is such that its rotation axis coincides with the x-axis. The surface will then be generated by a curve $\alpha(t) = (x(t),\, y(t),\, 0)$. If the function $x(t)$ is constant, then the surface will be a piece of a plane orthogonal to the x-axis.

This can be shown by noticing that the Gaussian curvature corresponding to α is given by

$$K = \frac{x'(x''y' - y''x')}{y((x')^2 + (y')^2)^2}. \tag{4.2.2}$$

We can easily see that, if the function $x(t)$ is constant, then the Gauss curvature, K is identically zero everywhere. It follows that any minimal surface of revolution in $\mathbb{R}^3$ is up to a rigid motion, part of a catenoid or part of a plane.

Otherwise, there exists a point t_0 such that $x' \neq 0$ in a neighborhood of t_0. We may then represent α by $(x,\, y(x),\, 0)$ in neighborhood of the point $\alpha(t_0)$. The part of the surface obtained by rotating this piece of curve can be parameterized by

$$\mathbf{X}(x,\, y) = (x,\, y(x)\cos v,\, y(x)\sin v).$$

It is a simple computation to show that

$$H = \frac{1}{2}\left(\frac{-yy'' + 1 + (y')^2}{y(1+(y')^2)^{3/2}}\right), \quad \text{is equivalent to}$$

$$-\,y\,y + 1 + (y')^2 = 0 \qquad \text{for minimal surfaces, that is,} \quad H = 0. \tag{4.2.3}$$

This equation can be integrated once by using the transformation

$$p = \frac{dy}{dx}, \quad \text{from which we get} \quad \frac{d^2y}{dx^2} = \frac{dp}{dy}\,\frac{dy}{dx} = p\,\frac{dp}{dy}.$$

Substitution of this into Equation (4.2.3) yields

$$-\,y\,p\frac{dp}{dy} + 1 + p^2 = 0,$$

which can be integrated to give

$$y = a\sqrt{1+p^2},$$

where a is the constant of integration. Notice that

$$p = \frac{dy}{dx} = \sqrt{\frac{y^2}{a^2} - 1}.$$

A second integration gives

$$\cosh^{-1}\left(\frac{y}{a}\right) = \frac{x}{a} + b.$$

Therefore,

$$y = a\cosh\left(\frac{x}{a} + b\right). \tag{4.2.4}$$

Since minimal surfaces are real analytic, so is α. It follows that the curve α must coincide everywhere with the graphic of the function $y(x)$. Hence the theorem is proved. The catenoid is shown in Figures 9.1c and 9.1d. $\square$

4.3 Scherk's first surface

In 1835 Scherk discovered another minimal surface by solving Equation (4.0.6) for functions of the type $f(x, y) = g(x) + h(y)$. This however, seems like a natural algebraic condition to impose on a function. It is also one of the standard multiplicative substitutions for solving partial differential equations.

If we substitute $f(x, y) = g(x) + h(y)$ into Equation (4.0.6), we get

$$(1 + {h'}^2(y))\, g''(x) + (1 + g'^2(x))\, h''(y) = 0, \tag{4.3.1}$$

where the primes denote derivatives with respect to the appropriate variables. On rearranging Equation (4.3.1), we get the following separable ordinary differential equations

$$-\frac{g''(x)}{1 + g'^2(x)} = \frac{h''(y)}{1 + h'^2(y)}, \tag{4.3.2}$$

where x and y are independent variables, each side of Equation (4.3.2) is equal to the same constant, a. On solving Equation (4.3.2) with this requirement, we get,

$$g(x) = \frac{1}{a}\log_e(\cos ax) \text{ and } h(y) = -\frac{1}{a}\log_e(\cos ay). \tag{4.3.3}$$

Hence

$$\begin{aligned} f(x, y) &= g(x) + h(y) = \frac{1}{a}\log_e\,(\cos ax) - \frac{1}{a}\log_e\,(\cos ay), \\ &= \frac{1}{a}\log_e\left(\frac{\cos ax}{\cos ay}\right). \end{aligned} \tag{4.3.4}$$

The surface formed by Equation (4.3.4) is known as Scherk's first minimal surface. Note that Scherk's minimal surface is only defined for $\left(\frac{\cos ax}{\cos ay}\right) > 0$. If, for example, we take $a = 1$ for convenience, a piece of Scherk's minimal surface will be defined over the square $(-\pi/2 < x < \pi/2,\ -\pi/2 < y < \pi/2)$. The pieces of Scherk's surface fit together according to the Schwartz reflection principles. The Scherk's first surface is shown in Figure 9.3c.

Chapter 5

Complex analysis

In this chapter, we shall present some complex number theory that are relevant to the study of minimal surfaces. Some of the materials in this chapter were adapted from [4].

Suppose that $D \subseteq \mathbb{C}$ is a domain, where $\mathbb{C}$ is the complex plane. A function $f : D \to \mathbb{C}$ is said to be differentiable at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. Then we write

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{5.0.1}$$

and call $f'(z_0)$ the derivative of f at z_0.

If $z \neq z_0$, then

$$f(z) = \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0) + f(z_0).$$

It follows from Equation (5.0.1) and the arithmetic of limits that if $f'(z_0)$ exists, then $f(z) \to f(z_0)$ as $z \to z_0$, so that f is continuous at z_0. In other words, differentiability at z_0 implies continuity at z_0.

Note that the argument here is the same as in the case of a real valued function of a real variable. In fact, the similarity in argument extends to the arithmetic of limits. Indeed, if the functions $f : D \to \mathbb{C}$ and $g : D \to \mathbb{C}$ are both differentiable at $z_0 \in D$, then both $f + g$ and fg are differentiable at z_0, and

$$(f + g)'(z_0) = f'(z_0) + g'(z_0) \text{ and } (fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0).$$

If the extra condition $g'(z_0) \neq 0$ holds, then f/g is differentiable at z_0, and

$$\left(\frac{f}{g} \right)' (z_0) = \frac{g(z_0)f'(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}.$$

One can also establish the Chain rule for differentiation as in real analysis. More precisely, suppose that the function f is differentiable at z_0 and the function g is differentiable at $w_0 = f(z_0)$. Then the function $g \circ f$ is differentiable at $z = z_0$, and

$$(g \circ f)'(z_0) = g'(w_0) f'(z_0).$$

Example 4.1 Consider the function $f(z) = \bar{z}$, where for every $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate of z. Suppose that $z_0 \in \mathbb{C}$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{\overline{z - z_0}}{z - z_0}. \tag{5.0.2}$$

If $z - z_0 = h$ is real and non-zero, then Equation (5.0.2) takes the value 1. On the other hand, if $z - z_0 = ik$ is purely imaginary, then Equation (5.0.2) takes the value -1. It follows that this function is not differentiable anywhere in $\mathbb{C}$, although its real and imaginary parts are rather well behaved.

5.1 The Cauchy-Riemann equations

If we use the notation

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h},$$

and then examine the behavior of the ratio

$$\frac{f(z + h) - f(z)}{h},$$

first as $h \to 0$ through real values and then through imaginary values. Indeed, for the derivative to exist, it is essential that these two limiting processes produce the same limit $f'(z)$. Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. If h is real, then the two limiting processes above correspond to

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{u(x + h, y) - u(x, y)}{h} + i \lim_{h \to 0} \frac{v(x + h, y) - v(x, y)}{h} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

and

$$\lim_{h \to 0} \frac{f(z + ih) - f(z)}{ih} = \lim_{h \to 0} \frac{u(x, y + h) - u(x, y)}{ih} + i \lim_{h \to 0} \frac{v(x, y + h) - v(x, y)}{ih} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

respectively. Equating real and imaginary parts, we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{5.1.1}$$

Note that while the existence of the derivative in real analysis is a mild smoothness condition, the existence of the derivative in complex analysis leads to a pair of partial differential equations.

Definition 5.1.1. The partial differential equations (5.1.1) are called the Cauchy–Riemann equations.

With a view to when we shall consider a parametrization $\mathbf{X}(u,\, v)$ with complex coordinates, we write $z = u + i\,v$, and $\overline{z} = u - i\,v$ and introduce the following notation for complex partial differentiation

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\,\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right).$$

These are motivated by the equations $x = (z + \overline{z})/2$, and $y = (z - \overline{z})/(2i)$, which, if z and $\overline{z}$ were independent variables, would give $\partial x/\partial z = \frac{1}{2}$, and $\partial y/\partial z = \frac{i}{2}$, etc.

In terms of these, the Cauchy–Riemann equations are exactly equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0, \quad \text{which is also equivalent to} \quad \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}.$$

Thus, if f is analytic on D then the partial derivatives of f exist and are continuous on D, and the Cauchy – Riemann equations are satisfied there. The converse is true as well. One of the advantages of this notation is that it provides an easy test for f to be holomorphic.

Theorem 5.1.1. *If the partial derivatives of f exist and are continuous on D and the Cauchy–Riemann equations are satisfied there, then f is analytic on D.*

Remark Theorem 5.1.1 can be weakened to say that if f is continuous on D and the partial derivatives exist and satisfy the Cauchy-Riemann equations there (without assuming that the partial derivatives are continuous), then the complex derivative of f exists on D (which is equivalent to f being analytic on D). This is the Looman–Menchoff Theorem. We do need at least continuity, since otherwise we could take f to be the characteristic function of the coordinate axes.

Note that

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \frac{1}{4}\Delta, \qquad \text{where} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

This shows that any analytic function is harmonic (equivalently, its real and imaginary parts are harmonic). It also shows that the conjugate of an analytic function, while not analytic, is harmonic.

Theorem 5.1.2. *Suppose that $f(z) = u(x,\, y) + iv(x,\, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that $f'(z)$ exists. Then the four partial derivatives in Equations (5.1.1) exist, and the Cauchy-Riemann Equations (5.1.1) hold. Furthermore, we have*

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \qquad \text{and} \qquad f'(z) = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}. \tag{5.1.2}$$

A natural question to ask is whether the Cauchy-Riemann equations are sufficient to guarantee the existence of the derivative. We shall show next that we require also the continuity of the partial derivatives in Equations (5.1.1).

Theorem 5.1.3. *Suppose that $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, and u and v are real valued functions. Suppose further that the four partial derivatives in Equation (5.1.1) are continuous and satisfy the Cauchy-Riemann Equations (5.1.1) at z_0. Then f is differentiable at z_0, and the derivative $f'(z_0)$ is given by the Equations (5.1.2) evaluated at z_0.*

Proof. Write $z_0 = x_0 + iy_0$. Then

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{(u(x, y) - u(x_0, y_0)) + i(v(x, y) - v(x_0, y_0))}{z - z_0}.$$

We can write

$$u(x,\, y) - u(x_0, y_0) = (x - x_0)\left(\frac{\partial u}{\partial x}\right)_{z_0} + (y - y_0)\left(\frac{\partial u}{\partial y}\right)_{z_0} + |z - z_0|\epsilon_1(z),$$

and

$$v(x,\, y) - v(x_0, y_0) = (x - x_0)\left(\frac{\partial v}{\partial x}\right)_{z_0} + (y - y_0)\left(\frac{\partial v}{\partial y}\right)_{z_0} + |z - z_0|\epsilon_2(z).$$

If the four partial derivatives in Equation (5.1.1) are continuous at z_0, then,

$$\lim_{z\to z_0} \epsilon_1(z) = 0 \qquad \text{and} \qquad \lim_{z\to z_0} \epsilon_2(z) = 0.$$

In view of the Cauchy-Riemann Equations (5.1.1), we have:

$$u(x, y) - u(x_0, y_0) + i\,(v(x, y) - v(x_0, y_0))=$$

$$\begin{aligned}
&= (x - x_0)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} + (y - y_0)\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + i\epsilon_2(z)),\\
&= (x - x_0)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} + (y - y_0)\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + i\epsilon_2(z)),\\
&= (x - x_0)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} + i(y - y_0)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + i\epsilon_2(z)),
\end{aligned}$$

$$= (z - z_0)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} + |z - z_0|(\epsilon_1(z) + i\epsilon_2(z)).$$

Hence, as $z \to z_0$,, we have

$$\frac{f(z) - f(z_0)}{z - z_0} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0} + \left(\frac{|z - z_0|}{z - z_0}\right)(\epsilon_1(z) + i\epsilon_2(z)) \to \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)_{z_0},$$

which leads to the desired results. $\square$

5.2 Analytic (Holomorphic) functions

In the previous chapter, we have shown that differentiability in complex analysis leads to a pair of partial differential equations. Now partial differential equations are seldom of interest at a single point, but rather in a region. Therefore it seems reasonable to make the following definition.

Definition 5.2.1. A function f is said to be analytic or holomorphic at a point $z_0 \in \mathbb{C}$ if it is differentiable at every z in some ϵ-neighborhood of the point z_0. The function f is said to be analytic or holomorphic in a region if it is analytic at every point in the region. The function f is said to be entire if it is analytic in $\mathbb{C}$.

Example 4.2.1 Consider the function $f(z) = |z|^2$.

In our usual notation, we clearly have; $u = x^2 + y^2$ and $v = 0$.

The Cauchy-Riemann equations are: $2x = 0$ and $2y = 0$.

We see that they can only be satisfied at $z = 0$. It follows that the function is differentiable only at the point $z = 0$, and is therefore analytic nowhere.

Example 4.2.2 The function $f(z) = z^2$ is entire. Since $u = x^2 - y^2$ and $v = 2xy$.

Theorem 5.2.1. *If f is holomorphic, then*

$$\int_\gamma f' = f(b) - f(a). \tag{5.2.1}$$

Proof. Let $f(z) = \rho(u,\, v) + i\,\tau(u,\, v)$, so that $f'(z) = \rho_u + i\,\tau_u$. Furthermore, let $\gamma(t) = u(t) + i\,v(t)$ be a curve in the complex plane with $(a \leq t \leq b)$. Then, using the Cauchy-Riemann equations $\rho_u = \tau_v,\;\; \rho_v = -\,\tau_u$ and the usual Fundamental Theorem of Calculus, we have □

$$\begin{aligned}
\int_\gamma f' = \int_a^b f'(\gamma(t))\,\gamma'(t)\,dt \quad &\Rightarrow \quad \int_a^b (\rho_u + i\,\tau_u)(u' + i\,v')\,dt, \\
&\Rightarrow \quad \int_a^b (\rho_u u' - \tau_u v') + i\,(\tau_u u' + \rho_u v')\,dt, \\
&\Rightarrow \quad \int_a^b \rho_u u' + \rho_v v' + i\,(\tau_u u' + \tau_v v')\,dt, \\
&\Rightarrow \quad \int_a^b \left(\frac{d\rho}{dt} + i\,\frac{d\tau}{dt}\right)\,dt,
\end{aligned}$$

$$\Rightarrow \quad \rho(u(b),\, v(b)) + i\,\tau(u(b),\, v(b)) - \rho(u(a),\, v(a)) - i\,\tau(u(a),\, v(a)) \quad \Rightarrow \quad f(b) - f(a).$$

The last line means that f is evaluated at the end points of γ. Since the Fundamental Theorem of Calculus is valid for complex analysis, many of the properties from real analysis are applicable to complex analysis as well. This will allow the calculation of complex integrals in the Weierstrass-Enneper representation later.

Remark 5.2.2. If f is holomorphic with a continuous derivative on and inside a closed curve γ, we can use Green's theorem to show that $\int_\gamma f' = 0$. This is a weak version of Cauchy's theorem. As shown above, we see that the integrals of holomorphic functions only depend on the endpoints and not on the paths chosen over which to integrate.

5.3 Meromorphic functions

A complex meromorphic function g, is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (that is, these exceptional points must be poles and not essential singularities). A simpler definition states that a meromorphic function is a function $g(z)$ of the form

$$g(z) = \frac{\mathcal{P}(z)}{\mathcal{Q}(z)}, \quad \text{where } \mathcal{P}(z) \text{ and } \mathcal{Q}(z) \text{ are entire functions with } \{z : \mathcal{Q}(z) \neq 0\}.$$

A meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain. A meromorphic function with an infinite number of poles is exemplified by cosecant $(1/z)$ on the punctured disk $U = D \backslash \{0\}$, where D is the open unit disk.

An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. Meromorphic functions play essential roles in the analysis of minimal surfaces when Weierstrass-Enneper method is employed.

Theorem 5.3.1. *If g is meromorphic in $\mathbb{C}$, has an isolated singularity at infinity (∞), and has a pole or removable singularity at infinity (∞), then g is a rational function.*

This is a type of uniqueness theorem; it describes all solutions to a certain problem. Liouville's theorem is the special case where g is assumed to be both differentiable on $\mathbb{C}$ and bounded, hence has a removable singularity at infinity (∞).

Proof. Let $\{z_j\}$ be the locations of the poles of g and $\{m_j\}$ be their orders. Then $f(z) = g(z)\prod_j (z - z_j)^{m_j}$ has no poles. The assumptions concerning the singularity at ∞ imply that $|f(z)| \leq M + M|z|^N$ for some finite M, N. By the generalized version of Liouville's theorem, f is a polynomial and hence g is rational. $\square$

5.4 Laurent series expansion

Let g have an isolated singularity at z_0. Let (a_n) be the coefficients in its Laurent expansion $g(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ about z_0.

Discussion. Recall that one function can have more than one Laurent expansion. But by the following theorem

Theorem 5.4.1. *If g is differentiable in punctured disk $(0 < |z - z_0| < \delta)$ then there is only one Laurent expansion in powers of $z - z_0$ which is valid in this disk. Moreover, given two different outer radii $\delta_1 < \delta_2$, we get two different Laurent series representations, with coefficients $(a_n^{(1)})$ and $(a_n^{(2)})$, one for each punctured disk. However, since the Laurent series expansion in $\{z : 0 < |z - z_0| < \delta_2\}$ converges to g in the smaller punctured disk $\{z : 0 < |z - z_0| < \delta_1\}$, the uniqueness part of Laurent's theorem guarantees that $(a_n^{(1)}) = (a_n^{(2)})$ for all n. Thus the coefficients (a_n) are unambiguously defined.*

Recall that g has a removable singularity at z_0 if and only if it has a differentiable extension to some open set which contains a neighborhood of z_0.

Fact. g has a removable singularity at z_0 if and only if $a_n = 0$ for all $n < 0$.

Proof. If the singularity is removable then there exists a differentiable extension f of g to an open disk $D = \{z : |z - z_0| < r\}$ for some $r > 0$. f has a power series expansion $\sum_{n=0}^{\infty} b_n(z - z_0)^n$, valid in D. This series converges to g in the punctured disk $D\backslash\{z_0\}$. Therefore $\sum_{n=0}^{\infty} b_n(z - z_0)^n$, is the Laurent series expansion for g in D. There are no negative powers in this expansion.

Conversely, if $a_n = 0$ for all $n < 0$ then the Laurent series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, is a power series, and it converges to g in $\{z : 0 < |z - z_0| < \delta\}$ for some $\delta > 0$. Thus this power series has positive radius of convergence. Therefore its sum, which is well defined in the disk $D = \{z : |z - z_0| < \delta\}$, converges to a differentiable function in this disk. The sum of this series defines the required extension of g to this neighborhood D of z_0. □

Definition 3.1 g has a pole at z_0 if there are only finitely many negative n satisfying $a_n \neq 0$. g has a pole of order $m \in \{1, 2, 3, \ldots\}$ at z_0 if $a_n = 0$ for all $n < -m$, and $a_{-m} \neq 0$.

Definition 3.2 g has an essential singularity at z_0 if there are infinitely many $n < 0$ for which $a_n \neq 0$.

Proposition 5.4.2. *g has a pole of order m at z_0 if and only if $\lim_{z \to z_0} (z - z_0)^m f(z)$ exists and is $\neq 0$.*

Proposition 5.4.3. *If g has an essential singularity at z_0, then for any $\omega \in \mathbb{C}$ and any ϵ, $\delta > 0$ there exists $\zeta \in \mathbb{C}$ such that $0 < |\zeta - z_0| < \delta$ and $|f(\zeta) - \omega| < \epsilon$.*

Corollary 5.4.4. *If there exists $n < 0$ such that the coefficient a_n in the Laurent series expansion of g about z_0 is not zero, then for any $\delta > 0$, no matter how small, $|g|$ is unbounded in any punctured disk $\{z : 0 < |z - z_0| < \delta\}$.*

Finally, the Laurent series expansion of $g(z)$ about z_0 is given by

$$g(z) = \frac{a_n}{(z - z_0)^n} + \cdots + \frac{a_1}{(z - z_0)} + \sum_{j=0}^{\infty} b_j(z - z_0)^j,$$

for some finite n with coefficients determined by $g(z)$.

We shall not go into the proofs of these propositions since they will not impede our understanding of minimal surfaces. They have been included here only to help us understand some of the terms that were used in the definition of meromorphic functions.

5.5 Periodicity and its consequences

One of the fundamental differences between real and complex analysis is that the exponential function is periodic in $\mathbb{C}$.

Definition 5.5.1. A function f is periodic in $\mathbb{C}$ if there is some fixed non-zero $\omega \in \mathbb{C}$ such that the identity $f(z+\omega) = f(z)$ holds for every $z \in \mathbb{C}$. Any constant $\omega \in \mathbb{C}$ with this property is called a period of f.

Theorem 5.5.1. *The exponential function e^z is periodic in $\mathbb{C}$ with period $2\pi i$. Furthermore, any period $\omega \in \mathbb{C}$ of e^z is of the form $\omega = 2\pi k i$, where $k \in \mathbb{N}$ is non-zero.*

Proof. The first assertion follows easily from the observation

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1.$$

Suppose now that $\omega \in \mathbb{C}$. Clearly $e^{z+\omega} = e^z$ implies $e^\omega = 1$. Write $\omega = \alpha + i\,\beta$, where $\alpha,\ \beta \in \mathbb{C}$.

Then $\qquad e^\alpha(\cos \beta + i \sin \beta) = 1.$

Taking modulus, we conclude that $e^\alpha = 1$, so that $\alpha = 0$. It then follows that $\cos \beta + i \sin \beta = 1$. Equating real and imaginary parts, we conclude that $\cos \beta = 1$ and $\sin \beta = 0$, so that $\beta = 2\pi k$, where $k \in \mathbb{N}$. The second assertion follows. □ □

5.6 Laplace's equation and harmonic conjugates

We have shown that for any function $f = u + iv$, the existence of the derivative f' leads to the Cauchy-Riemann equations. More precisely, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{5.6.1}$$

Furthermore,

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}. \tag{5.6.2}$$

Suppose now that the second derivative f'' also exists. Then f' satisfies the Cauchy-Riemann equations.

The Cauchy-Riemann equations corresponding to the expression (5.6.2) are

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) \qquad \text{and} \qquad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right). \tag{5.6.3}$$

Substituting Equations 5.6.1 into Equations 5.6.3, we obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \tag{5.6.4}$$

We also obtain

$$\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}. \tag{5.6.5}$$

Definition 5.6.1. A real-valued continuous function $\Phi(x,\, y)$ that satisfies Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2}+\frac{\partial^2 \Phi}{\partial y^2}=0,$$

in a domain $D \subseteq \mathbb{C}$ is said to be harmonic in D.

Theorem 5.6.1. *If $f(z)=\phi(x,\, y)+i\,\psi(x,\, y)$ is holomorphic, then both $\phi(x,\, y)$ and $\psi(x,\, y)$ are harmonic.*

In practice, we may use Equation (5.6.1) as follows. Suppose that ϕ is a real-valued harmonic function in a domain D. We could write

$$g(z)=\frac{\partial \phi}{\partial x}-i\,\frac{\partial \phi}{\partial y}. \tag{5.6.6}$$

Then Cauchy-Riemann equations for g are

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)=-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right),$$

which clearly hold. It follows that g is analytic in D. Suppose now that ϕ is the real part of an analytic function f in D. Then $f'(z)$ agrees with the right hand side of Equation (5.6.6) in view of Equation (5.1.1) and Equation (5.1.2). Hence $f'=g$ in D. The issue of course is to find this f. If we successfully find f, then the imaginary part ψ is a harmonic conjugate of the harmonic function ϕ.

In order to show how special harmonic functions are, we state one of their important properties. Let $U \subseteq \mathbb{R}^2$ be a bounded open set with closure $\overline{U}$ having boundary $\partial U=\overline{U}-U$.

Theorem 5.6.2. *Let $\phi : \overline{U} \to \mathbb{R}$ be a harmonic function which is continuous on $\overline{U}$ and differentiable on U. Then ϕ takes its maximum and minimum values on the boundary ∂U.*

Furthermore, if ϕ is a twice differentiable harmonic function of x and y, then on some open set there is another harmonic function ψ such that $f=\phi+i\,\psi$ is holomorphic. Harmonic functions ϕ and ψ which give such an f are said to be *harmonic conjugates.*

5.7 Some facts about harmonic functions

Theorem 5.7.1. Mean Value Properties. *Suppose u is harmonic in $D \subseteq \mathbb{R}^2$. Then u satisfies the following mean value properties.*

If $B_r(\mathbf{x}_0)$ is any ball with $\overline{B}_r(\mathbf{x}_0) \subseteq D$, then

$$u(\mathbf{x}_0) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{x}_0)} u\, ds, \tag{a}$$

and

$$u(\mathbf{x}_0) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{x}_0)} u(x, y)\, dx\, dy. \tag{b}$$

Remark: 1 The integral on the left hand side of (a) is a line integral with respect to arc length.

Thus if $\mathbf{x}_0 = (a,\, b)$,

Then
$$\int_{\partial B_r(\mathbf{x}_0)} u\, ds \equiv \int_0^{2\pi} u(a + r\,\cos\, t, b + r\,\sin\, t)\, r\, dt.$$

Remark: 2 The theorem says that the value of a harmonic function at a point is equal to both the average value of the function over a circle around that point (as in part (a)) and also to the average value of the function over a disk centered at that point (as in part (b)).

Proof. Since $\Delta u = \mathrm{div}(D\, u)$, we have by Green's theorem that

$$\int\limits_{\partial B_\rho\,(\mathbf{x}_0)} \Delta\, u\, dx\, dy = \int\limits_{\partial B_\rho(\mathbf{x}_0)} -u_y\, dx + u_x\, dy,$$

Since $\Delta u = 0$,

$$0 = \int\limits_{\partial B_\rho\,(\mathbf{x}_0)} -u_y\, dx + u_x\, dy,$$

$$= \rho \int_0^{2\pi} [u_y(a + \rho\,\cos\, t,\, b + \rho\,\sin\, t)\,\sin\, t + u_x(a + \rho\,\cos\, t,\, b + \rho\,\sin\, t)\,\cos\, t]\, dt,$$

$$= \int_0^{2\pi} \frac{d}{d\,\rho}\, u(a + \rho\,\cos\, t,\, b + \rho\,\sin\, t)\, dt,$$

$$= \frac{d}{d\rho} \int_0^{2\pi} u\,(a + \rho \cos t,\, b + \rho \sin t)\, dt.$$

Integrating this with respect to ρ from 0 to r, we have that

$$\int_0^{2\pi} u(a + r \cos t,\, b + r \sin t)\, dt = 2\pi u\,(\mathbf{x}_0),$$

or, equivalently,

$$u(\mathbf{x}_0) = \frac{1}{2\pi r} \int\limits_{\partial B_r(\mathbf{x}_0)} u\, ds.$$

This is part (a). To prove part (b), note that by changing variables to polar coordinates we have for any continuous function f that

$$\int\limits_{B_r(\mathbf{x}_0)} f(x,\, y)\, dx\, dy = \int_0^r \int_0^{2\pi} = f(a + \rho \cos t,\, b + \rho \sin t)\rho\, dt\, d\rho,$$

or, equivalently, that

$$\int_0^r \int\limits_{\partial B_\rho(\mathbf{x}_0)} f\, ds\, d\rho = \int\limits_{B_\rho(\mathbf{x}_0)} f(x,\, y) dx\, dy. \tag{5.7.1}$$

Now replace r with ρ in the identity of part (a) and multiply the resulting identity by $2\pi\rho$ and integrate both sides with respect to ρ over $[0, r]$. In view of Equation (5.7.1), this gives exactly the identity of part (b). □

Lemma 5.7.2. Gradient estimate for non-negative harmonic functions

Suppose u is non-negative and harmonic in D. If $\overline{B}_r(\mathbf{x}_0) \subseteq D$, then

$$|Du(\mathbf{x}_0)| \leq \frac{2\sqrt{2}}{r} u(\mathbf{x}_0). \tag{5.7.2}$$

Proof. Note first that if u is harmonic, then each of its partial derivatives u_x and u_y is also harmonic. This follows directly by differentiating the equation $\Delta u = 0$. Hence by the mean value identity (b) of Theorem 5.7.1, we have that

$$u_x(\mathbf{x}_0) = \frac{1}{\pi r^2} \int\limits_{B_r(\mathbf{x}_0)} u_x(x,\, y)\, dx\, dy. \tag{5.7.3}$$

By Green's theorem

$$\int\limits_{B_r(\mathbf{x}_0)} u_x\, dx\, dy = \int\limits_{B_r(\mathbf{x}_0)} u\, dy = \int_0^{2\pi} u(a + r\,\cos\, t,\, b + r\,\sin\, t) r\,\cos\, t\, dt.$$

Using this in Equation (5.7.3) and keeping in mind that u is non-negative, we get,

$$|u_x(\mathbf{x}_0)| \le \frac{1}{\pi r^2} \int_0^{2\pi} u(a + r\cos\, t,\, b + r\sin\, t) r\, dt = \frac{1}{\pi r^2} \int\limits_{B_r(\mathbf{x}_0)} u\, ds.$$

Now using the mean value property (a) of Theorem 5.7.1 we deduce from the above that

$$|u_x(\mathbf{x}_0)| \le \frac{2}{r}\, u(\mathbf{x}_0).$$

Similarly, we also have

$$|u_y(\mathbf{x}_0)| \le \frac{2}{r}\, u(\mathbf{x}_0),$$

and combining these two estimates, we conclude that

$$|D\, u(\mathbf{x}_0)| \le \frac{2\sqrt{2}}{r}\, u(\mathbf{x}_0).$$

□

Corollary 5.7.3. Joseph Liouville. *If a harmonic function on the entire plane $\mathbb{R}^2$ is bounded above or below, then it must be a constant. An alternative way of stating this, is the following – A bounded complex function $f : \mathbb{C} \to \mathbb{C}$ which is holomorphic on the entire complex plane is a constant function.*

Proof. Suppose u is harmonic everywhere in $\mathbb{R}^2$ and that $u(\mathbf{x}) \le M$ for some constant M and all $\mathbf{x} \in \mathbb{R}^2$. Then $v(\mathbf{x}) = M - u(\mathbf{x})$ is harmonic in $\mathbb{R}^2$ and non-negative. Thus for any point $\mathbf{x} \in \mathbb{R}^2$, we have by Lemma 5.7.2 (with v in place of u and $\mathbf{x}$ in place of $\mathbf{x}_0$) that

$$|D\, u(\mathbf{x})| \le \frac{2\sqrt{2}}{r}\, (M - u(\mathbf{x})),$$

for all $r > 0$. Letting $r \to \infty$, we conclude that $Du(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}^2$, and hence that u is a constant. □

A similar argument applies to the case when u is bounded below.

5.8 Harmonic functions and minimal surfaces

When a minimal surface is parameterized by an isothermal parametrization $\mathbf{X}(u, v)$, there is a close relationship between the Laplace operator $\Delta\mathbf{X} = \mathbf{X}_{uu} + \mathbf{X}_{vv}$ and the mean curvature, H.

Theorem 5.8.1. *If the parametrization* $\mathbf{X}$ *is isothermal, then* $\Delta\mathbf{X} = \mathbf{X}_{uu} + \mathbf{X}_{vv} = (2EH)\mathbf{N}$.

Proof.

$$\begin{aligned}
\mathbf{X}_{uu} + \mathbf{X}_{vv} &= \left(\frac{E_u}{2E}\mathbf{X}_u - \frac{E_v}{2G}\mathbf{X}_v + e\,\mathbf{N}\right) + \left(-\frac{G_u}{2E}\mathbf{X}_u + \frac{G_v}{2G}\mathbf{X}_v + g\,\mathbf{N}\right), \\
&= \frac{E_u}{2E}\mathbf{X}_u - \frac{E_v}{2G}\mathbf{X}_v + e\,\mathbf{N} - \frac{E_u}{2E}\mathbf{X}_u + \frac{E_v}{2G}\mathbf{X}_v + g\,\mathbf{N}, \\
&= (e+g)\mathbf{N} = 2E\left(\frac{e+g}{2E}\right)\mathbf{N}.
\end{aligned}$$

By examining Equation (3.4.5) for mean curvature, when $E = G$ and $F = 0$, we see that

$$H = \frac{Ee + Eg}{2E^2} = \frac{e+g}{2E}.$$

Therefore,

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = (2EH)\mathbf{N}.$$

□

Corollary 5.8.2. *A surface* M *with an isothermal parametrization* $\mathbf{X}(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$ *is minimal if and only if* x^1, x^2 *and* x^3 *are harmonic functions.*

Proof. If M is a minimal surface, then $H = 0$ and, by Theorem 5.8.1,

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0.$$

Therefore, the coordinate functions of $\mathbf{X}$ are harmonic.

Conversely, suppose x_1, x_2 and x_3 are harmonic functions, Then

$$\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0,$$

and by Theorem 5.8.1, $(2EH)\mathbf{N} = 0$. Therefore, since $\mathbf{N}$ is the unit normal and

$$E = \langle \mathbf{X}_u\,, \mathbf{X}_u \rangle \neq 0,$$

we have $H = 0$, and M is a minimal surface. □

This result is the bridge between the theory of minimal surface and complex analysis.

Theorem 5.8.3. *If M is a minimal surface without boundary (that is, a generalized minimal surface), then M cannot be compact.*

Proof. Suppose M is an isothermal parametrization. According to Corollary 5.8.2, the coordinate functions of the parametrization are harmonic functions. If M were compact, then each coordinate function $\mathbf{X}(u, v)$ would attain its maximum and minimum. According to Theorem 5.6.2, this must take place on the boundary of M. Furthermore by Corollary 5.7.3 would be constant. Since M has no boundary, this contradicts compactness. □

Chapter 6

Weierstrass-Enneper representations I

The Weierstrass-Enneper system has proved to be a very useful and suitable tool in the study of minimal surfaces in $\mathbb{R}^3$. The formulation of Weierstrass and Enneper of a system inducing minimal surfaces can be briefly presented as follows.

Let $D(u, v) \subset \mathbb{R}^2$, where $u, v \in \mathbb{R}^2$ be a simply connected open set and $\phi : D \to \mathbb{R}^3$ be an immersion of class C^k $(k \geq 2)$ or real analytic. The mapping ϕ describes a parametric surface in $\mathbb{R}^3$. If

$$|\phi_u| = |\phi_v| \quad \text{and} \quad \langle \phi_u \, , \, \phi_v \rangle = 0, \tag{6.0.1}$$

then ϕ is a conformal mapping (that is, it preserves angles) which induces the metric

$$ds^2 = \lambda^2 \, (du^2 + dv^2), \tag{6.0.2}$$

where $\lambda = |\phi_u| = |\phi_v|$. We then say that (u, v) are isothermal parameters for the surface described by ϕ.

Theorem 6.0.1. Existence of isothermal parameters. *Let D be a simply connected open set and let $\phi : D \to \mathbb{R}^3$ be an immersion of class C^k $(k \geq 2)$ or real analytic. Then there exists a diffeomorphism $\varphi : D \to D$ of class C^k $(k \geq 2)$ or real analytic such that $\tilde{\phi} = \langle \phi \, , \, \varphi \rangle$ is a conformal mapping.*

A proof of this theorem is given in Lemma 8.1.2.

Let M be a surface (that is, a 2-dimensional manifold of class C^k $(k \geq 2)$). Suppose M is connected and orientable and let $\mathbf{X}(u, v) : M \to \mathbb{R}^3$ be an immersion of class C^k. By Theorem 6.0.1 each point $p \in M$ has a neighborhood in which isothermal parameters (u, v) are defined. The metric induced on M by $\mathbf{X}$ will be represented locally in terms of such parameters by

$$ds^2 = \lambda^2 \, |dz|^2, \tag{6.0.3}$$

where $z = u + i\, v$. Clearly, a change of coordinates of such parameters is a conformal mapping.

Since M is orientable, we can restrict ourselves to a family of isothermal parameters whose change of coordinates preserve the orientation plane. In terms of the variable $z = x + i\,y$, this means that such changes of coordinates are holomorphic. A surface M together with such a family of isothermal parameters is called a Riemann surface.

We can extend the notion of holomorphic mapping to such surfaces as follows: if M and $\overline{M}$ are Riemann surfaces, we could say that $f : M \to \overline{M}$ is holomorphic when every of its representation, in terms of local isothermal parameters (in M and $\overline{M}$), is a holomorphic function.

In a Riemann surface we consider, locally the operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\,\frac{\partial}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\,\frac{\partial}{\partial v}\right), \tag{6.0.4}$$

the definition of these operators is such that, if $f : M \to \mathbb{C}$ is a complex valued differentiable function, then

$$df = \frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z}. \tag{6.0.5}$$

The function f is holomorphic if and only if $\frac{\partial f}{d\bar{z}} = 0$; and if $\frac{\partial f}{dz} = 0$, we say that f is anti-holomorphic.

The evaluation of many quantities in the study of surfaces are simplified when considered in reference to the Riemann surfaces. For example, the Laplacian operator becomes

$$\Delta = \frac{1}{\lambda^2}\left(\frac{\partial}{\partial u^2} + \frac{\partial}{\partial v^2}\right) = \frac{4}{\lambda^2}\,\frac{\partial}{\partial z}\,\frac{\partial}{\partial \bar{z}}. \tag{6.0.6}$$

Another example is the Gaussian curvature of M which is then given by

$$K = -\,\Delta \log_e \lambda. \tag{6.0.7}$$

For an immersion $\mathbf{X} = (X_1,\, X_2,\, X_3) : M \to \mathbb{R}^3$, and if we define $\Delta\mathbf{X}$ as the vectored valued function $(\Delta X_1,\, \Delta X_2,\, \Delta X_3)$, then,

$$\Delta = 2\,H\,N. \tag{6.0.8}$$

where H is the mean curvature of the immersion and $N : M \to S^2(1)$ is its Gauss map.

The proof of Equation (6.0.8) is as follows: Notice that $\Delta\mathbf{X}$ is normal to M. From Equation (6.0.1), we have

$$\langle \mathbf{X}_u\,,\, \mathbf{X}_u\rangle = \langle \mathbf{X}_v\,,\, \mathbf{X}_v\rangle \quad \text{and} \quad \langle \mathbf{X}_u\,,\, \mathbf{X}_v\rangle = 0. \tag{6.0.9}$$

On differentiating Equation (6.0.9), with respect to u and v separately, we get

$$\langle \mathbf{X}_{uu}\,,\, \mathbf{X}_u\rangle = \langle \mathbf{X}_{uv}\,,\, \mathbf{X}_v\rangle \quad \text{and} \quad \langle \mathbf{X}_{vu}\,,\, \mathbf{X}_v\rangle + \langle \mathbf{X}_u\,,\, \mathbf{X}_{vv}\rangle = 0. \tag{6.0.10}$$

Hence

$$\langle (\mathbf{X}_{uu} + \mathbf{X}_{vv}) \,,\, \mathbf{X}_u \rangle = \langle \mathbf{X}_{uv} \,,\, \mathbf{X}_v \rangle - \langle \mathbf{X}_{vu} \,,\, \mathbf{X}_v \rangle = 0. \tag{6.0.11}$$

Similarly, we can show that,

$$\langle (\mathbf{X}_{uu} + \mathbf{X}_{vv}) \,,\, \mathbf{X}_v \rangle = 0. \tag{6.0.12}$$

and from this we conclude that $\Delta \mathbf{X}$ is normal to the surface M.

Recall that if $\mathbf{N}$ is the Gauss mapping of the surface $\mathbf{X}$, and $\mathbf{N}_u = a_{11}\mathbf{X}_u + a_{12}\mathbf{X}_v$ and $\mathbf{N}_v = a_{21}\mathbf{X}_u + a_{22}\mathbf{X}_v$, the mean curvature becomes

$$H = -\frac{1}{2}(a_{11} + a_{22}). \tag{6.0.13}$$

By using Equation (6.0.6), we get,

$$\begin{aligned} \lambda^2 \langle \Delta \mathbf{X} \,,\, \mathbf{N} \rangle &= \langle (\mathbf{X}_{uu} + \mathbf{X}_{vv}) \,,\, \mathbf{N} \rangle = -\langle \mathbf{X}_u \,,\, \mathbf{N}_u \rangle - \langle \mathbf{X}_v \,,\, \mathbf{N}_v \rangle - \\ & \quad a_{11}|\mathbf{X}_u|^2 - a_{22}|\mathbf{X}_v|^2 = -(a_{11} + a_{22})\lambda^2 = 2HN, \end{aligned} \tag{6.0.14}$$

this proves Equation (6.0.8).

The proposition below is an immediate corollary of Equation (6.0.8).

Proposition 6.0.2. *A mapping $\mathbf{X} : M \to \mathbb{R}^3$ is a minimal surface if and only if $\mathbf{X}$ is harmonic.*

Lemma 6.0.3. *Let us define $\phi = \frac{\partial \mathbf{X}}{\partial z}$. From Equation (6.0.6), we get $\frac{\partial \phi}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \frac{\partial \mathbf{X}}{\partial z} = \frac{\lambda^2}{4} \Delta \mathbf{X}$. Therefore, ϕ is holomorphic if and only if $\mathbf{X}$ is harmonic.*

Note that ϕ is a function defined locally on M with values in $\mathbb{C}^3$. Its image is contained in a quadric $\mathcal{Q}$ of $\mathbb{C}^3$ and is given by:

$${Z^2}_1 + {Z^2}_2 + {Z^2}_3 = 0, \qquad (Z_1,\, Z_2,\, Z_3) \in \mathbb{C}^3. \tag{6.0.15}$$

To see this, observe that if $\phi = (\phi_1,\, \phi_2,\, \phi_3)$, then

$$\phi_k = \frac{\partial \mathbf{X}_k}{\partial u} - i\, \frac{\partial \mathbf{X}_k}{\partial v}, \qquad z = u + iv.$$

Hence $\sum_{k=1}^{3} \phi_k^2 = 0$ and $\sum_{k=1}^{3} |\phi_k|^2 > 0$. Thus $|\phi| > 0$. The proofs are given in Lemma 8.1.5.

Notice that we have a mapping ϕ defined in terms of isothermal parameters, in some neighborhood of each point of M. If $z = x + i\, y$ and $w = r + i\, s$ are isothermal parameters around some point in M, then the change of coordinates $w = w(z)$ is holomorphic with $\partial w / \partial z \neq 0$. It follows that $\tilde{\phi} = \frac{\partial x}{\partial z}$ is related to ϕ by

$$\phi = \frac{\partial x}{\partial z} = \frac{\partial x}{\partial w}\, \frac{\partial w}{\partial z} = \frac{\partial w}{\partial z}\, \tilde{\phi}.$$

Therefore, if we consider the vector valued differential forms $\alpha = \phi\, dz$ and $\tilde{\alpha} = \tilde{\phi}\, dw$, we get

$$\alpha = \phi dz = \tilde{\phi}\,\frac{\partial w}{\partial z}\, dz = \tilde{\phi}\, dw = \tilde{\alpha}.$$

This means that we have a vector valued differential form α globally defined on M, whose local expression is $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3)$, with

$$\alpha_k = \phi_k\, dz, \qquad 1 \leq k \leq 3. \tag{6.0.16}$$

Equation (6.0.16) together with Proposition 6.0.2 and Lemma 6.0.3 prove the following

Lemma 6.0.4. *Let $\mathbf{X} : M \to \mathbb{R}^3$ be an immersion. Then, $\alpha = \phi\, dz$ is a vector valued holomorphic form on M if and only if $\mathbf{X}$ is a minimal surface. Furthermore,*

$$\mathbf{X}_k(z) = Re\left(\int^z \phi\, dz\right), \quad k = 1,\, 2,\, 3. \tag{6.0.17}$$

where the integral is taken along any path from a fixed point to z on M.

When the real part of the integral of α along any closed path is zero, we say α has no real periods. The non-existence of real periods for α is easily seen to be equivalent to $\mathrm{Re}\left(\int^z \phi\, dz\right)$ be independent of the path on M.

Theorem 6.0.5. Weierstrass-Enneper representation. *Let α_1, α_2, and α_3 be holomorphic differentials on M such that*

1. *$\sum_{k=1}^3 \phi_k^2 = 0$ (that is, locally $\alpha_k = \phi_k\, dz$ and $\sum_{k=1}^3 \phi_k^2 = 0$);*
2. *$\sum_{k=1}^3 |\phi_k|^2 > 0$ and*
3. *each α_k has no real periods on M.*

Then the mapping $\mathbf{X} : M \to \mathbb{R}^3$ defined by $\mathbf{X} = (X_1,\, X_2,\, X_3)$, with $\mathbf{X}_k(z) = Re\left(\int_{p_0}^z \alpha_k\, dz\right)$, $k = 1,\, 2,\, 3$ is a minimal surface.

Condition (3) of the theorem is necessary in order to guarantee that $\mathrm{Re}\left(\int_{p_0}^z \alpha_k\, dz\right)$ depends only on the final point z. Therefore, each $\mathbf{X}_k$ is well defined independently of the path from p_0 to z. It is obvious that $\phi = \frac{\partial \mathbf{X}}{\partial z}$ is holomorphic and so $\mathbf{X}$ is harmonic. Hence $\mathbf{X}$ is a minimal surface. Condition (2) guarantees that $\mathbf{X}$ is a surface.

It is possible to give a simple description of all solutions of equation $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$ on M. In order to do this, assume $\alpha_1 \neq i\,\alpha_2$. (If $\alpha_1 = i\,\alpha_2$, then $\alpha_3 = 0$ and the resulting minimal surface is a plane.) Let us define a holomorphic form ω and a meromorphic function g as follows

$$\begin{aligned} \omega &= \alpha_1 - i\,\alpha_2, \quad \text{and,} \\ g &= \frac{\alpha_3}{\alpha_1 - i\,\alpha_2}. \end{aligned} \tag{6.0.18}$$

Locally, if $\alpha_k = \phi_k\, dz$, then $\omega = f\, dz$, where f is a holomorphic function and we have

$$\begin{aligned} f &= \phi_1 - i\,\phi_2, \quad \text{and,} \\ g &= \frac{\phi_3}{\phi_1 - i\,\phi_2}. \end{aligned} \tag{6.0.19}$$

In terms of g and ω, α_1, α_2 and α_3 are given by

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\,(1-g^2)\,\omega, \\ \alpha_2 &= \frac{i}{2}\,(1+g^2)\,\omega, \quad \text{and,} \\ \alpha_3 &= g\,\omega. \end{aligned} \tag{6.0.20}$$

Therefore, the minimal immersion $\mathbf{X}$ of Theorem 6.0.5 is given by

$$\begin{aligned} X_1 &= \mathrm{Re}\left(\int^z \alpha_1\right) = \mathrm{Re}\left(\int^z \frac{1}{2}\,(1-g^2)\,\omega\right), \\ X_2 &= \mathrm{Re}\left(\int^z \alpha_2\right) = \mathrm{Re}\left(\int^z \frac{i}{2}\,(1+g^2)\,\omega\right) \text{and,} \\ X_3 &= \mathrm{Re}\left(\int^z \alpha_3\right) = \mathrm{Re}\left(\int^z g\,\omega\right). \end{aligned} \tag{6.0.21}$$

If z_0 is a point where g has a pole of order m, then from Equation (6.0.20) it is clear that ω must have a zero of order exactly $2m$ at z_0, in order to have condition (2) given above satisfied and each α_k holomorphic.

Lemma 6.0.6. *Conversely, suppose we have defined on M a meromorphic function g and a holomorphic form ω, whose zeros coincide with the poles of g, in such a way that each zero of order m of ω corresponds to a pole of order $2m$ of g. Then the forms α_1, α_2 and α_3, as defined above are holomorphic on M, satisfy $\alpha_1^2+\alpha_2^2+\alpha_3^2 = 0$ and*

$$|\phi|^2\,|ds|^2 = \sum_{k=1}^{3} |\alpha_k|^2 = \frac{1}{2}\,(1+|g|^2)^2\,|\omega|^2 > 0. \tag{6.0.22}$$

Furthermore, if such forms α_k $(1 \leq k \leq 3,)$ do not have real periods, then we may apply Equation (6.0.21) to obtain a minimal surface - $\mathbf{X} : M \to \mathbb{R}^3$.

Equations 6.0.21 are called the Weierstrass-Enneper representation formulas for minimal surfaces in $\mathbf{X}$. This representation provides us with a useful tool for studying many types of minimal surfaces. The metric obtained *a posteriori* by such representation is given by

$$ds^2 = \frac{1}{2}\,|f|^2\,(1+|g|^2)^2\,|dz|^2. \tag{6.0.23}$$

Examples of classical minimal surfaces obtained with this formulation are given in Figures 9.1 to 9.4.

6.1 The Gauss map

The Gauss map G, of a surface M with parametrization $\mathbf{X}(u,\,v)$ is a mapping from the surface M to the unit sphere $S^2(1) \subset \mathbb{R}^3$, denoted by $G : M \to S^2$ and given by $G(p) = \mathbf{N}_p$, where $\mathbf{N}_p$ is the unit normal vector to M at p. In terms of parametrization, we may write $G(\mathbf{X}(u,\,v)) = \mathbf{N}(u,\,v)$ and, for a small piece of M, think of $\mathbf{N}(u,\,v)$ as a parametrization of part of the sphere S^2. There is also an induced linear transformation of tangent planes given for the basis $\{\mathbf{X}_u,\,\mathbf{X}_v\}$ by $G_\star(\mathbf{X}_u) = \mathbf{N}_u$ and $G_\star(\mathbf{X}_v) = \mathbf{N}_v$. To understand this, we note the following. A tangent vector on M is the velocity vector of some curve on M, so by taking $\mathbf{N}$ only along the curve, we create a new curve on the sphere S^2. The tangent vector of the new spherical curve is then, by definition, the image under $G_\star$ of the original curve's tangent vector.

On applying this line of thought to the parameter curves, we see that $G_\star(\mathbf{X}_u) = \mathbf{N}_u$ and $G_\star(\mathbf{X}_v) = \mathbf{N}_v$.

It is pertinent to mention that the Gauss map of any surface is closely related to the Gauss curvature. In order to see this, let us proceed thus:

Recall the following acceleration formulas

$$\begin{aligned} \mathbf{N}_u &= -\frac{e}{E}\,\mathbf{X}_u - \frac{f}{G}\,\mathbf{N}_v \quad \text{and,} \\ \mathbf{N}_v &= -\frac{f}{E}\,\mathbf{X}_u - \frac{g}{G}\,\mathbf{N}_v. \end{aligned} \tag{6.1.1}$$

Let the quantity $\mathcal{T}$ be defined as

$$\mathcal{T} = \frac{|G_\star(\mathbf{X}_u)\,\times\,G_\star(\mathbf{X}_v)|}{|\mathbf{X}_u\,\times\,\mathbf{X}_v|}. \tag{6.1.2}$$

Now if we substitute for $\mathbf{N}_u$ and $\mathbf{N}_v$ from Equation (6.1.1) into Equation (6.1.2), and remembering that for minimal isothermal surfaces $E = G$ and $F = 0$, and $e = -\,g$ we get

$$\mathcal{T} = \frac{e\,g - f^2}{EG} = \frac{-\,e^2 - f^2}{E^2}. \tag{6.1.3}$$

Notice that $\mathcal{T}$ is the Gauss curvature K. This shows an example of the close relationship between Gauss map and Gauss curvature as mentioned above.

Let us investigate another important characteristic of the Gauss map. Before stating this property, let us recall that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal if, for fixed $\rho > 0$,

$$T(x) \cdot T(y) = \rho^2\, x \cdot y,$$

for all vectors $x,\,y \in \mathbb{R}$.

Proposition 6.1.1. *T is conformal if and only if, for a basis $\{v_1,\,v_2\}$ of $\mathbb{R}^2$, $|T(v_1)| = \rho|v_1|$, $|T(v_2)| = \rho|v_2|$ and $T(v_1) \cdot T(v_2) = \rho^2\, v_1 \cdot v_2$ for $\rho > 0$.*

Proof. The implication ($\Rightarrow$) is trivial. For ($\Leftarrow$), we let $x = av_1 + bv_2$ and $y = cv_1 + dv_2$ with $x \cdot y = ac\,|v_1|^2 + (\,bc + ad\,)v_1 \cdot v_2 + bd\,|v_2|^2$. The linearity of T gives $T(x) = aT(v_1) + bT(v_2)$ and $T(y) = cT(v_1) + dT(v_2)$ with,

$$\begin{aligned} T(x) \cdot T(y) &= ac\,|\,T(v_1)|^2 + (\,bc + ad\,)T(v_1) \cdot T(v_2) + bd\,|\,T(v_2)|^2, \\ &= ac\,\rho^2\,|v_1|^2 + (\,bc + ad\,)\rho^2\,v_1 \cdot v_2 + bd\,\rho^2\,|v_2|^2, \\ &= \rho^2\,x \cdot y. \end{aligned} \tag{6.1.4}$$

□

Lemma 6.1.2. *When $F : M \to N$ is a mapping of one surface to another, F is said to be conformal when the induced map on tangent vectors $F_\star$ is conformal for each point on M. In this case, the factor ρ varies from point to point and is therefore a function of the surface parameters u and v. We then write $\rho(u,\, v)$ and call it the scaling factor. On each tangent plane, however, ρ is constant.*

For example consider the helicoid $\mathbf{X}(u,\, v) = (u\,\cos\,v,\; u\,\sin\,v,\; v)$. In order to compute the Gauss map, we first have to calculate $G_\star(\mathbf{X}_u) = \mathbf{N}_u$ and $G_\star(\mathbf{X}_v) = \mathbf{N}_v$, we therefore proceed as follows

$$\mathbf{X}_u = (\cos\,v,\; \sin\,v,\; 0), \qquad \mathbf{X}_v = (-\,u\,\sin\,v,\; u\,\cos\,v,\; 1).$$

Therefore, $\langle \mathbf{X}_u\,,\, \mathbf{X}_v \rangle = 0$ and $|\mathbf{X}_u| = 1$ and $|\mathbf{X}_v| = \sqrt{1 + u^2}$. The unit normal vector is given by

$$\mathbf{N} = \frac{\mathbf{X}_u\,\times\,\mathbf{X}_v}{|\mathbf{X}_u\,\times\,\mathbf{X}_v|} = \frac{(\sin\,v,\; -\,\cos\,v,\; u)}{\sqrt{1 + u^2}}.$$

On taking partial derivatives of $\mathbf{N}$ with respect to u and v, we get

$$\mathbf{N}_u = \frac{(-\,u\,\sin\,v,\; u\,\cos\,v,\; 1)}{(1 + u^2)^{3/2}} \qquad \text{and} \qquad \mathbf{N}_v = \frac{(\cos\,v,\; \sin\,v,\; 0)}{\sqrt{1 + u^2}},$$

with the moduli of $\mathbf{N}_u$ and $\mathbf{N}_v$ computed as

$$\begin{aligned} |\mathbf{N}_u| &= \frac{1}{1 + u^2} = \frac{1}{1 + u^2}\,|\mathbf{X}_u| \quad \text{and,} \\ |\mathbf{N}_v| &= \frac{1}{\sqrt{1 + u^2}} = \frac{1}{1 + u^2}\,|\mathbf{X}_v|. \end{aligned}$$

Furthermore, $\langle G_\star(\mathbf{X}_u)\,,\, G_\star(\mathbf{X}_v) \rangle = \langle \mathbf{N}_u\,,\, \mathbf{N}_v \rangle = 0$, as required for the Gauss map G to be conformal with scaling factor $\rho(u,\, v) = 1/(1 + u^2)$.

Proposition 6.1.3. *Let M be a minimal surface with isothermal parametrization $\mathbf{X}(u,\, v)$. Then the Gauss map of M is a conformal map.*

Proof. In order to show that G is conformal, we only have to show that $|G_\star(\mathbf{X}_u)| = \rho(u,\,v)|\mathbf{X}_u|$, $|G_\star(\mathbf{X}_v)| = \rho(u,\,v)|\mathbf{X}_v|$ and $\langle G_\star(\mathbf{X}_u)\,,\,G_\star(\mathbf{X}_v)\rangle = \rho^2(u,\,v)\langle \mathbf{X}_u\,,\,\mathbf{X}_v\rangle$. Since isothermal parameters have $E = G$ and $F = 0$, we have $H = (e+g)/2E$ as well as,

$$\begin{aligned} G_\star(\mathbf{X}_u) &= \mathbf{N}_u = -\,\frac{e}{E}\,\mathbf{X}_u - \frac{f}{E}\,\mathbf{X}_v, \quad \text{and,} \\ G_\star(\mathbf{X}_v) &= \mathbf{N}_v = -\,\frac{f}{E}\,\mathbf{X}_u - \frac{g}{E}\,\mathbf{X}_v. \end{aligned}$$

Using isothermal coordinates and taking dot products, we have

$$|\mathbf{N}_u|^2 = \frac{1}{E}\,\left[e^2 + f^2\right], \qquad |\mathbf{N}_v|^2 = \frac{1}{E}\,\left[f^2 + g^2\right], \qquad \langle \mathbf{N}_u\,,\,\mathbf{N}_v\rangle = \frac{f}{E}\,[e+g]\,.$$

Since M is a minimal surface, $H = 0$ and this implies $e = -\,g$ by Equation (3.4.5).

It follows that

$$|\mathbf{N}_u|^2 = \frac{1}{E}\,\left[e^2 + f^2\right] = |\mathbf{N}_v|^2 \qquad \text{and} \qquad \langle \mathbf{N}_u\,,\,\mathbf{N}_v\rangle = 0.$$

Since $|\mathbf{X}_u| = \sqrt{E} = |\mathbf{X}_v|$ and $\langle \mathbf{X}_u\,,\,\mathbf{X}_v\rangle = 0$, we see that the Gauss map G is conformal with scaling factor $\sqrt{e^2 + f^2}/E$. □

Proposition 6.1.4. *Let M be a surface parameterized by $\mathbf{X}(u,\,v)$ whose Gauss map $G : M \to S^2$ is conformal. Then either M is part of a sphere or M is a minimal surface.*

Proof. Let us assume that the parametrization $\mathbf{X}(u,\,v)$ has the property $F = 0$. Since Gauss map is conformal and $F = \langle \mathbf{X}_u\,,\,\mathbf{X}_v\rangle = 0$, then $\langle \mathbf{N}_u\,,\,\mathbf{N}_v\rangle = 0$ as well.

From the formulas in Equation (6.1.1) the foregoing imply $f(Ge+Eg) = 0$. Therefore, either $f = 0$ or $Ge + Eg = 0$. Since $F = 0$, the second condition imply $H = 0$ and M is minimal. If this condition does not hold, then $f = 0$. Let us use $f = 0$, conformality and the formulas in Equation (6.1.1), we have:

$$\frac{e^2}{E} = \langle \mathbf{N}_u\,,\,\mathbf{N}_u\rangle = \rho^2\,E \qquad \text{and} \qquad \frac{g^2}{G} = \langle \mathbf{N}_v\,,\,\mathbf{N}_v\rangle = \rho^2\,G.$$

On simplifying and rearranging, we have:

$$\frac{e^2}{E^2} = \rho^2 = \frac{g^2}{G^2} \qquad \Longrightarrow \qquad \frac{e}{E} = \pm\,\frac{g}{G}. \tag{6.1.5}$$

Suppose $e/E = -\,g/G$. Then $(Ge + Eg)/EG = 0$. This shows that $H = 0$ and M is a minimal surface.

Suppose that $e/E = g/G = k$ at every point of M. Together with $f = 0$ and the formulas in Equation (6.1.1), this implies

$$-\,\mathbf{N}_u = k\,\mathbf{X}_u \quad \text{and,} \tag{6.1.6}$$

$$-\,\mathbf{N}_v = k\,\mathbf{X}_v. \tag{6.1.7}$$

Equations (6.1.6) and (6.1.7) show that at each point of M, the tangent vectors $\mathbf{X}_u$ and $\mathbf{X}_v$ are eigenvectors of the operator called the shape operator, which is defined by taking the a kind of directional derivative of the unit normal vector, $\mathbf{N}$. The eigenvalues associated with the eigenvectors of the shape operator are known to be the maximum and minimum normal curvatures at a point. For the situation above, both eigenvalues are equal to k, so all normal curvatures at a point are constant. Such a point on a surface is called an umbilic point. Therefore when we let $f = 0$ and $e/E = g/G = k$, we are saying that every point of the surface is an umbilic point. Such a non-planar surface is known to be the sphere. Hence the proposition is proved. □

Proposition 6.1.4 can be restated as follows and the corresponding proof is in [31].

Lemma 6.1.5. R. Osserman [31]. *If $\mathbf{X}(\zeta) : \mathcal{D} \to \mathbb{R}^3$ defines a regular minimal surface in isothermal coordinates, then the Gauss map $G(\zeta)$ defines a complex analytic map of $\mathcal{D}$ into the unit sphere considered as the Riemann sphere.*

6.2 Stereographic projection

The Gauss map for a minimal surface has a description directly in terms of the Weierstrass-Enneper representation. Stereographic projection, St from the North pole $\mathcal{N}$ is denoted by

$$\begin{aligned} St &: S^2 - \{\mathcal{N}\} \to \mathbb{R}^2 \quad \text{or,} \\ St &: S^2 - \{(0,\,0,\,1)\} \to \mathbb{R}^2, \end{aligned}$$

and defined as

$$St(\cos\,u\,\cos\,v,\,\sin\,u\,\cos\,v,\,\sin\,v) = \left(\frac{\cos\,u\,\cos\,v}{1-\sin\,v},\,\frac{\sin\,u\,\cos\,v}{1-\sin\,v},\,0\right). \tag{6.2.1}$$

Let us take the induced mapping on tangent vectors by differentiating with respect to u and v,

$$St_\star(\mathbf{X}_u) = \frac{\partial}{\partial u}\left(\frac{\cos\,u\,\cos\,v}{1-\sin\,v},\,\frac{\sin\,u\,\cos\,v}{1-\sin\,v},\,0\right) = \left(\frac{-\sin\,u\,\cos\,v}{1-\sin\,v},\,\frac{\cos\,u\,\cos\,v}{1-\sin\,v},\,0\right),$$

$$St_\star(\mathbf{X}_v) = \frac{\partial}{\partial v}\left(\frac{\cos\,u\,\cos\,v}{1-\sin\,v},\,\frac{\sin\,u\,\cos\,v}{1-\sin\,v},\,0\right) = \left(\frac{\cos\,u}{1-\sin\,v},\,\frac{\sin\,u}{1-\sin\,v},\,0\right). \tag{6.2.2}$$

Therefore on taking dot product in $\mathbb{R}^3$, we have

$$\begin{aligned} \langle St_\star(\mathbf{X}_u)\,,\, St_\star(\mathbf{X}_u)\rangle &= \frac{\cos^2 v}{(1-\sin v)^2}, \\ &= \frac{1}{(1-\sin v)^2}\langle \mathbf{X}_u\,,\,\mathbf{X}_u\rangle, \end{aligned} \tag{6.2.3}$$

$$\begin{aligned} \langle St_\star(\mathbf{X}_v)\,,\, St_\star(\mathbf{X}_v)\rangle &= \frac{1}{(1-\sin v)^2}, \\ &= \frac{1}{(1-\sin v)^2}\langle \mathbf{X}_v\,,\,\mathbf{X}_v\rangle, \end{aligned} \tag{6.2.4}$$

$$\langle St_\star(\mathbf{X}_u)\,,\, St_\star(\mathbf{X}_v)\rangle = 0. \tag{6.2.5}$$

The factor $1/(1-\sin v)$ shows that stereographic projection is a conformal map with a scaling factor $1/(1-\sin v)$. That is, stereographic projection preserves angles. In Cartesian coordinate system, stereographic projection is given by

$$St(x,\,y,\,z) = \left(\frac{x}{1-z},\,\frac{y}{1-z},\,0\right). \tag{6.2.6}$$

The real plane $\mathbb{R}^2$ can be identified with the complex plane $\mathbb{C}$ and extend St to a one-to-one onto mapping $St : S^2 \to \mathbb{C}\cup\{\infty\}$ with the North pole mapping to infinity.

6.3 Meromorphic function

The meromorphic function $g : M \to \mathbb{C}\cup\{\infty\}$ which appears in the Weierstrass representation of a minimal surface $\mathbf{X} : M \to \mathbb{R}^3$ has an important geometrical meaning. In order to see this let us derive an expression for the Gauss mapping $G : M \to S^2(1)$ in terms of the Weierstrass representation for $\mathbf{X}$. Locally, at each point of M, $\alpha_k = \phi_k\, dz$ define the functions ϕ_k.

Theorem 6.3.1. *Let M be a minimal surface with isothermal parametrization $\mathbf{X}(u,\,v)$ and Weierstrass-Enneper representation $(f,\,g)$. Then the Gauss map of M, $G : M \to \mathbb{C}\cup\{\infty\}$, may be identified with the meromorphic function g.*

Proof. Recall that $\phi = \frac{\partial \mathbf{X}}{dz}$, $\bar\phi = \frac{\partial \mathbf{X}}{d\bar z}$ and

$$\phi_1 = \frac{1}{2}\, f(1-g^2), \quad \phi_2 = \frac{i}{2}\, f(1+g^2), \quad \phi_3 = f\,g.$$

First let us describe the Gauss map in terms of ϕ_1, ϕ_2 and ϕ_3. Next, we write

$$\begin{aligned} \mathbf{X}_u \times \mathbf{X}_v &= \left((\mathbf{X}_u \times \mathbf{X}_v)^1,\,(\mathbf{X}_u \times \mathbf{X}_v)^2,\,(\mathbf{X}_u \times \mathbf{X}_v)^3\right), \\ &= \left(X_u^2\,X_v^3 - X_u^3\,X_v^2,\; X_u^3\,X_v^1 - X_u^1\,X_v^3,\; X_u^1\,X_v^2 - X_u^2\,X_v^1\right), \end{aligned}$$

and let us consider the first component of $(\mathbf{X}_u \times \mathbf{X}_v)^1 = X_u^2\, X_v^3 - X_u^3\, X_v^2$. We have

$$\begin{aligned} X_u^2\, X_v^3 - X_u^3\, X_v^2 &= \operatorname{Im}[(X_u^2 - i\, X_v^2)(X_u^3 + i\, X_v^3)], \\ &= \operatorname{Im}[2(\partial X^2/\partial z \cdot 2(\partial X^3/\partial \bar{z})], \\ &= 4\operatorname{Im}(\phi_2\, \bar{\phi}_3). \end{aligned}$$

Similarly, $(\mathbf{X}_u \times \mathbf{X}_v)^2 = 4\operatorname{Im}(\phi_3\, \bar{\phi}_1)$ and $(\mathbf{X}_u \times \mathbf{X}_v)^3 = 4\operatorname{Im}(\phi_1\, \bar{\phi}_2)$ when these three components of $\mathbf{X}_u \times \mathbf{X}_v$ are taken together, we get

$$\begin{gathered} \mathbf{X}_u \times \mathbf{X}_v = 4\operatorname{Im}(\phi_2\, \bar{\phi}_3,\, \phi_3\, \bar{\phi}_1,\, \phi_1\, \bar{\phi}_2) = 2(\phi \times \bar{\phi}) = \\ \tfrac{1}{4}\, |f|^2\, \left(|g|^2 + 1)(2\operatorname{Re}(g),\, 2\operatorname{Im}(g),\, (|g|^2 - 1)\right), \end{gathered}$$

where $2(\phi \times \bar{\phi})$ follows from $z - \bar{z} = 2\operatorname{Im} z$.

Since $\mathbf{X}(u,\, v)$ is isothermal, $|\mathbf{X}_u \times \mathbf{X}_v| = |\mathbf{X}_u| \cdot |\mathbf{X}_v| = |\mathbf{X}_u|^2 = E = 2|\phi|^2$ by Lemma 8.1.5.

Therefore, we have

$$\mathbf{N} = \frac{\mathbf{X}_u \times \mathbf{X}_v}{|\mathbf{X}_u \times \mathbf{X}_v|} = \frac{2(\phi \times \bar{\phi})}{2|\phi|^2} = \frac{\phi \times \bar{\phi}}{|\phi|^2} = \left(\frac{2\operatorname{Re}(g)}{|g|^2 + 1},\, \frac{2\operatorname{Im}(g)}{|g|^2 + 1},\, \frac{|g|^2 - 1}{|g|^2 + 1}\right). \tag{6.3.1}$$

The Gauss map $G : M \to \mathbb{C} \cup \{\infty\}$ may now be given in terms of the ϕ^i

$$\begin{aligned} G(\mathbf{X}(u,\, v)) &= St\,(\mathbf{N}(u,\, v)) = St\left(\frac{\phi \times \bar{\phi}}{|\phi|^2}\right), \\ &= St\left(\frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3,\, \phi_3\, \bar{\phi}_1,\, \phi_1\, \bar{\phi}_2)}{|\phi|^2}\right), \\ &= \left(\frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3)}{|\phi|^2 - 2\operatorname{Im}(\phi_1\, \bar{\phi}_2)},\, \frac{2\operatorname{Im}(\phi_3\, \bar{\phi}_1)}{|\phi|^2 - 2\operatorname{Im}(\phi_1 \bar{\phi}_2)},\, 0\right). \qquad (6.3.2) \end{aligned}$$

□

Equation (6.3.2) follows because

$$\begin{aligned} \frac{x}{1 - z} &= \frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3)}{|\phi|^2} \cdot \frac{1}{1 - \frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3)}{|\phi|^2}}, \\ &= \frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3)}{|\phi|^2} \cdot \frac{|\phi|^2}{|\phi|^2 - 2\operatorname{Im}(\phi_1\, \bar{\phi}_2)}, \\ &= \frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3)}{|\phi|^2 - 2\operatorname{Im}(\phi_1\, \bar{\phi}_2)}, \qquad (6.3.3) \end{aligned}$$

and similarly for $y/(1 - z)$. Identifying $(x,\, y) \in \mathbb{R}^2$ with $x + i\, y \in \mathbb{C}$ allows us to write

$$G(\mathbf{X}(u,\, v)) = \frac{2\operatorname{Im}(\phi_2\, \bar{\phi}_3) + 2\, i\operatorname{Im}(\phi_3\, \bar{\phi}_1)}{|\phi|^2 - 2\operatorname{Im}(\phi_1\, \bar{\phi}_2)}. \tag{6.3.4}$$

Now let us consider the numerator, $\mathcal{N}$ of Equation (6.3.4)

$$\begin{aligned}
\mathcal{N} &= 2\,\mathrm{Im}(\phi_2\,\bar{\phi_3}) + 2\,i\,\mathrm{Im}(\phi_3\,\bar{\phi_1}), \\
&= \frac{1}{i}\left[\phi_2\,\bar{\phi_3} - \bar{\phi_2}\,\phi_3 + i\,\phi_3\,\bar{\phi_1} - i\,\bar{\phi_3}\,\phi_1\right], \\
&= \phi_3\,(\bar{\phi_1} + i\,\bar{\phi_2}) - \bar{\phi_3}\,(\phi_1 + i\,\phi_2).
\end{aligned} \tag{6.3.5}$$

Recall that $(\phi)^2 = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 = (\phi_1 - i\,\phi_2)(\phi_1 + i\,\phi_2) + (\phi_3)^2 = 0.$

Therefore

$$\mathcal{N} = 2\,\mathrm{Im}(\phi_2\bar{\phi_3}) + 2\,i\,\mathrm{Im}(\phi_3\,\phi_1 + i\,\phi_2) = \frac{-(\phi_3)^2}{\phi_1 - i\,\phi_2}, \tag{6.3.6}$$

$\mathcal{N}$ can be simplified further as follows:

$$\begin{aligned}
\mathcal{N} &= \phi_3(\bar{\phi_1} - i\,\phi_2) + \bar{\phi_3}\,\frac{(\phi_3)^2}{\phi_1 - i\,\phi_2}, \\
&= \frac{\phi_3\left[\left((\phi_1 - i\,\phi_2)(\bar{\phi_1} + i\,\bar{\phi_2}) + |\phi_3|^2\right)\right]}{\phi_1 - i\,\phi_2}, \\
&= \frac{\phi_3}{\phi_1 - i\,\phi_2}\left[|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 + i\,(\bar{\phi_2}\,\phi_1 - \phi_2\,\bar{\phi_1})\right], \\
&= \frac{\phi_3\,\left[|\phi|^2 - 2\,\mathrm{Im}(\phi_1\,\bar{\phi_2})\right]}{\phi_1 - i\,\phi_2}.
\end{aligned} \tag{6.3.7}$$

The second factor in the numerator of Equation (6.3.7) cancels out the denominator of Equation (6.3.4), and we end up with

$$G(\mathbf{X}(u,\,v)) = \frac{\phi_3}{\phi_1 - i\,\phi_2}. \tag{6.3.8}$$

As will be shown below, the Gauss map, $G(\mathbf{X}(u,\,v))$ is the meromorphic function g.

6.4 Parametric surfaces in $\mathbb{R}^3$ and the Gauss map

Let us start with important observation that for the case $n = 3$, we can describe explicitly all solutions of the equation

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0. \tag{6.4.1}$$

Now let us consider the following lemmas

Lemma 6.4.1. *Every simply-connected minimal surface M has a reparametrization in the form $\mathbf{X}(z) : D \to \mathbb{R}^n$, where D is either the unit disk, $|z| < 1$, or the entire z-plane.*

Lemma 6.4.2. *Let D be a domain in the complex z-plane, $g(z)$ an arbitrary meromorphic function in D and $f(z)$ analytic function in D having the property that each point where $g(z)$ has a pole of order m, $f(z)$ has a zero of order at least $2m$. Then the functions*

$$\phi_1 = \frac{1}{2} f(1-g^2), \qquad \phi_2 = \frac{i}{2} f(1+g^2), \qquad \textit{and} \qquad \phi_3 = fg, \tag{6.4.2}$$

will be analytic in D and satisfy Equation (6.4.1). Conversely, every triple of analytic functions in D satisfying Equation (6.4.1) may be represented in the form of Equations 6.4.2, except for $\phi_1 \equiv i\,\phi_2$ and $\phi_3 \equiv 0$.

Proof. By substituting Equations (6.4.2) into Equation (6.4.1), verifies that Equations (6.4.2) satisfy Equation (6.4.1). Conversely, given any solution of Equation (6.4.1), we set

$$f = \phi_1 - i\,\phi_2, \text{ and } g = \frac{\phi_3}{\phi_1 - i\,\phi_2}. \tag{6.4.3}$$

If we write Equation (6.4.1) in the form

$$(\phi_1 - i\,\phi_2)(\phi_1 + i\,\phi_2) = -\,\phi_3^2, \tag{6.4.4}$$

we find that

$$\phi_1 + i\,\phi_2 = -\,\frac{\phi_3^2}{\phi_1 - i\,\phi_3} = -\,fg^2. \tag{6.4.5}$$

Combining Equations (6.4.3) and (6.4.5) gives Equation (6.4.2). It is pertinent to mention that the conditions relating the zeros of f and the poles of g must obviously be satisfied, since otherwise by Equation (6.4.5), $(\phi_1 + i\,\phi_2)$ would fail to be analytic. This representation can fail only if the denominator in the expression for for g in Equation (6.4.3) vanishes identically. In this case we have by Equation (6.4.4) that $\phi_3 \equiv 0$, which is the exceptional case mentioned.

Therefore the Gauss map of M, $G : M \to \mathbb{C} \cup \{\infty\}$, can be identified with the meromorphic function g of $\mathbf{X}$. In particular, G is a conformal map as already mentioned above, and this property characterizes minimal surfaces (besides the sphere). This concludes the proof of the theorem. □

6.4.1 More on stereographic projection

Let us write the stereographic projection, St again for quick and easy reference

$$St : \overline{\mathbb{C}} \to S^2, \qquad St(z) = \begin{cases} \frac{1}{|z|^2+1}\left(2\,\mathrm{Re}\, z,\ 2\,\mathrm{Im}\, z,\ |z|^2 - 1\right) & \text{for } z \in \mathbb{C}, \\ (0,\ 0,\ 1) & \text{for } z = \infty. \end{cases} \tag{6.4.6}$$

To derive Equation (6.4.6), let $\mathcal{N} = (0,0,1)$ be the north pole. A projection by straight line means $St(z) = \lambda z + (1-\lambda)N$ with $\lambda \neq 0$. It maps to S^2 if and only if

$$1 = |St(z)|^2 = \lambda^2|z|^2 + (1-\lambda)^2 = 1 - 2\lambda + \lambda^2(|z|^2+1).$$

Thus

$$0 = -2 + \lambda(|z|^2 + 1) \qquad \Longrightarrow \qquad \lambda = \frac{2}{|z|^2+1}, \tag{6.4.7}$$

so that indeed Equation (6.4.6) follows. It has been shown above that St is a diffeomorphism, and St is conformal.

We can now explain the geometric meaning of g and f.

Proposition 6.4.3. *Let $\mathbf{X}(z) : D \to \mathbb{R}^3$ be a minimal surface with Weierstrass data g and f.*

Then:

(i) $\mathbf{N}_p$ is stereographic projection of the Gauss map g of $\mathbf{X}(z)$, that is,

$$\mathbf{N}_p = St(g) = \frac{1}{|g|^2+1}\left(2\,Re\,g,\ 2\,Im\,g,\ |g|^2 - 1\right).$$

Moreover, $\mathbf{N}_p$ is conformal.

(ii) The function $Re\,\mathbf{X}(z)$ is the derivative of the height function $X_3 = \phi_3$, that is, $X'_3 = \phi'_3 = Re\,f$.

Proof. (i) It remains to show that $St(g)$ is perpendicular to the tangent space. Recall that $\mathbf{X}(z) = \mathrm{Re}\int_{z_0}^{z} \Phi(\omega)\,d\omega$.

Therefore

$$\frac{\partial \mathbf{X}(z)}{\partial x} - i\,\frac{\partial \mathbf{X}(z)}{\partial y} = \frac{\partial \mathbf{X}(z)}{\partial z} = \Phi = \mathrm{Re}\,\Phi + i\,\mathrm{Im}\,\Phi; \qquad z = x + i\,y \tag{6.4.8}$$

we conclude that

$$\frac{\partial \mathbf{X}(z)}{\partial x} = \mathrm{Re}\,\Phi \quad \text{and} \quad \frac{\partial \mathbf{X}(z)}{\partial y} = \mathrm{Im}\,\Phi, \tag{6.4.9}$$

meaning that real and imaginary parts of Φ span the tangent space. Noting that $St(g)$ is a real vector, we compute

$$\langle \mathrm{Re}\,\Phi\,,\, St(g)\rangle + i\,\langle \mathrm{Im}\,\Phi\,,\, St(g)\rangle = \langle \Phi\,,\, St(g)\rangle =$$

$$\begin{aligned}
&= \left\langle f\left(\frac{1}{2}\left(\frac{1}{g} - g\right), \frac{i}{2}\left(\frac{1}{g} + g\right), 1\right) \,,\, \frac{1}{|g|^2+1}\left(2\,\mathrm{Re}\,g,\, 2\,\mathrm{Im}\,g,\, |g|^2 - 1\right)\right\rangle, \\
&= \frac{f}{|g|^2+1}\left((\mathrm{Re}\,g)\left(\frac{1}{g} - g\right) + i\,(\mathrm{Im}\,g)\left(\frac{1}{g} + g\right) + |g|^2 - 1\right),
\end{aligned}$$

$$\frac{f}{|g|^2+1}\left(\frac{g}{g} - g\,\overline{g} + |g|^2 - 1\right) = 0. \tag{6.4.10}$$

Thus $\partial \mathbf{X}(z)/\partial x$ and $\partial \mathbf{X}(z)/\partial y$ are perpendicular to $St(g)$. Conformality follows from the fact that g is conformal and St is conformal.

(ii) is immediate from $\frac{\partial \mathbf{X}(z)}{\partial z} = \text{Re}\,\Phi$. □

Example. The catenoid and Enneper surface have $g(z) = z$, so that $\mathbf{N}(z) = St(z)$. Thus they are parameterized by (the stereographic projection of) the Gauss map. In particular, their Gauss image is $-S^2$ up to sets of measure zero, and so the total curvature is $-A(S^2) = -4\pi$.

6.5 Weierstrass-Enneper representation II

We can simplify Weierstrass-Enneper representations I even further so that we only have to choose one function instead of two. Suppose in Equation (6.0.21), g is holomorphic and has an inverse function g^{-1} (in a domain D) which is holomorphic as well. Then we can consider g as a new complex variable $\tau = g$, with $d\tau = g'\,dz$. let us define $F(\tau) = f/g'$ and obtain $F(\tau)\,d\tau = f\,dz$.

Therefore, if we replace g by τ and $f\,dz$ by $F(\tau)\,d\tau$, we obtain

Theorem 6.5.1. *For any holomorphic function $F(\tau)$, a minimal surface is defined by the parametrization $\mathbf{X}(z) = (x_1(z), x_2(z), x_3(z))$, where*

$$\begin{aligned} x_1(z) &= Re\left[\int^z \left(1-\tau^2\right) F(\tau)\,d\tau\right], \\ x_2(z) &= Re\left[\int^z i\left(1+\tau^2\right) F(\tau)\,d\tau\right], \\ x_3(z) &= Re\ 2\left[\int^z \tau F(\tau)\,d\tau\ \right]. \end{aligned}$$

Notice that the corresponding ϕ components are

$$\phi_1 = \frac{1}{2}(1-\tau^2)F(\tau), \quad \phi_2 = \frac{i}{2}(1+\tau^2)F(\tau), \quad \phi_3 = \tau F(\tau).$$

This representation tells us that any holomorphic function $F(\tau)$ defines a minimal surface. However, it is pertinent to mention that not every function will give a complex integral which evaluates nicely (in closed form). For example, if we choose $F(\tau) = i/2\tau^2$ and the substitute $\tau = e^z$ after integration, we get a form of helicoid.

From the foregoing, we see that the Weierstrass-Enneper representations enable us to analyze many aspects of a minimal surface directly from representing functions $(f,\,g)$ and $F(\tau)$. Surface whose Weierstrass-Enneper integrals can not be expressed in closed-form can also be analyzed using this method. For illustration purposes, let us use the Weierstrass-Enneper representations to compute the Gauss curvature, K of a minimal surface in terms of $F(\tau)$.

Recall from Theorem 3.5.2, that if we use isothermal parameters, we have

$$
\begin{aligned}
K &= -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)+\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right),\\
&= -\frac{1}{2E}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{E}\right)+\frac{\partial}{\partial u}\left(\frac{E_u}{E}\right)\right),\\
&= -\frac{1}{2E}\left(\frac{\partial^2}{\partial v^2}\ln E+\frac{\partial^2}{\partial u^2}\ln E\right),\\
&= -\frac{1}{2E}\Delta(\ln E),
\end{aligned}
$$

where Δ is the Laplace operator $\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}$. Recall that $E = 2|\phi|^2$ – we shall formally show this in Equation (8.1.21). We can solve for E by substituting ϕ_1, ϕ_2 and ϕ_3 into $E = 2|\phi|^2$.

It follows that

$$
\begin{aligned}
E &= 2\left[\left|\frac{1}{2}(1-\tau^2)F(\tau)\right|^2+\left|\frac{i}{2}(1+\tau^2)F(\tau)\right|^2+|\tau F(\tau)|^2\right],\\
&= \frac{1}{2}|F|^2\left[|\tau^2-1|^2+|\tau^2+1|^2+4|\tau|^2\right]. \qquad (6.5.1)
\end{aligned}
$$

Therefore, $\tau^2 = u^2-v^2+2iuv$, $\quad |\tau^2-1|^2 = (u^2-v^2-1)^2+4u^2v^2,$

$$|\tau^2+1|^2 = (u^2-v^2+1)^2+4u^2v^2, \qquad 4|\tau|^2 = 4(u^2+v^2).$$

On substituting these quantities into Equation (6.5.1), we get:

$$
\begin{aligned}
E &= \frac{1}{2}|F|^2\,2[(u^2-v^2)^2+1+4u^2v^2+2u^2+2v^2],\\
&= |F|^2\,[u^4+2u^2v^2+v^4+1+2u^2+2v^2],\\
&= |F|^2\,[1+u^2+v^2]^2. \qquad (6.5.2)
\end{aligned}
$$

Recall that,

$$\frac{E}{2} = |\phi|^2 = \frac{1}{4}|f|^2\left((1-g^2)(1-\bar{g}^2)+(1+g^2)(1+\bar{g}^2)+4g\bar{g}\right) \Rightarrow E = |f|^2\,[1+|g|^2]^2,$$

$$\ln E = \ln|F|^2+2\ln(1+u^2+v^2) \Rightarrow \Delta(2\ln(1+u^2+v^2)) = \frac{8}{(1+u^2+v^2)},$$

$$\Delta(\ln|F|^2) = \Delta(\ln F\,\bar{F}) = \Delta(\ln F+\ln\bar{F}),$$

Recall from Equation (5.1.2) that $\Delta = 4\,\partial^2/\partial\bar{z}\partial z$. Since F is holomorphic, $\bar{F}$ cannot be, therefore

$$\partial\bar{F}/\partial z = 0 \qquad \Rightarrow \qquad \partial(\ln\bar{F})/\partial z = 0,$$

$$\Delta(\ln F) = 4\,\frac{\partial^2(\ln F)}{\partial\bar{z}\partial z} = 4\,\frac{\partial(F'/F)}{\partial\bar{z}} = 0.$$

Since F, $\bar{F}$ and hence F'/F are holomorphic.

Therefore, $\quad \Delta(\ln|F|^2) = 0 \quad$ and $\quad \Delta(\ln E) = 8/(1+u^2+v^2)^2$.

Theorem 6.5.2. *The Gauss curvature of the minimal surface determined by the Weierstrass-Enneper representation II is*

$$K = -\,\frac{4}{|f|^2(1+u^2+v^2)^4}. \tag{6.5.3}$$

Proof. From the foregoing,

$$\begin{aligned} K &= -\,\frac{1}{2E}\,\Delta(\ln E), \\ &= -\,\frac{8}{2|F|^2(1+u^2+v^2)^4}, \\ &= -\,\frac{4}{|F|^2(1+u^2+v^2)^4}. \qquad (6.5.4) \end{aligned}$$

□

Notice that if substitute $F = f/g'$ in Equation (6.5.3), we get

$$K = -\,\frac{4|g'|^2}{|f|^2(1+|g|^2)^4}. \tag{6.5.5}$$

Equation (6.5.3) gives the value of Gauss curvature, K in terms of Weierstrass-Enneper representation II, while Equation (6.5.5) gives K in terms of Weierstrass-Enneper representation I.

Corollary 6.5.3. *The Gauss curvature of a minimal surface is non-positive and unless the surface is a plane, it can have only isolated zeros.*

Proof. From Equation (6.5.5), we see that K is zero only when g' is zero. Since g is meromorphic, g' is also meromorphic. This means that unless g' is identically zero everywhere, g' has a finite number of poles. g' is not identically zero everywhere for that would mean M is a part of the plane, which is contrary to our initial assumption, which is; M is not a plane. As mentioned in Theorem 5.3.1, meromorphic function have the property that their zeros are isolated, so g' has isolated zeros which implies that K does as well. □ □

6.5.1 Comments

1. We noticed that the values of Gauss curvature, K given by Equations (6.5.3) and (6.5.5) are different. The first formula never allows $K = 0$ while the second allows $K = 0$ at points where $g' = 0$. This discrepancy is due to assumption we made in transforming from Weierstrass-Enneper representation I to Weierstrass-Enneper representation II by allowing g to be holomorphic and to have an inverse g^{-1} which is holomorphic in a domain D as well. Refer to the proof of Corollary 6.5.3 for more details.

2. A minimal surface which is described by $(f,\, g)$ or $F(\tau)$ has an associated *family* of minimal surfaces given by $(e^{it}f,\, g)$ or $(e^{it}F(\tau))$ respectively. Two surfaces of such family described by t_0 and t_1 are said to be *adjoint* if $t_1 - t_0 = \pi/2$. It is pertinent to mention that $E = \mathbf{X}_u \cdot \mathbf{X}_u$ remains the same no matter what t is taken. Since we have isothermal parameters, this is enough to show that all the surfaces of an associated family are locally isometric, Oprea [29].

6.6 Classical complete embedded minimal surfaces using Weierstrass - Enneper data

In this chapter, we present the construction of classical complete embedded minimal surfaces (helicoid, catenoid, Scherk's, Enneper's and the Henneberg's surfaces) using Weierstrass - Enneper data. The first four surfaces are complete in the induced metric, only the Henneberg's surface is not complete.

6.6.1 The Helicoid

Take $M = \mathbb{C}$, $\quad g(z) = -\,ie^{z}$ and $\omega = e^{-z}\,dz$. Observe that g has no poles and ω has no zeros in $\mathbb{C}$. From Equation (6.0.20), we have

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\,(1-g^2)\omega = \cosh{(z)}\,dz, \\ \alpha_2 &= \frac{1}{2}\,(1+g^2)\omega = -i\sinh{(z)}\,dz, \\ \alpha_3 &= g\,\omega = -\,i\,dz. \end{aligned} \tag{6.6.1}$$

Since $\cosh(z)$, $\sinh(z)$ and multiplication by a constant are holomorphic functions in $\mathbb{C}$, then we have $\int_\gamma \alpha_k = 0$, for every closed path γ in $\mathbb{C}$ and $k = 1,\, 2,\, 3$. That is, the forms α_k do not have periods. From Equation (6.0.21), we get

$$\begin{aligned} X_1 &= \mathrm{Re}\int_0^z \cosh(z)\,dz = \mathrm{Re}\,(\sinh(z)) = \cos{(v)}\sinh{(u)}, \\ X_2 &= \mathrm{Re}\int_0^z -\,i\sinh(z)\,dz = \mathrm{Re}\,(-\,i\cosh(z)+i) = \sin{(v)}\sinh{(u)}, \\ X_3 &= \mathrm{Re}\int_0^z -\,i\,dz = \mathrm{Re}\,(-i\,z) = v. \end{aligned} \tag{6.6.2}$$

Therefore, $\mathbf{X}(u,\, v) = (\cos{(v)}\,\sin{(v)}\sinh{(u)},\, v)$ describes a minimal surface – The helicoid. Figures 9.1a and 9.1b show such a helicoid.

6.6.2 The Catenoid

Take $M = \mathbb{C}$, $\quad g(z) = -\,e^z$ and $\omega = -\,e^{-z}\,dz$. Observe that g has no poles and ω has no zeros in $\mathbb{C}$. From Equation (6.0.20), we have

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\,(1-g^2)\omega = \sinh(z)\,dz, \\ \alpha_2 &= \frac{1}{2}\,(1+g^2)\omega = -i\cosh(z)\,dz, \\ \alpha_3 &= dz. \end{aligned} \tag{6.6.3}$$

Since $\cosh(z)$, $\sinh(z)$ and multiplication by a constant are holomorphic functions in $\mathbb{C}$, then we have $\int_\gamma \alpha_k = 0$, for every closed path γ in $\mathbb{C}$ and $k = 1, 2, 3$. That is, the forms α_k do not have periods. From Equation (6.0.21), we get

$$\begin{aligned} X_1 &= \operatorname{Re}\int_0^z \sinh(z)\,dz = \operatorname{Re}\,(\cosh(z)-1) = \cos(v) - 1, \\ X_2 &= \operatorname{Re}\int_0^z -\,i\cosh(z)\,dz = \operatorname{Re}\,(-\,i\sinh(z)) = \sin(v)\cosh(u), \\ X_3 &= \operatorname{Re}\int_0^z dz = \operatorname{Re}\,(z) = u. \end{aligned} \tag{6.6.4}$$

Therefore, $\mathbf{X}(u,\,v) = (\cos(v)\cosh(u),\ \sin(v)\cosh(u),\ u) - (1,\,0,\,0)$. This is up to a translation, the parametrization of the catenoid shown in Figures 9.1c and 9.1d. Such a parametrization wraps the plane $\mathbb{C}$ around the catenoid infinitely many times.

Another way of obtaining the catenoid is the following: Take $M = \mathbb{C} - \{0\}$, $\quad g(z) = z$ and $\omega = dz/z^2$. Then

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\,(1-z^2)\frac{dz}{z^2} = \frac{1}{2}\left(\frac{1}{z^2} - 1\right)dz, \\ \alpha_2 &= \frac{i}{2}\,(1+z^2)\frac{dz}{z^2} = \frac{1}{2}\left(\frac{1}{z^2} + 1\right)dz, \\ \alpha_3 &= \frac{1}{z}\,dz. \end{aligned} \tag{6.6.5}$$

The forms α_1 and α_2 do not have periods, but α_3 has only a purely imaginary period, which can be calculated thus

$$\oint \frac{1}{z}\,d\,z = \int_0^{2\pi} \frac{1}{r}\,e^{-\theta\,i}\,d(r\,e^{\theta\,i}) = 2\pi i,$$

$$\text{where} \quad z = u + i\,v, \quad r = \sqrt{u^2+v^2} \quad \text{and} \quad \tan\theta = \frac{v}{u}.$$

From Equation (6.0.21), we get,

$$
\begin{aligned}
X_1 &= -\frac{u}{2}\left(1+\frac{1}{u^2+v^2}\right)+1,\\
X_2 &= -\frac{v}{2}\left(1+\frac{1}{u^2+v^2}\right),\\
X_3 &= \frac{1}{2}\log_e\,(u^2+v^2).
\end{aligned}
\tag{6.6.6}
$$

Equations 6.6.6 describe the catenoid, up to a translation. In order to see this, set:

$$\rho = \frac{1}{2}\log_e\,(u^2+v^2) \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{u}{v}\right) - \pi.$$

6.6.3 The Enneper's surface

The simplest choice that one can make for M, g and ω is to take $M = \mathbb{C}$, $g(z) = z$ and $\omega = dz$. This choice results in a minimal surface $\mathbf{X} : \mathbb{C} \to \mathbb{R}^3$ given by

$$\mathbf{X}(u,\,v) = \frac{1}{2}\left(u - \frac{u^3}{3} + u\,v^2,\ -v + \frac{v^3}{3} - u^2\,v,\ u^2 - v^2\right), \tag{6.6.7}$$

which describes the Enneper's surface shown in Figures 9.3a and 9.3b. This is a complete minimal surface. Its Gauss curvature is

$$K = -\frac{16}{(1+|z|^2)^4}, \qquad \text{where} \qquad z = u + i\,v. \tag{6.6.8}$$

We note however, that Enneper's surface, within a certain radius boundary curve, is minimal but not area minimizing. This is illustrated in Figure 9.4.

6.6.4 The Scherk's surface

Consider the unit disk $\mathcal{D} = \{z \in \mathbb{C};\quad |z| < 1\}$. Take $M = \mathcal{D}$, $\quad g(z) = z$ and $\omega = \frac{4\,dz}{(1-z^4)}$. From Equation (6.0.20), we have,

$$
\begin{aligned}
\alpha_1 &= \frac{2\,dz}{1+z^2} = \left(\frac{i}{z+i} - \frac{i}{z-i}\,dz\right)dz,\\
\alpha_2 &= \frac{2\,i\,dz}{1-z^2} = \left(\frac{i}{z+1} - \frac{i}{z-1}\,dz\right)dz,\\
\alpha_3 &= \frac{4z\,dz}{1-z^4} = \left(\frac{2z}{z^2+1} - \frac{2z}{z^2-1}\right)dz.
\end{aligned}
\tag{6.6.9}
$$

It is obvious that $\alpha_1,\ \alpha_2$, and α_3 have no periods in $\mathcal{D}$. From Equation (6.0.21), we get

$$\begin{aligned} X_1 &= \operatorname{Re}\left(i\log_e\left(\frac{z+i}{z-i}\right)\right) = -\arg\left(\frac{z+i}{z-i}\right), \\ X_2 &= \operatorname{Re}\left(i\log_e\left(\frac{z+1}{z-1}\right)\right) = -\arg\left(\frac{z+1}{z-1}\right), \\ X_3 &= \operatorname{Re}\left(\log_e\left(\frac{z^2+1}{z^2-1}\right)\right) = \log_e\left|\frac{z^2+1}{z^2-1}\right|. \end{aligned} \tag{6.6.10}$$

It is easy to see that

$$\frac{z+i}{z-i} = \frac{|z|^2-1}{|z-i|^2} + i\,\frac{z+\bar{z}}{|z-i|^2},$$

and

$$\frac{z+1}{z-1} = \frac{|z|^2-1}{|z-1|^2} + \frac{z-\bar{z}}{|z-1|^2}.$$

Since $|z|^2-1<1$ in $\mathcal{D}$, we have

$$-\frac{3\pi}{2} \le x_j \le -\frac{\pi}{2},\quad j=1,\,2,\quad z = x_1 + i\,x_2.$$

From the expressions given above, we have

$$\cos\,x_1 = \frac{|z|^2-1}{z^2+1} \qquad \text{and} \qquad \cos\,x_2 = \frac{|z|^2-1}{z^2-1},$$

which gives,

$$x_3 = \log_e\left(\frac{\cos\,x_2}{\cos\,x_1}\right),\ \text{ where }\ (x_1,\,x_2)\ \text{ is restricted to}\left(-\frac{3\pi}{2},\,-\frac{\pi}{2}\right)\left(-\frac{3\pi}{2},\,-\frac{\pi}{2}\right).$$

Thus the the surface $\mathbf{X}:\mathcal{D}\to\mathbb{R}^3$ describes a piece of Scherk's minimal surface, which is shown in Figure 9.3c.

For us to obtain the whole surface, we consider g and ω defined in $M = \mathbb{C} - \{1,\,-1,\,i,\,-i\}$. In this case, the resulting forms α_1 and α_2 have real periods.

Let $\prod:\overline{M}\to M$ be the universal covering of M and define

$$\tilde{x}_k = \operatorname{Re}\int^z \prod^\star \alpha_k, \qquad\qquad k = 1,\,2,\,3 \tag{6.6.11}$$

Since $\overline{M}$ is simply connected, the forms $\prod^\star\ \alpha_k$ have no periods and therefore, the functions $\bar{x}_k:\overline{M}\to\mathbb{R}$ are well defined, $k=1,\,2,\,3$. The image $\tilde{x}(\tilde{M})$ can be obtained from the previous functions x_k if we now allow $x_1 = -\arg\left(\frac{z+i}{z-i}\right)$ and $x_2 = -\arg\left(\frac{z+1}{z-1}\right)$ to assume all possible values under the only restriction that

$$\frac{\cos\,x_2}{\cos\,x_1} > 0.$$

This is equivalent to considering the entire graphic of the real function

$$x_3 = \log_e\left(\frac{\cos\,x_2}{\cos\,x_1}\right).$$

6.6.5 The Henneberg's surface

$M = \mathbb{C} - \{0\}$, $g(z) = z$ and $\omega = 2\left(1 - \frac{1}{z^4}\right) dz$. This gives

$$\begin{aligned} \alpha_1 &= \left(-\frac{1}{z^4} + \frac{1}{z^2} + 1 - z^2\right) dz, \\ \alpha_2 &= i\left(-\frac{1}{z^4} - \frac{1}{z^2} + 1 - z^2\right) dz, \\ \alpha_3 &= 2\left(z - \frac{1}{z^3}\right) dz. \end{aligned} \tag{6.6.12}$$

Notice that α_1, α_2 and α_3 have no periods in M. We then substitute for the α's in Equation (6.0.21), to get

$$\begin{aligned} X_1 &= \operatorname{Re}\int_1^z \alpha_1\, dz = \operatorname{Re}\left(\int_1^z \frac{(1-z^2)^3}{3z^3}\right) dz = \operatorname{Re}\left(\frac{(\bar{z} - |z|^2)^3}{3|z|^6}\right), \\ X_2 &= \operatorname{Re}\int_1^z \alpha_2\, dz = \operatorname{Re}\left(\int_1^z \frac{i\,(1+z^2)^3}{3z^3} - \frac{8\,i}{3}\right) dz = -\operatorname{Im}\left(\frac{(\bar{z} + |z|^2 z)^3}{3|z|^6}\right), \end{aligned}$$

$$X_3 = \operatorname{Re}\int_1^z \alpha_3\, dz = \operatorname{Re}\left(\int_1^z \frac{(z^2-1)^2}{z^2}\right) dz = \operatorname{Re}\left(\frac{(z\,|z|^2 - \bar{z})^2}{|z|^4}\right), \tag{6.6.13}$$

resulting in the following

$$\begin{aligned} X_1 &= \frac{u^3\,(1-u^2-v^2)^3 - 3u\,v^2(1-u^2-v^2)(1+u^2+v^2)^2}{3(u^2+v^2)^3}, \\ X_2 &= \frac{3u^2\,v(1+u^2+v^2)^2\,(1-u^2-v^2) - v^3(1-u^2-v^2)^3}{3(u^2+v^2)^3}, \\ X_3 &= \frac{(1-u^2-v^2)^2\,u^2 - (1+u^2+v^2)\,v^2}{(u^2+v^2)^2}. \end{aligned} \tag{6.6.14}$$

Now let $\varphi(z) = (1-z^2)/z$ and $\psi(z) = (1+z^2)/z$. It is easy to verify that

$$\varphi\left(-\frac{1}{\bar{z}}\right) = \overline{\varphi}(z) \qquad \text{and} \qquad \psi = \left(-\frac{1}{\bar{z}}\right) = \overline{\psi}(z). \tag{6.6.15}$$

Since $X_1 = \frac{1}{3}\operatorname{Re}(\varphi(z)^3)$, $x_2 = \frac{1}{3}\operatorname{Re}\ i\,(\psi(z)^3)$ and $X_3 = \operatorname{Re}(\varphi(z)^2)$, we have that $x_k(-1/\bar{z}) = x_k(z)$, $k = 1, 2, 3$. If we identify M with the unit sphere minus two points through the stereographic projection, then z and $-1/\bar{z}$ correspond to antipodal points on the sphere.

From Equation (6.6.15) we conclude that $\mathbf{X} = (X_1, X_2, X_3)$ can be thought of as a mapping from the projective plane into $\mathbb{R}^3$. Therefore $\mathbf{X}(M)$ is a Möbius strip in $\mathbb{R}^3$.

It is pertinent to mention that $\mathbf{X}$ is not regular at every point; in fact, we have $\sum |\alpha_k|^2 = 0$ at the points $\pm\,1$ and $\pm\,i$, which are the only singular points of $\mathbf{X}$. Since these represent two pairs of antipodal points, we then have that $\mathbf{X}$, considered as a mapping on the projective plane, is singular at exactly two points.

Therefore, $\mathbf{X}$ restricted to $\mathbb{C} - \{0, 1, -1, i, -i\}$ represents a minimal surface which is not complete and whose image is a Mobius strip minus two points. Since $g(z) = z$, its Gauss mapping covers each point of $S^2(1)$ just once, with exception of six points. Therefore, its total curvature is -4π. The Henneberg's surface is shown in Figure 9.3d.

6.6.6 The Costa's surface

The Weierstrass-Enneper representation for Costa's surface can be achieved by choosing the relevant parameters as follows:

$$g = \frac{2\sqrt{2\,\pi}\,\wp(\frac{1}{2})}{\wp'(\zeta)} \qquad \text{and} \qquad \eta = \wp(\zeta)\,d\zeta.$$

The Weierstrass elliptic functions (or Weierstrass $\wp$-functions), are elliptic functions which, unlike the Jacobi elliptic functions, have a second-order pole at $z = 0$. To specify $\wp(z)$ completely, its half-periods ω_1 and ω_2 or elliptic invariants g_2 and g_3 must be specified. These two cases are denoted by $\wp(z \mid \omega_1, \omega_2)$ and $\wp(z; g_2, g_3)$ respectively.

The domain used was the unit square torus with points at 0, $\frac{1}{2}$, $\frac{i}{2}$ removed.

With the Weierstrass-Enneper representation, Ferguson and Gray [14, 17], discovered that the Costa surface can be represented parametrically and explicitly by:

$$\begin{aligned}
X_1 &= \frac{1}{2}\,\mathtt{Re}\left\{-\zeta\,(u+iv) + \pi\,u + \frac{\pi^2}{4\,e_1} + \frac{\pi}{2\,e_1}\left[\zeta\left(u+iv-\frac{1}{2}\right) - \zeta\left(u+iv-\frac{1}{2}\,i\right)\right]\right\},\\
X_2 &= \frac{1}{2}\,\mathtt{Re}\left\{-i\zeta\,(u+iv) + \pi\,v + \frac{\pi^2}{4\,e_1} - \frac{\pi}{2\,e_1}\left[i\,\zeta\left(u+iv-\frac{1}{2}\right) - i\,\zeta\left(u+iv-\frac{1}{2}\,i\right)\right]\right\},
\end{aligned}$$

$$X_3 = \frac{1}{4}\,\sqrt{2\,\pi}\,\mathtt{ln}\left|\frac{\wp(u+iv)-e_1}{\wp(u+iv)+e_1}\right|, \tag{6.6.16}$$

where $\zeta(z)$ is the Weierstrass zeta function, $\wp(g_2, g_3; z)$ is the Weierstrass elliptic function with $(g_2, g_3) = (189.072772\ldots, 0)$, the invariants correspond to the half-periods $1/2$ and $i/2$, and first root:

$$e_1 = \wp\left(\frac{1}{2}; 0, g_3\right) = \wp\left(\frac{1}{2}\,\middle|\,\frac{1}{2}, \frac{1}{2}\,i\right) \approx 6.87519,$$

where $\wp(z; g_2, g_3) = \wp(z \mid \omega_1, \omega_2)$ is the Weierstrass elliptic function.

The Costa surface (torus) [9] is probably the most celebrated complete minimal surface in $\mathbb{R}^3$ since the classical examples from the nineteenth century. It is a thrice punctured torus with total curvature -12π, two catenoidal ends and one planar middle end.

Costa [9] demonstrated the existence of this surface but only proved its embeddedness outside a ball in $\mathbb{R}^3$. Hoffman and Meeks [18] demonstrated its global embeddedness, thereby disproving the widely accepted erroneous conjecture that the only complete, embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type were the plane, catenoid and helicoid. The Costa surface contains two horizontal straight lines that intersect orthogonally, and has vertical planes of symmetry bisecting the right angles made by these two line. The complete Costa's surface is shown in Figure 9.5a. Whereas its depiction as been the union of a catenoid with a plane through its waist circle, with two pairs of "tunnels" reminiscent of Scherk's Second Surface passing between the plane and the catenoid ends is shown in Figures 9.5b, 9.5c and 9.5d.

Chapter 7

Minimal surfaces – area minimizing or not area minimizing

It is a well known fact that minimal surfaces do not always minimize area. This has been demonstrated by forming a catenoid between two rings (boundaries) whose axis are offset by a certain amount from one another. In this chapter of the survey an approach due to Schwartz which tells us when we have minimal non-area-minimizing surfaces [29] is presented. Before showing this, it would be instructive to use one simple example to illustrate where areas compared using the usual tools of vector analysis.

Let $z = f(x, y)$ be a function of two variables which satisfy the minimal surface equation given in Equation (4.0.6). Let the graph of f be parameterized by $\mathbf{X}(x, y) = (x, y, f(x, y))$ over a closed disc for example, in the xy-plane with boundary curve $\mathbf{C}$. Take any other function $z = g(x, y)$ on the disc with $g|_{\mathbf{C}} = f|_{\mathbf{C}}$, and suppose for convenience that the union of the two graphs along the common boundary $\mathbf{C}$ forms a surface with no self-intersections. That is, if $\mathbf{Y}(x, y) = (x, y, g(x, y))$ is a parametrization for the graph of g, then the surface $S = \mathbf{X} \cup_{\mathbf{C}} \mathbf{Y}$ encloses a volume in $\mathbb{R}^3$. Let $\mathbf{N}$ be the unit normal vector field for $\mathbf{X}$

$$\mathbf{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Although we normally think of $\mathbf{N}$ as a vector field on $\mathbf{X}$, there is no reason not to consider it at every point in $\mathbb{R}^3$ above the disc where f is defined. Notice that $\mathbf{N}$ does not depend on z. Without loss of generality, we assume the graph of f lies above that of g, then $\mathbf{N}$ is the outer normal of $\mathbf{X}$ and is pointed inside M on $\mathbf{Y}$.

Let us compute the divergence of the vector field $\mathbf{N}$

$$\langle \nabla\,,\, \mathbf{N}\rangle = -\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1+f_x^2+f_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1+f_x^2+f_y^2}}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\sqrt{1+f_x^2+f_y^2}}\right),$$

$$= -\frac{f_{xx}(1+f_x^2+f_y^2) - f_x^2 f_{xx} - f_x f_y f_{xy}}{(1+f_x^2+f_y^2)^{3/2}} - \frac{f_{yy}(1+f_x^2+f_y^2) - f_y f_x f_{xy} - f_y^2 f_{yy}}{(1+f_x^2+f_y^2)^{3/2}},$$

$$\Longrightarrow \quad -\frac{f_{xx}(1+f_y^2) - 2\, f_x f_y f_{xy} + f_{yy}(1+f_x^2)}{(1+f_x^2+f_y^2)^{3/2}} = 0, \tag{7.0.1}$$

since f satisfies the minimal surface equation. Recall that the divergence theorem states that

$$\iiint_\Omega \langle \nabla\,,\, V\rangle\, d\Omega = \iint_M \langle V\,,\, \mathbf{N}\rangle\,,$$

for a surface M enclosing a volume Ω. The divergence vector $V = (V_1, V_2, V_3)$ is defined as $\langle \nabla\,,\, V\rangle = \partial V_1/\partial x_1 + \partial V_2/\partial x_2 + \partial V_3/\partial x_3$ and $\mathbf{N}$ is the unit outward normal of M. Using the directions of the vector field mentioned above and the downward unit normal $\mathcal{N}$ of $\mathbf{Y}$ (corresponding to the outer normal on M), the divergence theorem together with the computation of $\langle \nabla\,,\, \mathbf{N}\rangle$ gives,

$$\begin{aligned}
0 &= \iiint \langle \nabla\,,\, \mathbf{N}\rangle\, dx\, dy\, dz = \iint_S \langle \mathbf{N}\,,\, dA_S\rangle = \iint_{\mathbf{X}} \langle \mathbf{N}\,,\, dA_{\mathbf{X}}\rangle + \iint_{\mathbb{Y}} \langle \mathbf{N}\,,\, dA_{\mathbb{Y}}\rangle,\\
&= \iint_{\mathbf{X}} \langle \mathbf{N}\,,\, \mathbf{N}\rangle \sqrt{1+f_x^2+f_y^2}\, dx\, dy + \iint_{\mathbb{Y}} \langle \mathbf{N}\,,\, \mathcal{N}\rangle \sqrt{1+g_x^2+g_y^2}\, dx\, dy,\\
&= \iint_{\mathbf{X}} \sqrt{1+f_x^2+f_y^2}\, dx\, dy + \iint_{\mathbb{Y}} \cos(\theta) \sqrt{1+g_x^2+g_y^2}\, dx\, dy,
\end{aligned}$$

where θ is the angle between the unit normal vectors $\mathbf{N}$ and $\mathcal{N}$.

Recall that

$$\begin{aligned}
\text{Area}\,(\mathbf{X}) &= \left|\iint \sqrt{1+f_x^2+f_y^2}\, dx\, dy\right| = \left|\iint \cos(\theta) \sqrt{1+g_x^2+g_y^2}\, dx\, dy\right|,\\
&\le \iint |\cos(\theta)| \sqrt{1+g_x^2+g_y^2}\, dx\, dy \;\le\; \iint \sqrt{1+g_x^2+g_y^2}\, dx\, dy, \qquad (7.0.2)\\
&= \text{Area}\,(\mathbf{Y}).
\end{aligned}$$

Therefore satisfying the minimal surface equation can guarantee minimum surface area when compared to the right other surface.

We have done enough groundwork to prepare us for the general case. Let us consider a minimal surface M bounded by a curve $\mathbf{C}$ and by Lemma 8.1.2, let the parametrization $\mathbf{X}(u, v)$ of M be isothermal. Therefore, all the consequences of having isothermal parameters follow; that is,

$E = G$, $F = 0$, $e = -g$, $K = -(e^2 + f^2)/E^2$, $\langle \mathbf{X}_u , \mathbf{X}_u \rangle = \langle \mathbf{X}_v , \mathbf{X}_v \rangle = (e^2 + f^2)/E$, $\langle \mathbf{X}_u , \mathbf{X}_v \rangle = 0$.

Furthermore, from the results of Equation (6.5.2) and Theorem 6.5.2, we have

$$(7.0.3) K = -\frac{4}{|F|^2(1 + u^2 + v^2)^4} \qquad \text{and} \qquad E = |F|^2\,[1 + u^2 + v^2]^2 = G.$$

Now let us take a variation $\mathbb{Y}^t(u, v) = \mathbf{X}(u, v) + t\,V(u, v)$, where $V(u, v) = \rho(u, v)\mathbf{N}(u, v)$ is a normal vector field on M of varying length $\rho(u, v)$ with $\rho(\mathbf{C}) = 0$. $\rho(\mathbf{C}) = 0$ means that ρ vanishes on the curve in the uv-plane which is carried to $\mathbf{C}$ by $\mathbf{X}(u, v)$. Note that $V(u, v) = \rho(u, v)\mathbf{N}(u, v)$ is an infinitesimal displacement of $\mathbf{X}(u, v)$ by a factor of t in the direction of $\mathbf{N}(u, v)$.

Next we need to calculate:

$$\begin{aligned} \mathbf{Y}_u^t &= \mathbf{X}_u + t\,V_u, \\ \mathbf{Y}_v^t &= \mathbf{X}_v + t\,V_v \quad \text{and,} \\ \mathbf{Y}_u^t \times \mathbf{Y}_v^t &= \mathbf{X}_u \times \mathbf{X}_v + t[\mathbf{X}_u \times V_v + V_u \times \mathbf{X}_v] + t^2(V_u \times V_v). \end{aligned}$$

In what follows, let

$$\begin{aligned} \mathcal{W} &= \langle (\mathbf{X}_u \times \mathbf{X}_v)\,,\,(\mathbf{X}_u \times V_v + V_u \times \mathbf{X}_v) \rangle, \\ \mathcal{W}\mathcal{W} &= 2\langle (\mathbf{X}_u \times \mathbf{X}_v)\,,\,(V_u \times V_v) \rangle + \langle (\mathbf{X}_u \times V_v)\,,\,(\mathbf{X}_u \times V_v) \rangle \\ &+ 2\langle (\mathbf{X}_u \times V_v)\,,\,(V_u \times \mathbf{X}_v) \rangle + \langle (V_u \times \mathbf{X}_v)\,,\,(V_u \times \mathbf{X}_v) \rangle, \\ S &= \sqrt{|\mathbf{X}_u \times \mathbf{X}_v|^2 + 2t\,\mathcal{W} + t^2\,\mathcal{W}\mathcal{W} + \mathcal{O}(t^3)}, \end{aligned}$$

where $\mathcal{O}(t^3)$ denote terms involving powers of t greater than or equal to three. With these notations, we see that the surface area $A(t)$ is given by

$$(7.0.4) \qquad A(t) = \int\!\!\int |\mathbf{Y}_u^t \times \mathbf{Y}_v^t|\,du\,dv = \int\!\!\int S\,du\,dv.$$

Let us assume that at $t = 0$ (corresponding to M) is a critical point for the area;

Therefore, $A'(0) = 0$.

Recall the Lagrange identity

$$\langle (\mathbf{v} \times \mathbf{w})\,,\,(\mathbf{a} \times \mathbf{b}) \rangle = (\langle \mathbf{v}\,,\,\mathbf{a} \rangle)(\langle \mathbf{w}\,,\,\mathbf{b} \rangle) - (\langle \mathbf{v}\,,\,\mathbf{b} \rangle)(\langle \mathbf{w}\,,\,\mathbf{a} \rangle).$$

Lemma 7.0.1. *If M is minimal, then $\mathcal{W} = 0$.*

Proof. Notice that

$$\begin{aligned} \mathcal{W} &= \langle (\mathbf{X}_u \times \mathbf{X}_v)\,,\,\langle (\mathbf{X}_u \times V_v) \rangle + \langle (\mathbf{X}_u \times \mathbf{X}_v)\,,\,(V_u \times \mathbf{X}_v) \rangle, \\ &= E\,\langle \mathbf{X}_v\,,\,V_v \rangle + E\,\langle \mathbf{X}_u\,,\,V_u \rangle, \end{aligned}$$

by Lagrange identity and using isothermal parameters.

Now, $V_u = \rho_u \mathbf{N} + \rho \mathbf{N}_u$ and $V_v = \rho_v \mathbf{N} + \rho \mathbf{N}_v$,

so we have

$$\begin{aligned}\langle \mathbf{X}_u\,,\,V_u\rangle &= \langle \mathbf{X}_u\,,\,(\rho_u \mathbf{N} + \rho \mathbf{N}_u)\rangle,\\ &= \langle \rho\,\mathbf{X}_u\,,\,\mathbf{N}_u\rangle = -\,\rho\,e, \quad \text{since } \langle \mathbf{X}_u\,,\,\mathbf{N}\rangle = 0,\end{aligned}$$

by the definition of e. Similarly, $\langle \mathbf{X}_u\,,\,V_v\rangle = -\,\rho\,f$, $\langle \mathbf{X}_v\,,\,V_u\rangle = -\,\rho\,f$, and $\langle \mathbf{X}_v\,,\,V_v\rangle = -\,\rho\,g = \rho\,e$.

Then we have; $\mathcal{W} = \rho\,E\,e - \rho\,E\,e = 0$. $\square$

Next, let us consider $\mathcal{WW}$. Employing Lagrange identity and the calculations stated above and recalling that:

$$\begin{aligned}V_u &= \rho_u \mathbf{N} + \rho \mathbf{N}_u,\\ V_v &= \rho_v \mathbf{N} + \rho \mathbf{N}_v,\\ \langle \mathbf{N}_u\,,\,\mathbf{N}_u\rangle &= \langle \mathbf{N}_v\,,\,\mathbf{N}_v\rangle = \frac{e^2+f^2}{E},\\ \langle \mathbf{N}_u\,,\,\mathbf{N}_v\rangle &= 0.\end{aligned}$$

From the foregoing, we have,

$$\begin{aligned}\mathcal{WW} &= 2\langle (\mathbf{X}_u \times \mathbf{X}_v)\,,\,(V_u \times V_v)\rangle + \langle (\mathbf{X}_u \times V_v)\,,\,(\mathbf{X}_u \times V_v)\rangle\\ &+ 2\langle (\mathbf{X}_u \times V_v)\,,\,(V_u \times \mathbf{X}_v)\rangle + \langle (V_u \times \mathbf{X}_v)\,,\,(V_u \times \mathbf{X}_v)\rangle,\\[1ex] &= 2[(\langle \mathbf{X}_u\,,\,V_u\rangle)(\langle \mathbf{X}_v\,,\,V_v\rangle) - (\langle \mathbf{X}_u\,,\,V_v\rangle)(\langle \mathbf{X}_v\,,\,V_u\rangle)]\\ &+ (\langle \mathbf{X}_u\,,\,\mathbf{X}_u\rangle)(\langle V_v\,,\,V_v\rangle) - (\langle \mathbf{X}_u\,,\,V_v\rangle)(\langle V_v\,,\,\mathbf{X}_u\rangle) + 2[(\langle \mathbf{X}_u\,,\,V_u\rangle)(\langle V_v\,,\,\mathbf{X}_v\rangle)\\ &\quad - (\langle \mathbf{X}_u\,,\,\mathbf{X}_v\rangle)(\langle V_v\,,\,V_u\rangle)] + (\langle V_u\,,\,V_u\rangle)(\langle \mathbf{X}_v\,,\,\mathbf{X}_v\rangle) - (\langle V_u\,,\,\mathbf{X}_v\rangle)(\langle \mathbf{X}_v\,,\,V_u\rangle),\\[1ex] &= 2[(-\,\rho e)(\rho e) - (-\,\rho f)(-\,\rho f)] + E(\rho_v^2 + \rho^2(e^2+f^2)/E)\\ &- \rho^2 f^2 + 2(-\,\rho e)(\rho e) + E(\rho_u^2 + \rho^2(e^2+f^2)/E) - \rho^2 f^2,\\[1ex] &= 4\rho^2[-\,e^2 - f^2] + \rho^2[e^2+f^2+e^2+f^2] + E(\rho_v^2+\rho_u^2),\\ &= 2\rho^2[-\,e^2 - f^2] + E(\rho_v^2+\rho_u^2) = 2\rho^2E^2K + E(\rho_u^2+\rho_v^2).\end{aligned}$$

Therefore, $\mathcal{S}$ simplifies to

$$\mathcal{S} = \sqrt{|\mathbf{X}_u \times \mathbf{X}_v|^2 + t^2\,\mathcal{WW} + \mathcal{O}(t^3)}.$$

On differentiating $A(t) = \int\int\ \mathcal{S}\,du\,dv$, we have

$$A'(t) = \int\int\ \frac{t(2\rho^2E^2K + E(\rho_u^2+\rho_v^2)) + \mathcal{O}(t^2)}{\mathcal{S}}\,du\,dv,$$

and

$$A''(t) = \int\int \frac{(2\rho^2 E^2 K + E(\rho_u^2 + \rho_v^2) + \mathcal{O}(t))\mathcal{S} - t(2\rho^2 E^2 K + E(\rho_u^2 + \rho_v^2))\,\mathcal{S}'}{\mathcal{S}^2}\,du\,dv.$$

Notice that $\mathcal{S}' = \frac{t\,\mathcal{W}\mathcal{W} + \mathcal{O}(t^2)}{\mathcal{S}}$ by Lemma 7.0.1.

Therefore, $\mathcal{S}|_{t=0} = |\mathbf{X}_u \times \mathbf{X}_v| = \sqrt{E^2} = E$ and $\mathcal{S}'|_{t=0} = 0$.

Hence

$$\begin{aligned} A''(0) &= \int\int \left[\frac{2\rho^2 E^2 K + E(\rho_u^2 + \rho_v^2)}{E}\right] du\,dv, \\ &= \int\int \left[2\rho^2 EK + \rho_u^2 + \rho_v^2\right] du\,dv. \end{aligned} \tag{7.0.5}$$

On substituting for K and E from Equations 7.0.3 into Equation (7.0.5), we get

$$\begin{aligned} A''(0) &= \int\int \left[\frac{(2\rho^2|F|^2(1+u^2+v^2)^2)(-4)}{|F|^2(1+u^2+v^2)^4} + \rho_u^2 + \rho_v^2\right] du\,dv, \\ &= \int\int \left[\frac{-8\rho^2}{(1+u^2+v^2)^2} + \rho_u^2 + \rho_v^2\right] du\,dv. \end{aligned} \tag{7.0.6}$$

The evaluation of $A''(0)$ is carried out over a region $\mathcal{R}$ in the $u\,v$-parameter plane, and the expression for $A''(0)$ does not depend on the Weierstrass-Enneper representation, but only on $\mathcal{R}$ and the choice of ρ on that region. As in calculus, if $t = 0$ gives a minimum, then the second derivative must be non-negative there. Therefore, if we can find a function ρ (with $\rho(\mathbf{C}) = 0$) defined on $\mathcal{R}$ such that $A''(0) < 0$, then the minimal surface M cannot have minimum area among surfaces spanning $\mathbf{C}$. Therefore, we have,

Theorem 7.0.2. Schwartz. *Let M be a minimal surface spanning a curve $\mathbf{C}$. if the closed unit disk $\mathcal{D} = \{(u, v)|u^2 + v^2 \leq 1\}$ is contained in the interior of $\mathcal{R}$, then a function ρ exists for which $A''(0) < 0$. Hence, M does not have minimum area among surfaces spanning $\mathbf{C}$.*

Proof. Let $\mathcal{D} = \{(u, v, r)\,|\,u^2 + v^2 \leq r^2\}$ be a domain bounded by the cone $r = \sqrt{u^2 + v^2}$. Let us define a function on $\mathcal{D}$, which is given by

$$\rho(u, v, r) = \frac{u^2 + v^2 - r^2}{u^2 + v^2 + r^2}, \tag{7.0.7}$$

and let us consider

$$\mathcal{A}(r) = \int\int_{\mathcal{D}(r)} \left[\frac{-8\rho^2}{(1+u^2+v^2)^2} + \rho_u^2 + \rho_v^2\right] du\,dv,$$

where $\mathcal{D}(r) = \{(u, v)\,|\,u^2 + v^2 < r^2\}$ is the open r-disk. This is $A''(0)$ when we are in $\mathcal{R}$, so our aim is to show that the choice of ρ above leads to $\mathcal{A}(r) < 0$ for certain values of r. Let us split the integral into two pieces and consider the integral of the last two terms of the integrand. If we let

$$P = -\,\rho\rho_v \quad \text{and} \quad Q = \rho\rho_u,$$

and apply Green's theorem, we get

$$\int_{u^2+v^2=r^2} -\,\rho\rho_v\,du + -\,\rho\rho_u\,dv = \int\!\!\int_{\mathcal{D}(r)} (\rho_u^2+\rho_v^2)\,du\,dv + \int\!\!\int_{\mathcal{D}(r)} \rho(\rho_{uu}+\rho_{vv})\,du\,dv.$$

The left hand side is zero because $\rho(u,\,v,\,r) = 0$ whenever $u^2+v^2=r^2$, therefore

$$\int\!\!\int_{\mathcal{D}(r)} (\rho_u^2+\rho_v^2)\,du\,dv = -\int\!\!\int_{\mathcal{D}(r)} \rho\,\Delta\rho\,du\,dv,$$

where $\Delta\,\rho = \rho_{uu}+\rho_{vv}$ is the Laplacian of ρ. Therefore

$$\mathcal{A}(r) = \int\!\!\int_{\mathcal{D}(r)} -\,\rho\,\left[\frac{-8\rho^2}{(1+u^2+v^2)^2} + \Delta\rho\right]\,du\,dv.$$

It is easy to see that for

$$\text{(7.0.8)}\qquad \rho = \rho(u,\,v,\,1) = \frac{u^2+v^2-1}{u^2+v^2+1} \quad\Longrightarrow\quad \frac{8\rho^2}{(1+u^2+v^2)^2} + \Delta\rho = 0,$$

since $u^2+v^2=1$. We see from Equation (7.0.8) that $\mathcal{A}(1)=0$. Recall that $r=1$

corresponds to the unit disk, we would like $\mathcal{A}(1)<0$ for r's slightly larger than 1. In order to show that this is true, we need to look at $\mathcal{A}'(1)$. In order to proceed, let us make the following change of variables

$$s = \frac{u}{r} \quad\text{and}\quad t = \frac{v}{r} \quad\text{with}\quad du = r ds \quad\text{and}\quad dv = r dt.$$

Then

$$\text{(7.0.9)}\qquad \mathcal{A}(r) \;=\; -\int\!\!\int \rho\left[\frac{8\rho^2}{(1+r^2(s^2+t^2))^2} + \Delta\rho\right] r^2\,ds\,dt,$$

$$\text{(7.0.10)}\qquad \rho(s,\,t,\,r) \;=\; \frac{r^2s^2+r^2t^2-r^2}{r^2s^2+r^2t^2+r^2} = \frac{s^2+t^2-1}{s^2+t^2+1} = \rho(s,\,t).$$

Therefore, in $(s,\,t)$ coordinates, ρ does not depend on r. Also, $\rho_u = \rho_s/r$ and $\rho_{uu} = \rho_{ss}/r^2$ by chain rule, and similarly for v. Hence, $\Delta_{u,\,v}\,(\rho) = \Delta_{s,\,t}\,\rho/r^2$. Now if we multiply by r^2 and replace the $u\,v$-Laplace operator in Equation (7.0.9) we get

$$\text{(7.0.11)}\qquad \mathcal{A}(r) \;=\; -\int\!\!\int \rho\left[\frac{8r^2\rho^2}{(1+r^2(s^2+t^2))^2} + \Delta_{s,\,t}\,\rho\right] r^2\,ds\,dt.$$

The fact that ρ and $\Delta_{s,\,t}\,\rho$ are independent of r allows us to easily take the derivative of $\mathcal{A}(r)$ with respect to r to obtain

$$\mathcal{A}'(r) = -\int\!\!\int \frac{16\rho^2 r(1+r^2(s^2+t^2))^2 - 32\rho^2r^3(1+r^2(s^2+t^2))(s^2+t^2)}{(1+r^2(s^2+t^2))^4}\,ds\,dt.$$

At $r = 1$, $s^2 + t^2 < 1$, therefore if we replace ρ by its definition in terms of s and t, we get:

$$\begin{aligned}
\mathcal{A}'(1) &= -\int\!\!\int \frac{16\rho^2(1+s^2+t^2) - 32\rho^2(s^2+t^2)}{(1+s^2+t^2)^3}\,ds\,dt,\\
&= -\int\!\!\int_{s^2+t^2<1} \frac{16\rho^2(1+s^2+t^2-2s^2-2t^2)}{(1+s^2+t^2)^3}\,ds\,dt,\\
&= -\int\!\!\int_{s^2+t^2<1} \frac{16(s^2+t^2-1)^2(1-s^2-t^2)}{(1+s^2+t^2)^5}\,ds\,dt,\\
&= -\int\!\!\int_{s^2+t^2<1} \frac{-16(s^2+t^2-1)^2(s^2+t^2-1)}{(1+s^2+t^2)^5}\,ds\,dt,\\
&= 16\int\!\!\int_{s^2+t^2<1} \frac{(s^2+t^2-1)^3}{(1+s^2+t^2)^5}\,ds\,dt. \qquad (7.0.12)
\end{aligned}$$

The numerator of Equation (7.0.12) is always negative, since ($s^2 + t^2 < 1$), therefore $\mathcal{A}'(1) < 0$. Since this means that $\mathcal{A}(r)$ is decreasing at $r = 1$ and we have seen previously that $\mathcal{A}(1) = 0$, it must be the case that $\mathcal{A}(r) < 0$ for $1 < r < \bar{r}$ (some $\bar{r}$).

Now suppose that the unit disk $\mathcal{D}$ is contained in the interior of the parameter domain $\mathcal{R}$. Then there is an r such that $1 < r < \bar{r}$ and $\{(u, v)\,|\,u^2 + v^2 \le r^2\} \subseteq \mathcal{R}$. Let us define

$$\rho(u, v) = \begin{cases} \rho(u, v, r) & \text{for } u^2 + v^2 \le r^2, \\ 0 & \text{for } u^2 + v^2 > r^2. \end{cases}$$

Note that $\rho|_{\partial\mathcal{R}} = 0$ and from the foregoing, $\mathcal{A}''(0) < 0$. Note that the partial derivatives ρ_u and ρ_v are not continuous on the boundary circle $\{(u, v)\,|\,u^2 + v^2 = r^2\}$, but may be *rounded-off* suitably there while keeping $\mathcal{A}''(0) < 0$. Therefore, the minimal surface given by the Weierstrass-Enneper representation which spans $\mathcal{R}$ is not area minimizing. This proofs the theorem. □

7.1 Geometric interpretation of Theorem 7.0.2

The point of emphasis here is that the Weierstrass-Enneper representation is not just a bunch of equations, rather it is a way to directly obtain geometrical information about the minimal surface from its representation. The parameters u and v are identified with the real and imaginary parts of the complex variable τ the moment M is described with the Weierstrass-Enneper II representation. Recall that τ itself is identified with the function g of the Weierstrass-Enneper I representation, and g is the Gauss map followed by stereographic projection (Theorem 5.3.1). Therefore since the stereographic projection from the North pole projects the lower hemisphere of S^2 onto the unit disk $\mathcal{D}$, $\mathcal{R}$ contains $\mathcal{D}$ in its interior precisely when the image of the Gauss map of M contains the lower hemisphere of S^2 in its interior.

There is nothing special about implementing stereographic projection from the North pole. Because it can be done from any point on the sphere, we have the following geometrical version of the Schwartz theorem [29].

Theorem 7.1.1. *Let M be a minimal surface spanning a curve* $\mathbf{C}$. *If the image of the Gauss map of M contains a hemisphere of S^2 in its interior, then M does not have minimum area among surfaces spanning* $\mathbf{C}$.

Enneper's surface $\mathbf{X}(u, v) = (u - u^3/3 + uv^2, -v + v^3/3 - vu^2, u^2 - v^2)$ has no self-intersections for $u^2 + v^2 < 3$, and its Gauss map (restricted to the disk $u^2 + v^2 < 3$) covers more than a hemisphere of S^2 see Figure 9.4d. By Theorem 7.1.1, it can therefore be inferred that Enneper's surface does not minimize area among all surfaces spanning the curve $\mathbf{C}$ given by applying the parametrization $\mathbf{X}$ to the parameter circle $u^2 + v^2 = R^2$, where $1 < \mathcal{R} < \sqrt{3}$. However, by the theorem of Douglas and Rado, there exists a least area [29] and hence a minimal surface spanning $\mathbf{C}$. This imply that there are at least two minimal surfaces spanning $\mathbf{C}$. This point goes to highlight the issues surrounding the uniqueness of the solution to Plateau's problem. The following theorem which we shall state without proof, provides some relief regarding the uniqueness of the solution to Plateau's problem.

Theorem 7.1.2. Ruchert [29] *For $0 < r \leq 1$, Enneper's surface is the unique solution to Plateau's problem for the curve given by applying the Enneper parametrization to a parameter circle of radius r.*

Although the result of Douglas and Rado shows that there are at least two minimal surfaces spanning the curve $\mathbf{C}$, only one of these is known explicitly, the Enneper's surface. Presently, there are no example(s) of two or more explicit minimal surfaces spanning a given curve, however, for two curves there is, Figures 9.1c and 9.1d.

General Comment It is pertinent to state again that the name *minimal surface* is sort of misleading. It is really not the surface with least area for the given boundary. The variation formula shown above only shows that $A'(0) = 0$, where $A(t)$ is the area of any surface for the given boundary and $A(0)$ is the surface area of the original surface. The only inference we can make from this is either $A(0)$ is maximum or minimum. It could be the case that a minimal surface has the largest area among those surfaces with the given boundary. The minimal surface with least area is called the *stable minimal surfaces.*

Chapter 8

S.N. Bernstein's and R. Osserman's theorems for minimal surfaces

The famous Bernstein's theorem (1915) is global in nature and it was one of the earliest theorems concerning solutions of the minimal surface equations (MSEs). The theorem states that only the trivial affine solutions $f(x,\, y) \equiv ax+by+c$ where (a, b, c are constants) can satisfy the minimal surface equation over the entire $\mathbb{R}^2$-plane. Bernstein proved this as a special consequence of his remarkable *geometric theorem* which states that any bounded $\mathcal{C}^2$ function on $\mathbb{R}^2$ with graph $z = f(x,\, y)$ having non-negative Gauss curvature (that is, with f satisfying $f_{xx}\, f_{yy} - f^2{}_{xy} \leq 0$) must necessarily have Gauss curvature identically zero. In particular bounded solutions $f \in \mathcal{C}^2\,(\mathbb{R}^2)$ of any equation of the form $af_{xx} + 2bf_{xy} + cf_{yy} = 0$, where $a\zeta^2 + 2b\zeta\eta + c\eta^2 \geq \zeta^2 + \eta^2$, $sup\{|a|,\, |b|,\, |c|\} < \infty$, and where the coefficients a, b, c may depend on f and its derivatives, are necessarily constant [42].

8.1 Proof of S. N. Bernstein's theorem for minimal graphs

The main objective of this chapter is to present a proof of Bernstein's theorem for minimal surfaces. The proof presented here was adapted from [31]. The proof of the theorem, will be carried out with the help of several independent lemmas, which are also very important in their own rights.

Theorem 8.1.1. Bernstein. *Suppose M is a surface given by the graph of a smooth function defined on all of $\mathbb{R}^2$. If M is minimal, then it is a plane.*

Remark 1: The theorem in fact holds under weaker regularity hypothesis but for simplicity we will assume that the graph is smooth.

Remark 2: Let M=graph f where $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth. Thus $M = \{(x,\, y,\, f(x,\, y)) : (x,\, y) \in \mathbb{R}^2\}$. In Section 4, we derived the conditions necessary for the surface M

to be minimal (that is, the mean curvature of M is zero everywhere). The minimal surface equation is recast below for easy reference.

$$(1+f_y^2)\, f_{xx} - 2 f_x\, f_y\, f_{xy} + (1+f_x^2)\, f_{yy} = 0. \tag{8.1.1}$$

In order to prove Bernstein's theorem, we have to prove that if f is a (smooth) solution of Equation (8.1.1) on $\mathbb{R}^2$, then $f(x,\, y) = ax + by + c$ for some constants a, b and c.

The following series of independent lemmas which are also very important will help accomplish this.

Lemma 8.1.2. Existence of isothermal parameters on a minimal surface. *Suppose M is a regular minimal surface in $\mathbb{R}^3$. (Not necessarily a graph.) If $p \in M$ is any point, there exists a parametrization $\mathbf{X} : D \to M$ with $p \in \mathbf{X}(D)$ such that $\mathbf{X}$ is isothermal; that is, $\langle \mathbf{X}_u\, ,\, \mathbf{X}_u \rangle \;=\; \langle \mathbf{X}_v\, ,\, \mathbf{X}_v \rangle$ and $\langle \mathbf{X}_u\, ,\, \mathbf{X}_v \rangle \;=\; 0$.*

Remark 8.1.1. The lemma is actually true without the minimality assumption. The proof for arbitrary regular surfaces however is much more involved than the proof given below for minimal surfaces. As is clear from the proof below, in case M is minimal, we can exploit the structure of the minimal surface Equation (8.1.1) to construct an isothermal parametrization.

Proof. Recall that in a neighborhood of any point of a regular surface, we may express the surface as a graph of a function defined over a domain in $x\,y$, $y\,z$ or $z\,x$ plane. Let us assume without loss of generality that we can do this over the $x\,y$ plane for some neighborhood of p. Thus, we can find a neighborhood $V \subseteq M$ with $p \in V$, an open ball $B_r(\mathbf{a}) \subset \mathbb{R}^2$ and a smooth function $f : B_r(\mathbf{a}) \to \mathbb{R}$ such that $V = \text{graph } f$.

Since M is minimal, f must be a solution of Equation (8.1.1) in $B_r(\mathbf{a})$. Now recall the First Fundamental Form with the usual notation, namely;

$$E(x,\, y) = 1 + f_x^2(x,\, y), \quad F(x,\, y) = f_x(x,\, y)\, f_y(x,\, y),$$

and

$$G(x,\, y) = 1 + f_y^2(x,\, y) \quad \text{for} \quad (x,\, y) \in B_r(\mathbf{a}).$$

Let

$$W = \sqrt{EG - F^2} \;=\; \sqrt{1 + f_x^2 + f_y^2}.$$

We claim that since f solves Equation (8.1.1), the following identities must hold in $B_r(\mathbf{a})$:

$$\frac{\partial}{\partial x}\left(\frac{F}{W}\right) = \frac{\partial}{\partial y}\left(\frac{E}{W}\right), \tag{8.1.2}$$

and

$$\frac{\partial}{\partial x}\left(\frac{G}{W}\right) = \frac{\partial}{\partial y}\left(\frac{F}{W}\right). \tag{8.1.3}$$

These can easily be checked by direct computation. For example to check Equation (8.1.2), we proceed as follows:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{f_x\, f_y}{\sqrt{1+f_x^2+f_y^2}}\right) &- \frac{\partial}{\partial y}\left(\frac{1+f^2{}_x}{\sqrt{1+f_x^2+f_y^2}}\right), \\
&= \frac{(1+f_x^2+f_y^2)(f_{xx}f_y+f_xf_{xy})-f_xf_y(f_xf_{xx}+f_yf_{yx})}{(1+f_x^2+f_y^2)^{3/2}}, \\
&- \frac{2(1+f_x^2+f_y^2)f_xf_{xy}-(1+f_x^2)(f_xf_{xy}+f_yf_{yy})}{(1+f_x^2+f_y^2)^{3/2}}, \\
&= \frac{f_y((1+f_y^2)f_{xx}-2f_xf_yf_{xy}+(1+f_x^2)f_{yy})}{(1+f_x^2+f_y^2)^{3/2}} = 0,
\end{aligned}
$$

by Equation (8.1.1).

The identity 8.1.3 can easily be checked by a similar computation.

Now recall the general fact that if, $\phi(x,\, y) = (\phi_1(x,\, y),\, \phi_2(x,\, y))$ is a smooth vector field in a ball $B_r(\mathbf{a})$ satisfying $\partial\phi_2/\partial x = \partial\phi_1/\partial y$, then ϕ is the gradient of a smooth function; that is, there exists a smooth function ψ on $B_r(\mathbf{a})$ such that $D\,\psi = (\phi_1,\, \phi_2)$.

In order to see this, simply define $\psi(x,\, y)$ to be the line integral of ϕ along the line segment from $\mathbf{a}$ to $(x,\, y) \in B_r(\mathbf{a})$.

Thus

$$\psi(x,\, y) = \int_0^1 x\phi_1((1-t)\mathbf{a}+t\mathbf{x}) + y\phi_2((1-t)\mathbf{a}+t\mathbf{x})\, dt.$$

Hence, the identity 8.1.2 implies that there exists a smooth function P on $B_r(\mathbf{a})$ such that

$$P_x = \frac{E}{W} \quad \text{and} \quad P_y = \frac{F}{W}. \tag{8.1.4}$$

Similarly, the identity 8.1.3 implies the existence of a smooth function Q on $B_r(\mathbf{a})$ with

$$Q_x = \frac{F}{W} \quad \text{and} \quad Q_y = \frac{G}{W}. \tag{8.1.5}$$

Set

$$u(x,\, y) = x + P(x,\, y) \quad \text{and} \quad v(x,\, y) = y + Q(x,\, y). \tag{8.1.6}$$

Then

$$\frac{\partial(u,\,v)}{\partial(x,\,y)} = (1+P_x)(1+Q_y) - P_yQ_x,$$
$$= \left(1+\frac{E}{W}\right)\left((1+\frac{G}{W}\right) - \frac{F^2}{W^2},$$
$$= 2+\frac{e+G}{W} > 0. \tag{8.1.7}$$

Hence by the inverse function theorem, the transformation $\eta : (x,\,y) \mapsto (u(x,\,y),\,v(x,\,y))$ given by Equation (8.1.6) has a smooth local inverse $\mu = \eta^{-1} : (u,\,v) \mapsto (x(u,\,v),\,y(u,\,v))$ with $d\mu(u,\,v) = (d\eta|_{\mu(u,\,v)})^{-1}$. Computing the inverse matrix on the right hand side of this, we see that

$$x_u = \frac{W+G}{2W+E+G},$$
$$x_v = y_u = -\frac{F}{2W+E+G},$$

and

$$y_v = \frac{W+E}{2W+E+G}. \tag{8.1.8}$$

Thus, there exists a neighborhood D of $\ = \eta(\mathbf{a})$ such that the map $\mathbf{X} : D \to M$, given by $\mathbf{X}(u,\,v) = (x(u,\,v),\,y(u,\,v),\,z(u,\,v))$, where $z(u,\,v) = f\circ\mu(u,\,v) = f(x(u,\,v),\,y(u,\,v))$ is a reparametrization of M near p.

We claim that $\mathbf{X}$ is isothermal. To see this we compute as follows:

First we have:

$$z_u = f_x x_u + f_y y_u = \frac{f_x(W+G) - f_yF}{2W+E+G}, \tag{8.1.9}$$

and,

$$z_v = f_x x_v + f_y y_v = \frac{f_y(W+E) - f_xF}{2W+E+G}. \tag{8.1.10}$$

Hence

$$\langle \mathbf{X}_u, \mathbf{X}_u\rangle = x_u^2 + y_u^2 + z_u^2,$$
$$= \frac{E(W+G)^2 + GF^2 - 2(W+G)F^2}{(2W+E+G)^2},$$
$$= \frac{W^2}{2W+E+G}. \tag{8.1.11}$$

$$\langle \mathbf{X}_v\,,\,\mathbf{X}_v\rangle = x_v^2 + y_v^2 + z_v^2,$$
$$= \frac{G(W+E)^2 + EF^2 - 2(W+E)F^2}{(2W+E+G)^2},$$
$$= \frac{W^2}{2W+E+G}, \tag{8.1.12}$$

and,

$$\langle \mathbf{X}_u\,,\,\mathbf{X}_v\rangle \;=\; x_u x_v + y_u y_v + z_u z_v,$$

$$= \frac{-F(W+G) - F(W+E) + F(W+G)(W+E) + F^3 - f_x^2 F(W+G) - f_y^2 F(W+E)}{(2W+E+G)^2},$$

$$= \frac{-EF(W+G) - GF(W+E) + F(W+G)(W+E) + F^3}{(2W+E+G)^2} = 0. \tag{8.1.13}$$

□

Lemma 8.1.3. A fact of linear algebra. *Suppose $A = (a_{ij})_{1\leq i,\, j\leq 2}$ is a (real) symmetric matrix. Then $a_{11} > 0$ and $\det A > 0$ if and only if*

$$\sum_{i,\,j=1}^{2} a_{ij}\,\xi_i\,\xi_j > 0,$$

for every vector $\xi = (\xi_1,\,\xi_2) \neq 0$. (A is called positive definite if these equivalent conditions hold.)

Proof. Note first that if $\xi = (\xi_1,\,\xi_2)$, then $\sum_{i,\,j=1}^{2} a_{ij}\,\xi_i\,\xi_j = \langle A\,\xi,\,\xi\rangle$. Since A is symmetric, $\langle A\,\xi,\,\zeta\rangle = \langle \xi,\,A\zeta\rangle$ for any two vectors $\xi,\,\zeta \in \mathbb{R}^2$. That is, A as a linear map is self-adjoint. Recall the spectral theorem - that $\lambda_1 \equiv max_{\{|\xi|=1\}}\langle A\xi,\,\xi\rangle$ and $\lambda_2 \equiv min_{\{|\xi|=1\}}\langle A\xi,\,\xi\rangle$ are the two eigenvalues of A. Thus $\sum_{i,\,j=1}^{2} a_{ij}\,\xi_i\,\xi_j > 0$ for each $\xi \neq 0$ if and only if $\lambda_2 > 0$. Since $\det A = \lambda_1\lambda_2$ and $a_{11} = \langle Ae_1,\,e_1\rangle$, it follows that $\lambda_2 > 0$ if and only if $\det A > 0$ and $a_{11} > 0$. □

Lemma 8.1.4. *Suppose f is a smooth function satisfying the minimal surface Equation (8.1.1) on the entire $\mathbb{R}^2$ plane. Then the map $\eta : (x,\,y) \mapsto (u(x,\,y),\,v(x,\,y))$ defined by Equation (8.1.6) is a diffeomorphism of the xy plane onto the uv plane.*

Proof. By Equation (8.1.7) we have that $\frac{\partial(u,\,v)}{\partial(x,\,y)} > 1$ everywhere, and hence η is a local diffeomorphism. We need to show that η is one-to-one and onto.

Since $P_y = Q_x$ by Equations 8.1.4 and 8.1.5, there exists a smooth function $R : \mathbb{R}^2 \to \mathbb{R}$ such that $P = R_x$ and $Q = R_y$. (See the paragraph preceding (Equations 8.1.4.)) Note then that $R_{xx} = P_x = \frac{E}{W} > 0$ and $R_{xx}R_{yy} - R_{xy}^2 = P_xQ_y - P_y^2 = \frac{EG-F^2}{W^2} \equiv 1$, and hence by Lemma 8.1.3, we have that, writing $(x,\,y) = (\nu_1,\,\nu_2)$ for notational convenience,

$$\sum_{i,\,j=1}^{2} R_{\nu_i\nu_j}\xi_i\,\xi_j > 0, \tag{8.1.14}$$

for every vector $(\xi_1,\,\xi_2) \neq 0$. Now let $(x_1,\,x_2)$ and $(y_1,\,y_2)$ be any two distinct points of $\mathbb{R}^2$, and let $\alpha(t) = R((1-t)x_1 + ty_1,\,(1-t)x_2 + ty_2)$ for $t \in [0, 1]$.

Then

$$\text{(8.1.15)} \quad \alpha'(t) = (y_1 - x_1)R_x(\mathbf{r}_t) + (y_2 - x_2)R_y(\mathbf{r}_t) = (y_1 - x_1)P(\mathbf{r}_t) + (y_2 - x_2)Q(\mathbf{r}_t),$$

where $\mathbf{r}_t = ((1-t)x_1 + ty_1, (1-t)x_2 + ty_2)$,

and

$$\alpha''(t) = \sum_{i,\,j=1}^{2} R_{\nu_i \nu_j}(\mathbf{r}_t)(y_i - x_i)(y_j - x_j).$$

Since $\alpha''(t) > 0$ by Equation (8.1.14), it follows that $\alpha'(1) > \alpha'(0)$. Thus, we have shown that for any two distinct points $\mathbf{p} = (x_1,\, x_2)$ and $\mathbf{q} = (y_1,\, y_2)$ satisfies:

$$\text{(8.1.16)} \qquad (y_1 - x_1)(P(\mathbf{q}) - P(\mathbf{p})) + (y_2 - x_2)(Q(\mathbf{q}) - Q(\mathbf{p})) > 0,$$

or, equivalently, that

$$\text{(8.1.17)} \qquad \langle (\eta(\mathbf{q}) - \eta(\mathbf{p}))\,,\, (\mathbf{q} - \mathbf{p}) \rangle > |\mathbf{q} - \mathbf{p}|^2.$$

Since $\langle \mathbf{a}\,,\, \mathbf{b} \rangle \le |\mathbf{a}||\mathbf{b}|$ this implies that

$$\text{(8.1.18)} \qquad \mathbf{q} - \mathbf{p} < |\eta(\mathbf{q}) - \eta(\mathbf{p})|,$$

for any two distinct points $\mathbf{p}$, $\mathbf{q} \in \mathbb{R}^2$. It follows immediately from this that η is one-to-one. Hence, since η is a local diffeomorphism, it follows that η is a global diffeomorphism of $\mathbb{R}^2$ (thought of as the xy plane) onto some open subset $D = \eta(\mathbb{R}^2)$ of $\mathbb{R}^2$ (thought of as the uv plane.)

It remains only to show that $D = \mathbb{R}^2$. If this is not true, then there exists a point $\mathbf{b} \in \overline{D} \setminus D$. Let $\mathbf{b}_j$ be a sequence of points in D converging to $\mathbf{b}$. Let $\mathbf{a}_j = \eta^{-1}(\mathbf{b}_j)$. Taking $\mathbf{p} = 0$ and $\mathbf{q} = \mathbf{a}_j$ in Equation (8.1.18), it follows that $|\mathbf{a}_j| < |\mathbf{b}_j - \eta(0)| \le 2|\mathbf{b}| + |\eta(0)|$ for all sufficiently large j. Thus the sequence $\{\mathbf{a}_j\}$ is bounded. Hence it must have a limit point $\mathbf{a}$. But this implies by continuity of η that $\mathbf{b} = \eta(\mathbf{a})$, contradicting the fact that $\mathbf{b} \notin \eta(\mathbf{R}^2)$. $\square$

8.1.1 Alternative proof of existence of isothermal parameters on minimal surfaces

We shall present an alternative proof of the existence of isothermal parameters on minimal surfaces. The proof given below is adapted from Dogamo [2]. Let $\mathbb{C}$ denote the complex plane, which is, as usual, identified with $\mathbb{R}^2$ by setting $\zeta = u + iv,\ \ \zeta \in \mathbb{C},\ (u,\, v) \in \mathbb{R}^2$. We recall that a function $f : D \subset \mathbb{C} \to \mathbb{C}$ is analytic when, by writing

$$f(\zeta) = f_1(u,\, v) + i f_2(u,\, v),$$

the real functions f_1 and f_2 have continuous partial derivatives of first order which satisfy the so-called Cauchy-Riemann equations

$$\frac{\partial f_1}{\partial u} = \frac{\partial f_2}{\partial v} \qquad \text{and} \qquad \frac{\partial f_1}{\partial v} = -\frac{\partial f_2}{\partial u}.$$

Now let $\mathbf{X} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a regular parameterized surface and define complex functions ϕ_1, ϕ_2, ϕ_3 by

$$\phi_1(\zeta) = \frac{\partial x}{\partial u} - i\,\frac{\partial x}{\partial v}, \qquad \phi_2(\zeta) = \frac{\partial y}{\partial u} - i\,\frac{\partial y}{\partial v} \quad \text{and} \quad \phi_3(\zeta) = \frac{\partial z}{\partial u} - i\,\frac{\partial z}{\partial v},$$

where x, y and z are the components functions of $\mathbf{X}$. The three complex functions ϕ_1, ϕ_2 and ϕ_3 can be written in the following shorthand form thus

$$\phi_k(\zeta) = \frac{\partial x_k}{\partial u} - i\,\frac{\partial x_k}{\partial v}; \qquad k = 1, 2, 3; \qquad \zeta = u + i\,v. \tag{8.1.19}$$

Lemma 8.1.5. *Let $\mathbf{X}(u, v) = (x(u, v), y(u, v), z(u, v) : D \subseteq \mathbb{R}^2 \to M)$ be a parametrization of a regular surface. Define complex valued functions $\phi_1 = x_u - ix_v$, $\phi_2 = y_u - iy_v$, $\phi_3 = z_u - iz_v$. Then $\mathbf{X}$ is isothermal if and only if $\phi_1^2 + \phi_2^2 + \phi_3^2 \equiv 0$. Furthermore, if $\mathbf{X}$ is isothermal, then $\mathbf{X}(D)$ is minimal if and only if ϕ_1, ϕ_2, ϕ_3 are analytic. Additionally, if u, v are isothermal parameters, then $\mathbf{X}$ is regular if and only if $|\phi_1^2| + |\phi_2^2| + |\phi_3^2| \neq 0$.*

Proof. From Equation (8.1.19), we have

$$\begin{aligned}
\sum_{k=1}^{3} \phi_k^2(\zeta) &= \frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 - \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2 - 2\,i\sum_{k=1}^{3}\frac{\partial x_k}{\partial u}\,\frac{\partial x_k}{\partial v}\right],\\
&= \frac{1}{4}\sum_{k=1}^{3}\left[\left|\frac{\partial \mathbf{X}}{\partial u}\right|^2 - \left|\frac{\partial \mathbf{X}}{\partial v}\right|^2 - 2i\,\frac{\partial \mathbf{X}}{\partial u}\cdot\frac{\partial \mathbf{X}}{\partial v}\right],\\
&= \frac{1}{4}\sum_{k=1}^{3}\left[\langle \mathbf{X}_u\,,\,\mathbf{X}_u\rangle - \langle \mathbf{X}_v\,,\,\mathbf{X}_v\rangle - 2i\,\langle \mathbf{X}_u\,,\,\mathbf{X}_v\rangle\right],
\end{aligned}$$

$$\sum_{k=1}^{3} \phi_k^2(\zeta) = \frac{1}{4}\left[\phi_1^2 + \phi_2^2 + \phi_3^2\right] = \frac{1}{4}\left[E - G - 2i\,F\right] = 0. \tag{8.1.20}$$

This proves the first part of Lemma 8.1.5, since $E = G$, and $F = 0$ for isothermal surfaces.

Moreover, $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$ if and only if:

$$\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial u}\right) = -\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial v}\left(\frac{\partial x}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\right),$$

$$\frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u}\right) = -\frac{\partial}{\partial v}\left(\frac{\partial y}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial v}\right),$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = -\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) \quad \text{and} \quad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right).$$

Since the three complex functions ϕ_1, ϕ_2 and ϕ_3 satisfy the Cauchy-Riemann conditions, we conclude that $\mathbf{X}_{uu} + \mathbf{X}_{vv} = 0$ if and only if ϕ_1, ϕ_2 and ϕ_3 are analytic. This proves the second portion of Lemma 8.1.5.

We now prove the last portion of Lemma 8.1.5, which states that if, u, v are isothermal parameters, then $\mathbf{X}$ is regular if and only if $|\phi_1^2| + |\phi_2^2| + |\phi_3^2| \neq 0$.

$$\begin{aligned}
\sum_{k=1}^{3} |\phi_k\,(\zeta)|^2 &= \frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u} - i\,\frac{\partial x_k}{\partial v}\right)\left(\frac{\partial x_k}{\partial u} + i\,\frac{\partial x_k}{\partial v}\right)\right], \\
&= \frac{1}{4}\left[\sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial u}\right)^2 + \sum_{k=1}^{3}\left(\frac{\partial x_k}{\partial v}\right)^2\right], \\
&= \frac{1}{4}\sum_{k=1}^{3}\left[\left|\frac{\partial \mathbf{X}}{\partial u}\right|^2 + \left|\frac{\partial \mathbf{X}}{\partial v}\right|^2\right] = \frac{1}{4}\sum_{k=1}^{3}\left[\langle \mathbf{X}_u\,,\,\mathbf{X}_u\rangle + \langle \mathbf{X}_v\,,\,\mathbf{X}_v\rangle\right],
\end{aligned}$$

$$\sum_{k=1}^{3} |\phi_k\,(\zeta)|^2 = \frac{1}{4}\,\left[|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2\right] = \frac{E+G}{4} = \frac{E}{2} \neq 0. \tag{8.1.21}$$

□

This concludes the proof of Lemma 8.1.5.

Proof. Having done all the preliminary work, we now give a proof of Bernstein's global theorem.

Suppose f is a smooth function satisfying the minimal surface Equation (8.1.1) on all of $\mathbb{R}^2$. By Lemma 8.1.4, the map η defined by Equation (8.1.6) is a diffeomorphism of the xy-plane onto the entire uv-plane. Hence by the argument in the proof of Lemma 8.1.2, the map $\mathbf{X}(u,\,v) = (x(u,\,v),\,y(u,\,v),\,z(u,\,v)) : \mathbb{R}^2 \to M =$ graph f,where $z(u,\,v) = f \circ \eta^{-1}(u,\,v)$ is a global isothermal parametrization of M.

Identifying the $u\,v$-plane with the complex plane $\mathbb{C}$, define complex valued functions ϕ_1, ϕ_2 and ϕ_3 by:

$$\phi_1(u + iv) = x_u - ix_v,$$

$$\phi_2(u + iv) = y_u - iy_v,$$

$$\phi_3(u + iv) = z_u - iz_v.$$

Since $\mathbf{X}$ is an isothermal parametrization of a minimal surface, by Lemma 8.1.5, the functions ϕ_1, ϕ_2 and ϕ_3 are analytic.

Now observe that

$$\operatorname{Im}(\overline{\phi_1}\,\phi_2) = -x_u y_v + x_v y_u = -\frac{\partial(x,y)}{\partial(u,\,v)} < 0, \tag{8.1.22}$$

where the last inequality follows from Equation (8.1.7). Hence $\phi_1 \neq 0$, so that the function $\phi = \phi_2/\phi_1$ is analytic. Furthermore,

$$\operatorname{Im}\phi = \frac{1}{|\phi_1|^2}\operatorname{Im}(\overline{\phi_1}\,\phi_2) < 0. \tag{8.1.23}$$

Thus ϕ is a complex analytic function on the entire complex plane with imaginary part bounded above, and hence by Corollary 5.7.3, $\phi = a + ib$ for some constants $a,\, b \in \mathbb{R}$. Note then that $b \neq 0$ by Equation (8.1.23).

Therefore,

$$y_u - iy_v = (a + ib)(x_u - ix_v),$$

or, equivalently,

$$y_u = ax_u + bx_v \qquad \text{and} \qquad y_v = ax_v - bx_u, \tag{8.1.24}$$

where $b \neq 0$. Now introduce the linear transformation:

$$s = bx, \qquad t = ax - y. \tag{8.1.25}$$

Note that since $b \neq 0$, this transformation is non-singular. By Equation (8.1.24),

$$s_u = bx_u, \quad s_v = bx_v, \quad t_u = ax_u - y_u = -\,bx_v \quad \text{and} \quad t_v = ax_v - y_v = bx_u,$$

so that

$$\frac{\partial(s,t)}{\partial(u,\,v)} = s_u t_v - s_v t_u = b^2(x_u^2 + x_v^2) \neq 0.$$

We claim that $(s,\,t)$ are also isothermal parameters on M. That is, that

$$\mathbf{Y}(s,\,t) = \mathbf{X}(u(s,\,t),\,v(s,\,t)),$$

is also a (global) isothermal parametrization of M. This can be verified directly by computing $\mathbf{Y}_s$, $\mathbf{Y}_t$ in terms of $\mathbf{X}_u$, $\mathbf{X}_v$, u_s, u_t, v_s and v_t, and checking that

$$\langle \mathbf{Y}_s\,,\,\mathbf{Y}_s\rangle \;=\; \langle \mathbf{Y}_t\,,\;\mathbf{Y}_t\rangle \quad\text{and}\quad \langle \mathbf{Y}_s\,,\,\mathbf{Y}_t\rangle \;=\; 0.$$

In summary, we have shown the following. Given that $M \equiv \{(x,\,y,\,f(x,\,y)) : (x,\,y) \in \mathbb{R}^2\}$ is a minimal graph over the entire plane, there exists a linear change of variables $(s,\,t) \mapsto (x,\,y)$ (given by Equation (8.1.25)) so that $(s,\,t)$ are isothermal parameters on M.

Now if we let

$$\tilde{\phi_1} = x_s - ix_t, \qquad \tilde{\phi_2} = y_s - iy_t \qquad \text{and} \qquad \tilde{\phi_3} = z_s - iz_t,$$

where $z(s,\,t) = f(x(s,\,t),\,y(s,\,t))$, then, since $(x(s,\,t),\,y(s,\,t), z(s,\,t))$ is isothermal, we have by Lemma 8.1.5 that

$$(\tilde{\phi_1})^2 + (\tilde{\phi_2})^2 + (\tilde{\phi_3})^2 \equiv 0. \tag{8.1.26}$$

However, by Equation (8.1.25), $\tilde{\phi_1}$ and $\tilde{\phi_2}$ are constant. Hence by Equation (8.1.26), $\tilde{\phi_3}$ is also constant. This means that z is a linear function of s and t, and hence f is a linear function of x and y. □

8.1.2 An alternative proof of Bernstein's theorem, Chern [5]

In this chapter, we present an alternative proof of Bernstein's Theorem which was given by Chern [5]. Recall that the classical Bernstein's theorem states that

Theorem 8.1.6. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function whose graph is a minimal surface, then f is linear; that is, M is planar.*

Remark 8.1.2. The theorem stated above fails in the higher codimension case in that M does not have to be planar. For example, given any entire function $w = w(z) : \mathbb{C} \to \mathbb{C},\;\; M = \{(z,\,w(z)) : z \in \mathbb{C}\} \subset \mathbb{C}^2 = \mathbb{R}^4$ is a minimal surface. Where $\mathbb{C}$ is the complex plane.

We now give an alternative proof of Bernstein theorem due to Chern [5].

Suppose $M = \{(u,\,v,\,f(u,\,v) \in \mathbb{R}^3;\;\; (u,\,v) \in \mathbb{R}^2\}$ is a minimal surface. Given local isothermal coordinates $(x,\,y)$ on M, we have

$$\Delta = -\frac{1}{h}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad K = \frac{1}{2}\,\Delta \log_e h, \tag{8.1.27}$$

where $ds^2 = h(dx^2 + dy^2)$ is the induced metric on $M \subset \mathbb{R}^3$, Δ is the Laplace-Beltrami operator of (M, ds^2), and K is the Gaussian curvature of (M, ds^2).

Let us put

$$J = \sqrt{EG - F^2} = \sqrt{1 + f_u^2 + f_v^2}.$$

Then a standard calculation gives

$$K = \Delta \log_e \frac{J}{J+1}. \tag{8.1.28}$$

On M introduce a new metric $d\tilde{s}^2 = \left(\frac{J+1}{J}\right)^2 ds^2$. $d\tilde{s}^2$ is conformally equivalent to ds^2 and its Gaussian curvature is identically zero. Since $1 \leq \left(\frac{J+1}{J}\right) \leq 2$ and ds^2 are complete, we see that $d\tilde{s}^2$ is also complete. It follows that $(M,\, d\tilde{s}^2)$ is isometric to the uv-plane with the flat metric $du^2 + dv^2$. Since $K \leq 0$, (that is, for minimal surfaces), we obtain

$$-\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \log_e \left(\frac{J}{J+1}\right) \leq 0. \tag{8.1.29}$$

Note that the Laplace-Beltrami operator of $(M,\, ds^2)$ is a multiple of $\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)$ since ds^2 and $d\tilde{s}^2$ are multiples of each other. Equation (8.1.29) says the function $\log_e \left(\frac{J}{J+1}\right)$ is a subharmonic negative function on the uv-plane. The parabolicity of uv-plane then implies that $\log_e \left(\frac{J}{J+1}\right)$ must be a constant. Hence by Equation (8.1.28), $K \equiv 0$ and therefore M is planar. This concludes Chern's method of proving Bernstein's theorem.

8.2 Osserman's theorem

One of the fundamental problems in the classical theory of minimal surfaces is to obtain Liouville type results for complete minimal surfaces. Robert Osserman started the systematic development of this theory, and in 1961 he proved that the Gauss map of a complete non-planar orientable minimal surface misses at most a set of logarithmic capacity zero. In 1981 F. Xavier [44] proved that the image of the Gauss map covers the sphere except at most six values, and finally in 1988 H. Fujimoto [16] discovered the best theorem yet, and proved that the number of points omitted by the Gauss map is at most four. An interesting extension of Fujimoto's results was proved in 1990 by X. Mo and R. Osserman [27]. They showed that if the Gauss map of a complete orientable minimal surface takes on five distinct values only a finite number of times, then the surface has finite total curvature.

There are many kinds of complete orientable minimal surfaces whose Gauss map omits four points of the sphere. Among these examples is the classical Scherk's doubly periodic. Under the additional hypothesis of finite total curvature, Osserman [32] proved that the number of exceptional values is at most three.

All the works cited above had two common objectives and they are, to find a generalization to Bernstein's theorem which had an essential flaw and to devise a method of characterizing minimal surfaces. Bernstein had stated that every solution of the minimal surface equation defined and regular for all finite values of $z = f(x,\, y)$ is a linear function. Notice that Bernstein's theorem had no constraints whatsoever imposed on such linear solutions. However, for harmonic functions the same conclusion would hold only under the additional assumption that $\varphi_x^2 + \varphi_y^2$ is uniformly bounded. The flaw in Bernstein's theorem was first solved in 1950 and ever since more proofs which removed the flaw have been provided.

In the following chapter, we shall provide three (Osserman [31], Xavier [44] and Fujimoto [16]) of the most recent generalizations to Bernstein's theorem. For its groundbreaking value, we shall give the proof of the classical theorem by Osserman [31]. The proof is adapted from [25] and Osserman [30]. On completion of the proof of Osserman, we will state, F. Xavier's [44] and H. Fujimoto's theorems [16] without proofs. We will then conclude this chapter by quoting several theorems on complete surface of finite total curvature in $\mathbb{R}^3$.

Theorem 8.2.1. Robert Osserman [31]. *Let M be a complete regular minimal surface in $\mathbb{R}^3$. Then either M is a plane, or else the set E omitted by the image of M under the Gauss map has logarithmic capacity zero.*

In order to present Osserman's proof, we shall restate Bernstein's theorem in a slightly different way, as follows

Theorem 8.2.2. *If M is a complete minimal surface in $\mathbb{R}^3$ whose normals form an acute angle with a fixed direction, then M is a plane.*

This reformulation of Bernstein's theorem was generalized by Osserman [31] and later by Xavier [44]. Osserman's result as well as Xavier were based on Weierstrass representation and a theorem in complex analysis known as Koebe Uniformization theorem. The definition of this theorem that will be useful to our proof is the following

Theorem 8.2.3. Koebe Uniformization Theorem. *Let M be a Riemann surface endowed with a complete metric ds^2. Let Δ represent any one of the following surfaces: the unit sphere or the complex plane $\mathbb{C}$, or the unit disk D. Then, there exists a locally invertible conformal mapping F from Δ onto M.*

If $\mathbf{X} : M \to \mathbb{R}^3$ is a complete minimal surface, then if we use Theorem 8.2.3, we could consider the mapping $\mathbf{X} \circ F : \Delta \to \mathbb{R}^3$, which would still remain a complete minimal surface in the induced metric.

Since minimal surfaces in $\mathbb{R}^3$ can not be compact (because $K \leq 0$), Δ can never be a sphere. Therefore, we eliminate the case of Δ being a sphere and restrict ourselves to the cases where $\Delta = \mathbb{C}$ or $\Delta = D$. Therefore, for $\mathbf{X} \circ F$, we will have a global Weierstrass representation on $\mathbb{C}$ or D.

Lemma 8.2.4. Osserman [31]. *Let $\mathbf{X}(\zeta) : D \to \mathbb{R}^3$ define a generalized minimal surface M, where D is the entire ζ-plane. Then either $\mathbf{X}(\zeta)$ lies on a plane, or else the normals to M take on all directions with at most two exceptions.*

Proof. To the surface M we associate the function $g(\zeta)$ which fails to be defined only if $\phi_1 = i\,\phi_2$, $\phi_3 = 0$. But in this case X_3 is constant and the surface lies in a plane. Otherwise $g(\zeta)$ is meromorphic in the entire ζ-plane, and by Picard's theorem it either takes on all values with at most two exceptions, or else is constant. But by Equation (6.3.1) the same alternative applies to the normal $\mathbf{N}$, and in the latter case M lies on a plane. $\square$

Lemma 8.2.5. *Let $f(z)$ be an analytic function in the unit disk D which has at most a finite number of zeros. Then there exists a divergent path C in D such that*

$$\int_C |f(z)|\,|dz| < \infty. \tag{8.2.1}$$

Proof. Suppose first that $f(z) \neq 0$ in D. Define $w = F(z) = \int_0^z f(\zeta)d\zeta$.

Then $F(z)$ maps $|z| < 1$ onto a Riemann surface which has no branch points. If we let $z = G(w)$ be that branch of the inverse function satisfying $G(0) = 0$, then since $|G(w)| < 1$, there is a largest disk $|w| < R < \infty$ in which $G(w)$ is defined. There must then be a point w_0 with $|w_0| = R$ such that $G(w)$ cannot be extended to a neighborhood of w_0.

Let ℓ be the line segment $w = tw_0,\ 0 \leq t < 1$, and let C be the image of ℓ under $G(w)$. Then C must be a divergent path, since otherwise there would be a sequence $t_n \to 1$, such that the corresponding sequence of points z_n on C would converge to a point z_0 in D. But then $F(z_0) = w_0$, and since $F'(z_0) = f(z_0) \neq 0$, the function $G(w)$ would be extendable to a neighborhood of w_0. Thus the path C is divergent, and we have:

$$\int_C |f(z)||dz| = \int_0^1 |f(z)| \left|\frac{dz}{dt}\right| dt = R < \infty.$$

This proves the lemma if $f(z)$ has no zeros. But if it has a finite number of zeros, say of order ν_k, at the points z_k, then the function

$$f_1(z) = f(z) \prod \left(\frac{1 - \bar{z}_k z}{z - z_k}\right)^{\nu_k},$$

never vanishes, and by the above argument there exists a divergent path C such that $\int_C |f_1(z)||dz| < \infty$. But $|f(z)| < |f_1(z)|$ throughout D, and Equation (8.2.1) follows. □

One application of this, is the proof of a generalization of Theorem 8.2.2, which can be restated as follows

Theorem 8.2.6. Osserman [30]. *If* $\mathbf{X} : M \to \mathbb{R}^3$ *is a complete minimal surface, then the image of the Gauss map* $\mathbf{N}$ *of* $\mathbf{X}$ *is dense in* $S^2(1)$*, unless* $\mathbf{X}(M)$ *is a plane.*

Proof. From the foregoing, we may assume that $\mathbf{X} : \Delta \to \mathbb{R}^3$, where $\Delta = \mathbb{C}$ or $\Delta = D$. Suppose the normals to M are not everywhere dense, then there exist an open set on the unit sphere which is not intersected by the image of M under the Gauss map. That is, if $\mathbf{N}(\Delta)$ is not dense in $S^2(1)$ there exists $P \in S^2(1)$ and g, $1 > g > 0$, such that:

$$\langle \mathbf{N}\,, \mathbf{P} \rangle \ \leq\ 1 - g. \tag{8.2.2}$$

By rotating coordinates in $\mathbb{R}^3$, we may assume that $P = (0,\, 0,\, 1)$.

Consider the Weierstrass representation of the surface. By using the expression for $\mathbf{N}$ given in Equation (6.3.1), we have,

$$\langle \mathbf{N}\,, \mathbf{P}\rangle = \left\langle \left(\frac{2\,\mathrm{Re}\,(g)}{1+|g|^2}, \frac{2\,\mathrm{Im}\,(g)}{1+|g|^2}, \frac{1-|g|^2}{1+|g|^2}\right), (0,\, 0,\, 1)\right\rangle = \frac{1-|g|^2}{|g|^2+1} \ \leq\ 1 - g. \tag{8.2.3}$$

From Equation (8.2.3), we conclude that $|g(z)| \leq A < \infty$. If $\Delta = \mathbb{C}$, by Corollary 5.7.3 (Liouville's theorem), g is constant. Hence, the Gauss map $\mathbf{N}$ is constant and $\mathbf{X}(M)$ describes a plane. The same is true for the universal covering surface $\hat{M}$ of M.

If $\Delta = D$, we can only conclude that g has no poles and by Lemma 6.0.6, f has no zeros. If α is a curve in D starting at the origin and going to the boundary of D, we have

$$(8.2.4)\, Length\,(\alpha) = \int_\alpha ds = \frac{1}{2}\int_\alpha |f|(1+|g|^2)\,|dz| \;<\; \frac{1+A^2}{2}\int_\alpha |f|\,|dz|.$$

We will now show that there exists α of finite length, thus contradicting the hypothesis of completeness of M. Let us proceed by considering the function

$$\mathbf{X} = \int_0^z f(\zeta)\,d\zeta. \tag{8.2.5}$$

Since $f \neq 0$ in D, $w : D \to \mathbb{C}$ is locally invertible. Let $z = G(w)$ be a local inverse function for w in a small disk around $w = 0$. Let R be the radius of the largest disk where G can be defined. Clearly, $R < \infty$ since the image of G lies in the disk $|z| < 1$. Therefore, there exists a point w_0 with $|w_0| = R$ such that G can not be extended to a neighborhood of w_0.

Set $\ell = \{t\,w_0 : 0 \le t < 1\}$ and $\alpha = G(\ell)$. The curve α so defined is divergent. If it were not, there would exist a sequence $\{t_n\}$ converging to 1 such that the corresponding sequence $\{z_n\}$ along α would converge to a point z_0 in D. By continuity, $G(w_0) = z_0$. But then, since the function w is invertible at z_0, G would be extendable to a neighborhood of w_0, a contraction. Therefore, α is divergent and we have

$$(8.2.6)\int_\alpha |f(z)|\,|dz| = \int_0^1 |f(z)|\left|\frac{dz}{dt}\right| dt = \int_0^1 \left|\frac{dw}{dt}\right| dt = \int_\ell |dw| = R < \infty.$$

Therefore, length$(\alpha) < \infty$ and so, with the induced metric, D is not complete, a contradiction. But if $f(z)$ has a finite number of zeros, say ν_k, at the points z_k, then the function

$$f_1(z) = f(z)\prod\left(\frac{1-\overline{z}_k\,z}{z-z_k}\right)^{\nu_k}, \tag{8.2.7}$$

never vanishes, and by the argument above there exist a divergent path C such that $\int_C |f_1(z)|\,|dz| < \infty$. But $|f(z)| < |f_1(z)|$ throughout D, and Equation (8.2.1) follows.

Thus D is the entire plane, and since normals omit more than two points, it follows from Lemma 8.2.4 that $\hat{M}$ must lie on a plane. The same is true of M, and since it is complete, M must be the whole plane. This completes the proof of Theorem 8.2.6. □

8.2.1 Remarks

Osserman's proof of Bernstein's theorem provided the answers to Nirenberg's conjectures. These conjectures are

1. A complete simply-connected minimal surface M must be a plane if the Gauss map, $G(M)$ of the surface omits a neighborhood of a point of a unit sphere

2. A complete simply-connected minimal surface M must be a plane if the Gauss map, $G(M)$ of the surface omits three (3) points of a unit sphere

From Theorem 8.2.6, under the condition of Nirenberg first conjecture, the surface M can not be conformally a disk or a sphere, hence it must be the conformed image of the whole plane, hence g is constant and M is a plane.

The next issue we need to address is, does a similar argument work for Nirenberg's second conjecture? The categoric answer is no and in fact, the conjecture is false, Osserman [33]. Using the Weierstrass-Enneper representations with suitable choice of functions f and g, we can give explicit examples of complete minimal surfaces whose normals omit not only three (3) but four (4) values.

After a lot of research work, the issue (Nirenberg's second conjecture) was finally resolved in 1988 by H. Fujimoto [16], who proved that if a complete minimal surface in $\mathbb{R}^3$ has a Gauss map that omits more than four values, then M is a plane. Fujimoto reasoned that even though the function, g will not be bounded, it may be possible to get some kind of growth condition that will guarantee the convergence of $\int_C |f|(1 + |g|^2)\,|dz|$ along some path C. In deed, by using the metric on the plane minus a finite number of points, with $K \equiv -1$, and applying Ahlfor's Lemma, Fujimoto obtained an upper bound on $|g'(z)|$ that yield the result [33]. A sharp form of Fujimoto's theorem was obtained by Xiaokang Mo [27].

We will now state F. Xavier [44], H. Fujimoto [16] and X. Mo and R. Osserman [27] theorems; but we will not be proving them, since the omission of their proofs would not detract from the fundamental objective of this project.

Theorem 8.2.7. F. Xavier [44]. *Let M be a complete regular minimal surface in $\mathbb{R}^3$, and $\mathbf{N} : M \to S^2(1)$ be its Gauss mapping. If $\mathbf{N}(M)$ omits seven or more points, then M is a plane.*

Theorem 8.2.8. H. Fujimoto [16]. *The plane is the only complete orientable minimal surface in $\mathbb{R}^3$ whose Gauss map omits at least five points of the sphere.*

Theorem 8.2.9. X. Mo and R. Osserman [27]. *If M is a complete minimal surface, not a plane, and if $G(M)$ omits four (4) values, then $G(M)$ not only takes on every other value, but does so infinitely often.*

Corollary 8.2.10. *Under the hypothesis, M has infinite total curvature, $\int_M |K|\,dA = \infty$.*

By Gauss definition of K, $\int_M |K|\,dA$ is precisely the area of the image under the Gauss map G. If the Gauss map covers the whole sphere infinitely often, omitting 4 values, then the area of the image is clearly infinite.The corollary was known much earlier.

8.2.2 Surfaces of finite total curvature

A great deal was known about the characteristics of surfaces of finite total curvature. Some of the characteristics are stated below

Theorem 8.2.11. *Let M be a complete surface of finite total curvature in $\mathbb{R}^3$. Then*

1. Osserman [31] proved that $G(M)$ can omit at most 3 values in the unit sphere unless M is a plane. However, there is an example of a complete surface of finite total curvature in $\mathbb{R}^3$ whose Gauss map, $G(M)$ omit 2 points Ru [45]. In view of this, the following question, namely, "under what circumstance(s) can the Gauss map, $G(M)$ of a complete surface of finite total curvature in $\mathbb{R}^3$ omit at most 2 points?", still remain unanswered. Researchers in this field have found out that it is a very difficult issue to resolve. It is the desire of Geometers to either prove it or find a complete surface of finite total curvature in $\mathbb{R}^3$ whose Gauss map, $G(M)$ omit 3 points.

2. M is conformally equivalent to a compact surface $\overline{M}$ with a finite number of points, $p_1, \ldots, p_k$ removed.

3. The normals to M approach a limit at each "end" corresponding to each p_j; in fact the function g extends to be meromorphic on $\overline{M}$.

4. $\int_M |K|\, dA = 4\pi m, \qquad m = 0,\, 1,\, 3,\, \ldots$

5. $m = 0$, if and only if, M is a plane. $m = 1$, if and only if M is a catenoid or a certain simply-connected surface called the Enneper's surface.

6. With χ = Euler characteristic of M, and k=number of ends, $\int_M |K|\, dA \leq 2\pi\,(\chi - k)$.

The last inequality is interesting in view of a theorem by Cohn-Voss that $\int_M |K|\, dA \leq 2\pi\chi$ for all surfaces. According to item number (4) equality never holds in Cohn-Voss's inequality if M is a minimal surface.

Finally, we may want to ask whether among complete minimal surfaces of finite total curvature, if there were embedded ones, that is, those without self-intersections other than the plane and the catenoid. Jorge and Meeks [22] showed that if there were, they would have to satisfy certain conditions, namely, the normals at the ends, $p_1, \ldots, p_k$ would be limited to two antipodal points on the sphere and inequality (4) would have to become an equality. Costa's work [9] resulted in an example with genus one and three ends and it satisfied these necessary conditions. Furthermore, Hoffman and Meeks [18] showed that Costa's surface was indeed embedded. They also showed that there were a lot of one-parameter family of deformations of Costa's surface that were also embedded, and that there were analogous surfaces with higher genus, or more ends or both.
It is pertinent to mention that, for generalized Gauss map of minimal surfaces in $\mathbb{R}^n$, where $n > 3$, one does not have a unique unit normal $\mathbf{N}$ anymore, [38, 39, 40, 34, 20, 21].

Chapter 9

Conclusions

A brief history of the development of the theory of minimal surfaces was presented. Some areas where the theory of minimal surfaces could be applied were also highlighted. A review of Differential Geometry of surfaces, Variational Calculus and Complex Number theory were presented. It was shown that the use of the Weierstrass equations provided a convenient artifice for computing minimal surfaces because these equations guarantee that the resulting surface(s) has/have zero mean curvature. Furthermore, once we know the relevant Weierstrass functions we can calculate important parameters for characterising the surface, namely, the metric, the normal vectors and the Gaussian curvature. In addition, the Weierstrass parametrization allow us to apply the Bonnet transformation for generating a variety of minimal surfaces which preserves the metric and the Gaussian curvature. However, a disadvantage of the Weierstrass equations in the computation of minimal surfaces is that in some surfaces, for example, in triply periodic minimal surfaces, we have to integrate very close to the singularities because important parts of the surface resides in the neighbourhood of these points. From the foregoing, it is highly recommended that the Weierstrass equations be considered as a method for finding new minimal surfaces which could be useful in describing the properties of biological, metallurgical, chemical, mechanical, architectural systems, etc.

The proofs of Bernstein's and Osserman's theorems were presented. In concluding the discussions on Bernstein's and Osserman's theorems, several theorems which are improved versions of the original Bernstein's and Osserman's theorems were stated. Finally, we concluded this report by presenting the characteristics of surfaces of finite total curvature. It was also pointed out that for generalized Gauss map of minimal surfaces in $\mathbb{R}^n$, where $n > 3$, one does not have a unique unit normal $\mathbf{N}$ anymore.

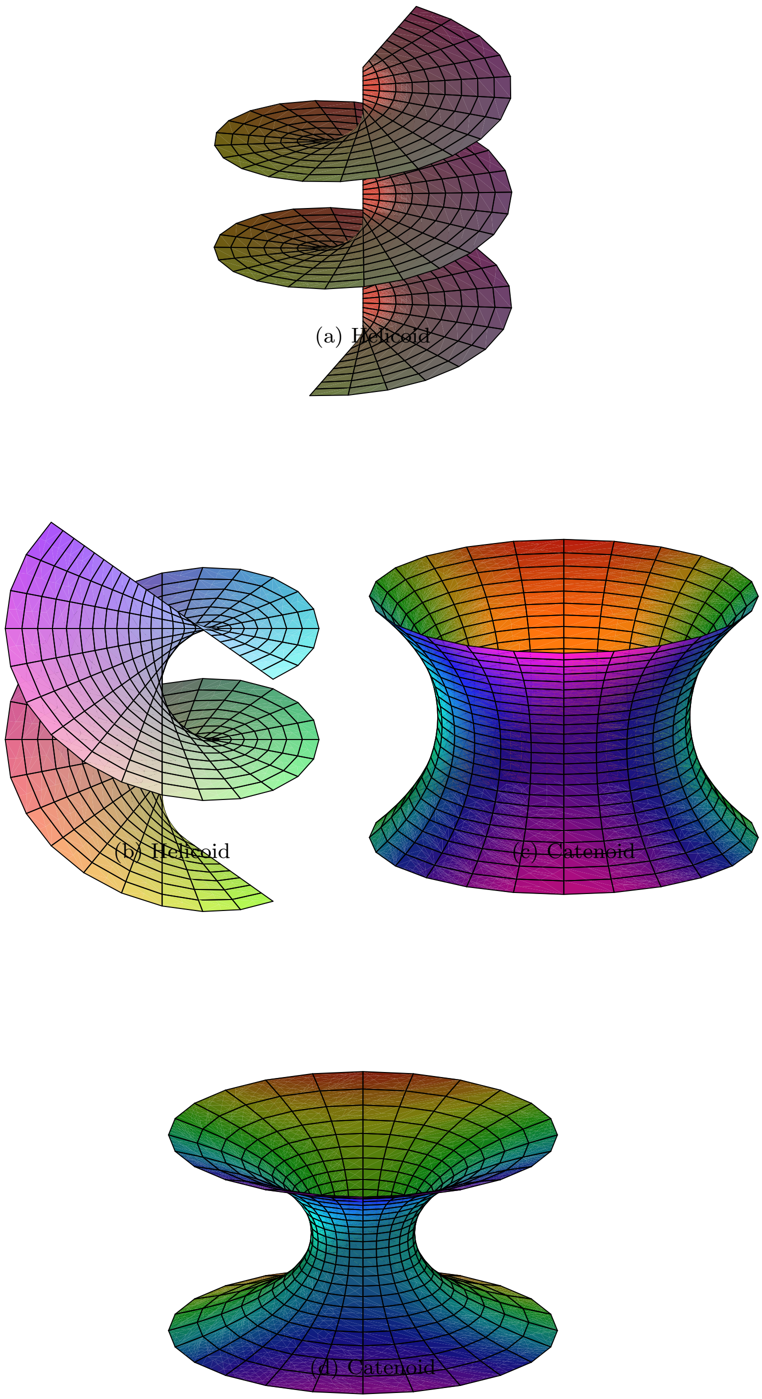
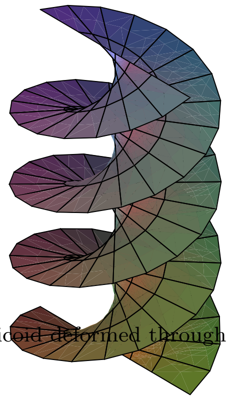
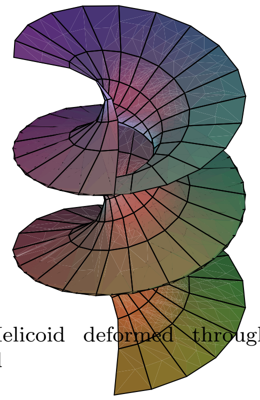


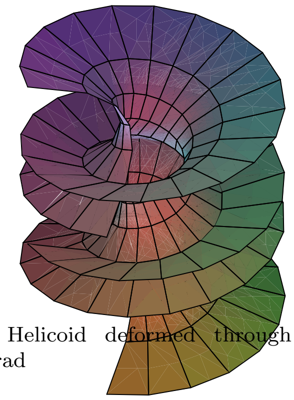
Figure 9.1: Diagrams of the helicoid and the catenoid



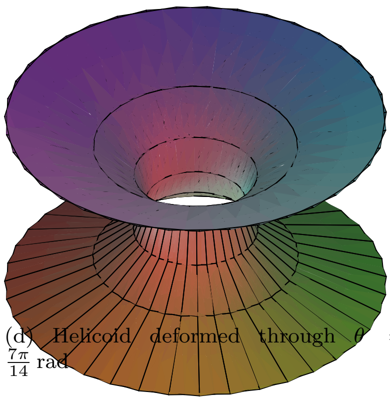
(a) Helicoid deformed through $\theta = 0\,\mathrm{rad}$



(b) Helicoid deformed through $\theta = \frac{2\pi}{7}\,\mathrm{rad}$

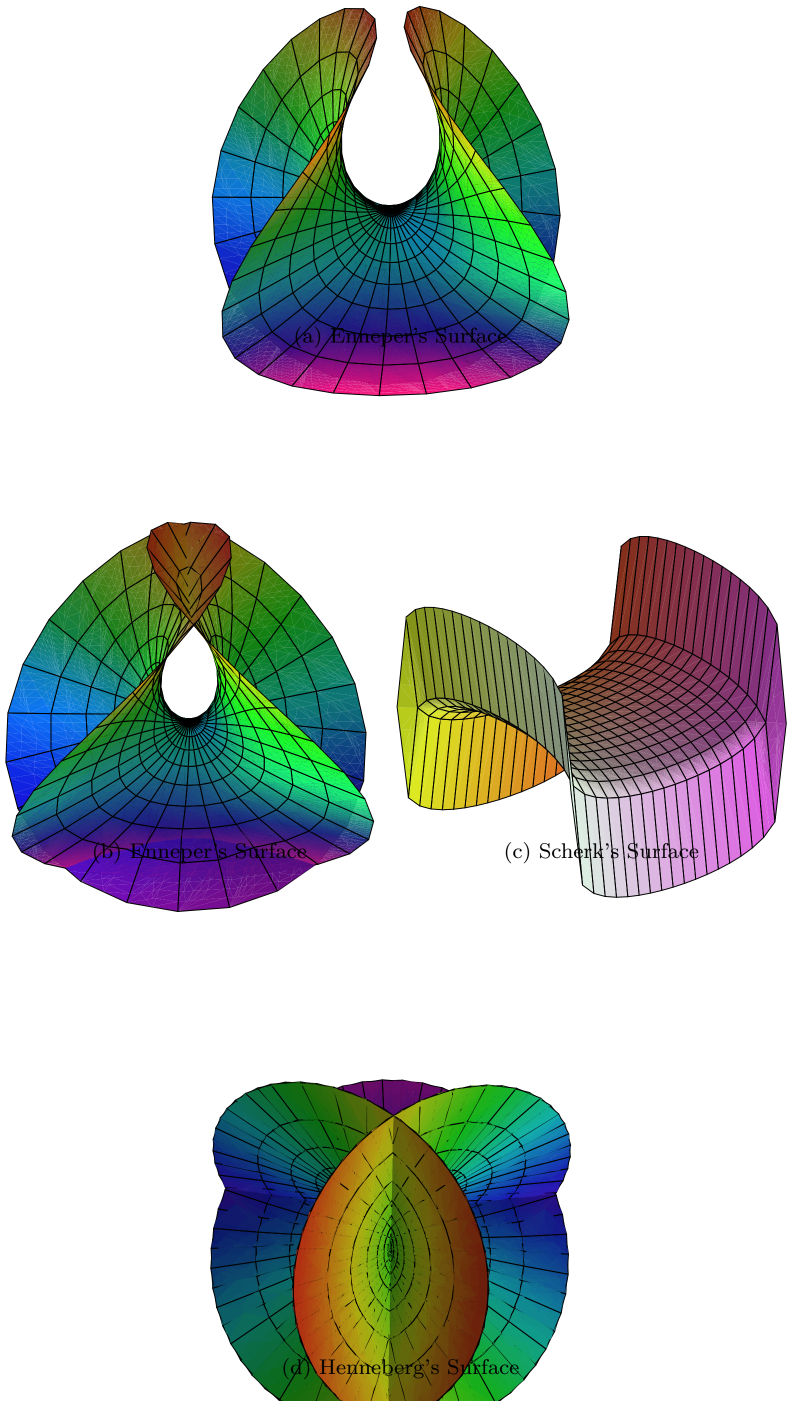


(c) Helicoid deformed through $\theta = \frac{5\pi}{14}\,\mathrm{rad}$



(d) Helicoid deformed through $\theta = \frac{7\pi}{14}\,\mathrm{rad}$

Figure 9.2: Continuous deformation of a helicoid into a catenoid



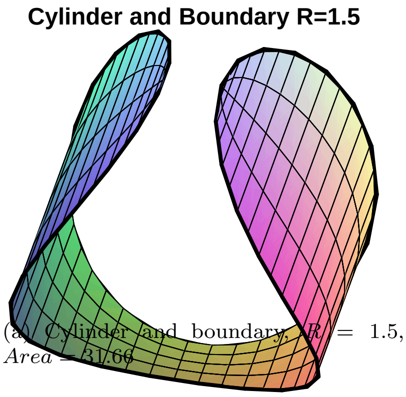
(a) Enneper's Surface

(b) Enneper's Surface

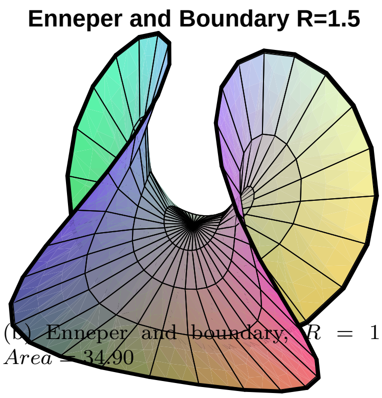
(c) Scherk's Surface

(d) Henneberg's Surface

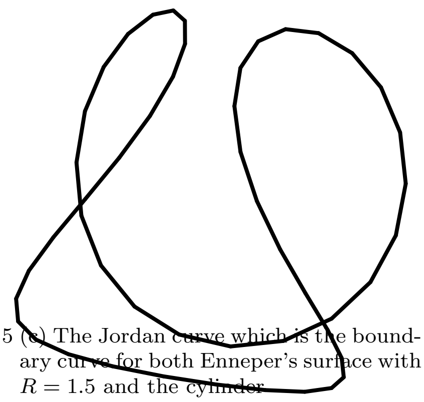
Figure 9.3: Diagrams of Enneper, Scherk's and Henneberg Surfaces



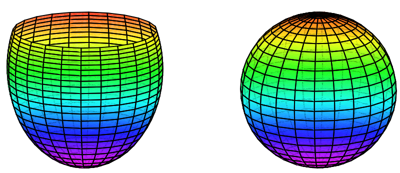
(a) Cylinder and boundary, $R = 1.5$, $Area = 31.66$



(b) Enneper and boundary, $R = 1.5$ $Area = 34.90$

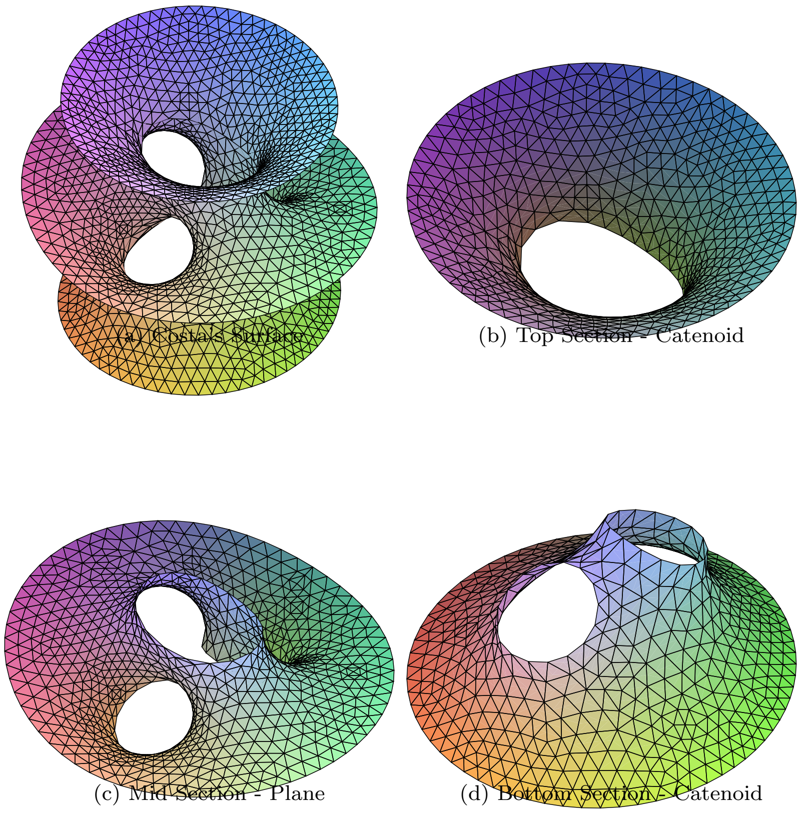


(c) The Jordan curve which is the boundary curve for both Enneper's surface with $R = 1.5$ and the cylinder



(d) Comparison of Gauss map of Enneper's Surface to unit sphere

Figure 9.4: Diagrams illustrating Schwarz Theorem



(a) Costa's Surface

(b) Top Section - Catenoid

(c) Mid Section - Plane

(d) Bottom Section - Catenoid

Figure 9.5: Costa Surface and its Disassembled Parts [43]

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