APPLICATIONS OF TERNARY RINGS TO C^*-ALGEBRAS

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ABSTRACT. We show that there is a functor from the category of positive admissible ternary rings to the category of ∗-algebras, which induces an isomorphism of partially ordered sets between the families of C^*-norms on the ternary ring and its corresponding ∗-algebra. We apply this functor to obtain Morita–Rieffel equivalence results between cross-sectional C^*-algebras of Fell bundles, and to extend the theory of tensor products of C^*-algebras to the larger category of full Hilbert C^*-modules. We prove that, like in the case of C^*-algebras, there exist maximal and minimal tensor products. As applications we give simple proofs of the invariance of nuclearity and exactness under Morita–Rieffel equivalence of C^*-algebras.

1. INTRODUCTION

An important tool in the study of C^*-algebras is Morita–Rieffel equivalence. When two C^*-algebras are Morita–Rieffel equivalent, they are related by a certain type of bimodule, from which one can see that these algebras share many properties. A Morita–Rieffel equivalence between two C^*-algebras implies that these algebras have many characteristics in common: they have the same K-theory, their spectra and primitive ideal spaces are homeomorphic, etc. In [12], Zettl introduced and studied C^*-ternary rings, and showed that these objects are essentially Morita–Rieffel equivalence bimodules. In fact, given a C^*-ternary ring

E, there exists essentially a unique structure of Morita–Rieffel equivalence bi-module on E compatible with its structure of ternary ring (perhaps after a minor change on the ternary product).

On the other hand, when dealing with constructions such as tensor products or any sort of crossed products of C^*-algebras, in general one has to follow two steps: first one defines some $*$-algebra, and then one takes the completion of that algebra with respect to a C^*-norm. A situation that appears frequently is that there is more than one reasonable C^*-norm to perform this second step. In many cases, for instance in several imprimitivity theorems, one is interested in finding a Morita–Rieffel equivalence between different C^*-completions of a given pair of $*$-algebras which are related by a certain bimodule. This is the situation we study in the present paper, adopting a viewpoint similar to that in Zettl's work, but starting from a more algebraic level.

More precisely, suppose E is an $A-B$ bimodule, where A and B are $*$-algebras, $\langle \, , \, \rangle_A : E \times E \to A$ and $\langle \, , \, \rangle_B : E \times E \to B$ satisfy all the algebraic properties of Hilbert bimodule inner products. In particular $\langle x, y \rangle_A z = x \langle y, z \rangle_B$, $\forall x, y, z \in E$. Then we can endow E with a $*$-ternary ring structure by defining a ternary product $(\, , \,) : E \times E \times E \to E$ such that $(x, y, z) = x \langle y, z \rangle_B$. We show that, under certain conditions, the partially ordered sets of C^*-norms on E and on the $*$-algebras A and B are isomorphic to each other, in such a way that the completions with respect to corresponding C^*-norms under these isomorphisms yields a Morita–Rieffel equivalence bimodule.

We think that the best way to do it is by using the above mentioned abstract characterization of equivalence bimodules given by Zettl in [12], under the name of C^*-ternary rings. Such an object is a Banach space with a ternary product on it, which implicitly carries all the structure of an equivalence bimodule. Natural morphisms between C^*-ternary rings are linear maps that preserve ternary products. With such morphisms, one obtains a C^*-category, which is very convenient for the study of properties invariant under Morita–Rieffel equivalence.

The structure of the paper is as follows. In the next section, working in a pure algebraic level, we define the category of admissible $*$-ternary rings, and we show there is a functor from this category to the category of $*$-algebras or, more precisely, to the category of right basic triples (see Definition 2.6). In Section 3, given an admissible ternary ring E with associated basic triple $(E, A, \langle \, , \, \rangle_A)$, we consider the lattice of C^*-seminorms on A that satisfy the Cauchy-Schwarz inequality $\|\langle x, y \rangle_A\|^2 \leq \|\langle x, x \rangle\| \, \|\langle y, y \rangle\|$, $\forall x, y \in E$. Then we prove that this lattice is isomorphic to the lattice of C^*-seminorms on E. In passing we obtain some of the results of [12] and [1] regarding C^*-ternary rings. Besides, since there is also a functor to the category of left basic triples, we obtain a fortiori an isomorphism between the lattices of C^*-seminorms (satisfying the Cauchy-Schwarz property) on the $*$-algebras associated to the left and to the right sides. The Hausdorff completions of corresponding C^*-seminorms under this isomorphism turn out to be Morita–Rieffel equivalent. In the last part of Section 3 we consider positive ternary rings, for which the C^*-seminorms on the associated $*$-algebras automatically satisfy the Cauchy-Schwarz inequality. In Section 4 we briefly study the case of C^*-ternary rings, in which basic triples are replaced by C^*-basic triples, that is,

Hilbert modules, and the functors from C^*-ternary rings to C^*-basic triples are shown to be exact. Finally, Section 5 is devoted to applications. We first refine a result from [2] concerning cross-sectional algebras of Fell bundles over groups. Then we consider tensor products of C^*-ternary rings, which is essentially the same as tensor products of Hilbert modules. We show that the theory of tensor products of C^*-algebras extends to this larger category, in the sense that there exist a maximal and a minimal tensor products. By using this theory we obtain easy and natural proofs of the known results of the Morita–Rieffel invariance of nuclearity and exactness of C^*-algebras.

2. Ternary rings

2.1. Ternary rings.

Definition 2.1. A $*$-ternary ring is a complex linear space E with a map $\mu : E \times E \times E \to E$, called $*$-ternary product on E, which is linear in the odd variables and conjugate linear in the second one, and such that:

$$\mu\big(\mu(x,y,z),u,v\big) = \mu\big(x,\mu(u,z,y),v\big) = \mu\big(x,y,\mu(z,u,v)\big), \ \forall x,y,z,u,v \in E$$

A homomorphism of $*$-ternary rings is a linear map $\phi : (E,\mu) \to (F,\nu)$ such that $\nu\big(\phi(x),\phi(y),\phi(z)\big) = \phi\big(\mu(x,y,z)\big), \forall x,y,z \in E$. Sometimes we will write (x,y,z) or $(x,y,z)_E$ instead of $\mu(x,y,z)$, and we will use the expression $*$-tring instead of $*$-ternary ring.

There is an inclusion of the category of $*$-algebras into the category of $*$-trings: if A is a $*$-algebra, then $(x,y,z) \mapsto xy^*z$ is a ternary product on A, and if $\pi : A \to A'$ is a homomorphism of $*$-algebras, then so is of $*$-trings.

Definition 2.2. If a subspace F of a $*$-tring E is invariant under the ternary product, we say that it is a sub-$*$-tring of E, or just a subring of E. A subring F is said to be hermetic in E if for $x \in E$ we have $(x,x,x) \in F \iff x \in F$.

Definition 2.3. A $*$-tring E will be called admissible if $\{0\}$ is hermetic in E. A $*$-algebra A will be called admissible if it is admissible as a $*$-tring.

Note that a $*$-algebra A is admissible if and only if the condition $a^*a = 0$ implies $a = 0$.

Definition 2.4. Let E be a $*$-tring and $F \subseteq E$ a subspace. We say that F is an ideal of E if $(E,E,F) + (E,F,E) + (F,E,E) \subseteq F$.

If $\pi : E \to F$ is a homomorphism into an admissible $*$-tring F, then $\ker \pi$ is an hermetic ideal of E:

$$\pi((x,x,x)) = 0 \iff (\pi(x),\pi(x),\pi(x)) = 0 \iff \pi(x) = 0$$

In case F is an ideal of E, then E/F has an obvious structure of $*$-tring for which the canonical map $q : E \to E/F$ is a homomorphism of $*$-trings. Note that E/F is admissible whenever F is hermetic. In particular if $\pi : E \to F$ is a homomorphism into an admissible $*$-tring F, then $E/\ker \pi$ is admissible

Suppose E is a complex vector space, and let E^* denote its complex conjugate linear space. If (E,μ) is a $*$-tring, then $\mu^* : E^* \times E^* \times E^* \to E^*$ given

by $\mu^*(x, y, z) = \mu(z, y, x)$, $\forall x, y, z \in E^*$, is a $*$-ternary product on E^*. We call (E^*, μ^*) the *adjoint* or *reverse* $*$-*tring* of (E, μ). If $\pi : E \to F$ is a homomorphism, then π remains a homomorphism $E^* \to F^*$, so it is clear that reversion is an autofunctor of order two of the category of $*$-trings, which moreover sends admissible $*$-trings into admissible $*$-trings. If A is a $*$-algebra considered as a $*$-tring as above, then its reverse $*$-tring A^* is the conjugate linear space of A^{op} considered as a $*$-tring.

Example 2.5 (Basic triples). Suppose $(E, A, \langle \, , \, \rangle)$ is a triple consisting of a $\mathbb{C}$-vector space E, a $*$-algebra A over which E is a right module, and a sesquilinear map $\langle \, , \, \rangle : E \times E \to A$ (conjugate linear in the first variable), such that $\langle x, y \rangle a = \langle x, ya \rangle$ and $\langle x, y \rangle^* = \langle y, x \rangle$, $\forall x, y \in E$, $a \in A$. Then $(\, , \, , \,) : E \times E \times E \to E$ given by $(x, y, z) \mapsto x \langle y, z \rangle$ is a ternary product. We will say that $(E, (\, , \, , \,))$ is the ternary ring associated with $(E, A, \langle \, , \, \rangle)$.

Definition 2.6. Triples as in Example 2.5 will be referred to as (right) basic triples. A basic triple $(E, A, \langle \, , \, \rangle_A)$ will be called admissible whenever A is admissible, and full if $\mathrm{span}\{\langle x, y \rangle_A : x, y \in E\} = A$. By a homomorphism from the basic triple $(E, A, \langle \, , \, \rangle_A)$ into the basic triple $(F, B, \langle \, , \, \rangle_B)$ we mean a pair (φ, ψ) of maps such that $\varphi : E \to F$ is linear, $\psi : A \to B$ is a homomorphism of $*$-algebras, and $\varphi(xa) = \varphi(x)\psi(a)$, $\forall x \in E$, $a \in A$.

Similarly we can define left basic triples, using left instead of right A-modules.

We will see soon that any admissible $*$-tring can be described in terms of basic triples as in 2.5.

Proposition 2.7. *Let $(E, A, \langle \, , \, \rangle)$ be a basic triple.*

(1) If A is admissible, and $\langle x, x \rangle = 0$ implies $x = 0$, then the $*$-tring E is admissible as well.

(2) If $(E, A, \langle \, , \, \rangle)$ is admissible and full, then E is faithful as an A-module.

Proof. If $x \in E$ is such that $x\langle x, x \rangle = 0$, then

$$\langle x, x \rangle^* \langle x, x \rangle = \langle x, x \rangle \langle x, x \rangle = \langle x, x \langle x, x \rangle \rangle = 0.$$

Now, if A is admissible, the latter equality implies $\langle x, x \rangle = 0$, so $x = 0$. As for the second statement suppose $(E, A, \langle \, , \, \rangle_A)$ is admissible and full, and $a \in A$ is such that $a = \sum_{j=1}^n \langle y_j, z_j \rangle$ and $ya = 0$, $\forall y \in F$. Then we have $a^*a = \sum_{j=1}^n \langle y_j, z_j a \rangle = 0$, so $a = 0$. Then E is a faithful A-module. $\square$

Lemma 2.8. *Suppose that $(E, A, \langle \, , \, \rangle_A)$ and $(F, B, \langle \, , \, \rangle_B)$ are basic triples, with the former full, and F admissible as $*$-tring and faithful as a B-module. Then, if $\varphi : (E, (\, , \, , \,)) \to (F, (\, , \, , \,))$ is a homomorphism between their associated $*$-trings, there exists a unique homomorphism of $*$-algebras $\psi : A \to B$ such that $\psi(\langle x, y \rangle_A) = \langle \varphi(x), \varphi(y) \rangle_B$, $\forall x, y \in E$. Besides we have $\varphi(xa) = \varphi(x)\psi(a)$, $\forall x \in E$, $a \in A$, and*

$$\ker \psi \subseteq \{a \in A : Ea \subseteq \ker \varphi\} \subseteq \{a \in A : \psi(a)^*\psi(a) = 0\}, \qquad (2.1)$$

both inclusions being equalities if B is admissible. If E is also a faithful A-module and φ is injective, then so is ψ.

Proof. We will suppose that $(F, B, \langle\,,\,\rangle_B)$ is full: otherwise we just replace B by $\mathrm{span}\langle F, F\rangle_B$. We concentrate in showing the existence of the map ψ, because its uniqueness is obvious. To this end suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are elements in E such that $\sum_{j=1}^n \langle x_j, y_j\rangle_A = 0$, and therefore also $\sum_{j=1}^n \langle y_j, x_j\rangle_A = 0$. Consider the element $c := \sum_{j=1}^n \langle \varphi(x_j), \varphi(y_j)\rangle_B$ of B. All we have to do is to show that $c = 0$. Now, if $x \in E$ and $u \in F$ we have

$$(\varphi(x), uc, uc) = \sum_k (\varphi(x), (u, \varphi(x_k), \varphi(y_k)), uc)$$

$$= \sum_k ((\varphi(x), \varphi(y_k), \varphi(x_k)), u, uc) = (\varphi(x \sum_k \langle y_k, x_k\rangle_A), u, uc) = 0.$$

Hence, if $u \in F$:

$$(uc, uc, uc) = \sum_j ((u, \varphi(x_j), \varphi(y_j)), uc, uc) = \sum_j (u, \varphi(x_j), (\varphi(y_j), uc, uc)) = 0$$

Since F is admissible, it follows that $uc = 0, \forall u \in F$, so $c = 0$ because F is a faithful B-module.

Suppose now that $a \in \ker \psi$. Then $\varphi(xa) = \varphi(x)\psi(a) = 0$, so $Ea \subseteq \ker \varphi$. On the other hand, if the element $a = \sum_j \langle x_j, y_j\rangle_A$ is such that $Ea \subseteq \ker \varphi$, then $\psi(a^*a) = \psi(\sum_{i,j} \langle y_i, x_i\langle x_j, y_j\rangle_A\rangle_A) = \sum_{i,j}\langle \varphi(y_i), \varphi(x_i)\psi(\langle x_j, y_j\rangle_A)\rangle_A$, so

$$\psi(a)^*\psi(a) = \sum_i \langle \varphi(y_i), \varphi(x_i)\psi(\sum_j \langle x_j, y_j\rangle_A)\rangle_A = \sum_i \langle \varphi(y_i), \varphi(x_ia)\rangle_A = 0,$$

because $\varphi(x_ia) = 0\ \forall i$. In case B is admissible we have $\psi(a)^*\psi(a) = 0$ if and only if $a \in \ker \psi$, so in this case the three considered sets agree. Finally, when E is faithful and $\ker \varphi = 0$, we have $\{a \in A : Ea \subseteq \ker \varphi\} = 0$, so $\ker \psi = 0$. $\qquad\square$

Given two modules E and F over a ring R, we denote by $\mathrm{Hom}_R(E, F)$ the abelian group of R-linear maps from E into F, and just by $\mathrm{End}_R(E)$ in case $E = F$. Let E be an admissible $*$-tring, and suppose $T \in \mathrm{End}_{\mathbb{C}}(E)$ is such that there exists $S \in \mathrm{End}_{\mathbb{C}}(E)$ that satisfies $(x, Ty, z) = (Sx, y, z), \forall x, y, z \in E$. Since $\{0\}$ is hermetic in E, given $T \in \mathrm{End}_{\mathbb{C}}(E)$, there exists at most one such endomorphism S; in this case we say that S is the adjoint of T to the left, and we denote it by T^*. The set $\mathscr{L}_l(E)$ of $\mathbb{C}$-linear endomorphisms of E that have an adjoint to the left is clearly a unital subalgebra of $\mathrm{End}_{\mathbb{C}}(E)$. Every pair of elements $y, z \in E$ gives rise to an endomorphism $\theta_{y,z} : E \to E$ given by $\theta_{y,z}(x) := (x, y, z)$. It is readily checked that $\theta_{y,z}$ is adjointable with adjoint $\theta_{z,y}$.

Proposition 2.9. *Let E be an admissible $*$-tring. Then the map $* : \mathscr{L}_l(E) \to \mathscr{L}_l(E)$, given by taking the adjoint, is an involution in $\mathscr{L}_l(E)$. Moreover, the $*$-algebra $\mathscr{L}_l(E)$ is an admissible $*$-tring, and $\mathrm{span}\{\theta_{y,z} : y, z \in E\}$ is a twosided ideal of $\mathscr{L}_l(E)$, which is essential in the sense that $T\theta_{y,z} = 0\ \forall y, z \in E$ or $\theta_{y,z}T = 0\ \forall y, z \in E$ implies $T = 0$.*

Proof. It is clear that the map $T \mapsto T^*$ is conjugate linear and antimultiplicative. On the other hand, if $T \in \mathscr{L}_l(E)$:

$$(u, (Tx, y, z), u) = (u, z, (y, T(x), u)) = (u, z, (T^*(y), x, u)) = (u, (x, T^*(y), z), u)$$

$\forall x, y, z, u \in E$ and $T \in \mathscr{L}_l(E)$, which shows that $T^{**} = T$. Now, if $x \in E$, and $T \in \mathscr{L}_l(E)$ is such that $T^*T = 0$: $(Tx, Tx, Tx) = (x, T^*Tx, Tx) = 0$, so $T(x) = 0$, and therefore $T = 0$. Finally, if $T \in \mathscr{L}_l(E)$ and $x, y, z \in E$: $\theta_{y,z}T(x) = (Tx, y, z) = (x, T^*y, z) = \theta_{T^*y,z}(x)$. Thus $T\theta_{y,z} = (\theta_{z,y}T^*)^* = \theta_{y,Tz}$. This shows that $\text{span}\{\theta_{y,z} : y, z \in E\}$ is an ideal of $\mathscr{L}_l(E)$. If $\theta_{y,z}T = 0 \ \forall y, z \in E$, then $0 = \theta_{Tx,Tx}T(x) = (Tx, Tx, Tx)$, $\forall x \in E$. Then $Tx = 0 \ \forall x \in E$ because E is admissible, so $T = 0$. $\qquad\square$

The next result shows that any admissible $*$-tring E gives rise to an admissible and full *right* basic triple $(E, E_0^r, \langle\ ,\ \rangle_r)$. In the same way one could show that E also defines a *left* basic triple $(E, E_0^l, \langle\ ,\ \rangle_l)$.

Theorem 2.10. *Let E and F be admissible $*$-trings. Then:*

(1) *There exists a pair $(E_0^r, \langle\ ,\ \rangle_r)$ such that $(E, E_0^r, \langle\ ,\ \rangle_r)$ is an admissible and full basic triple, whose associated $*$-tring is E.*

(2) *If $\pi : E \to F$ is a homomorphism of $*$-trings, and $(E_0^r, \langle\cdot,\cdot\rangle_r)$ and $(F_0^r, \langle\cdot,\cdot\rangle_r)$ are pairs like above for E and F respectively, there exists a unique homomorphism of $*$-algebras $\pi_0^r : E_0^r \to F_0^r$ such that*

$$\pi_0^r(\langle x, y\rangle_r) = \langle\pi(x), \pi(y)\rangle_r, \ \forall x, y \in E.$$

Moreover, $\pi(xb) = \pi(x)\pi_0^r(b)$, $\forall x \in E$, $b \in E_0^r$, that is, the pair (π, π_0^r) is a homomorphism of basic triples.

(3) *The pair $(E_0^r, \langle\cdot,\cdot\rangle_r)$ is the unique (up to canonical isomorphisms) such that the triple $(E, E_0^r, \langle\cdot,\cdot\rangle_r)$ is a full and admissible with E as associated $*$-tring.*

Proof. Note that E is a faithful right $\mathscr{L}_l(E)^{\mathrm{op}}$-module with $xT := T(x)$. Consider the ideal $E_0^r := \text{span}\{\theta_{y,z} : y, z \in E\}$ of $\mathscr{L}_l(E)^{\mathrm{op}}$ and let $\langle\ ,\ \rangle_r : E \times E \to E_0^r$ be given by $\langle x, y\rangle_r := \theta_{x,y}$. It is routine to verify that $(E, E_0^r, \langle\ ,\ \rangle_r)$ is a full and admissible basic triple whose associated $*$-tring is E. The second statement follows at once from 2.8 and 2.7, while the last assertion of the theorem follows immediately from the second one. $\qquad\square$

Corollary 2.11. *The assignment*

$$(E \xrightarrow{\pi} F) \longmapsto (E, E_0^r, \langle\ ,\ \rangle_r) \xrightarrow{(\pi, \pi_0^r)} (F, F_0^r, \langle\ ,\ \rangle_r)$$

defines a functor from the category of admissible $*$-trings into the category of admissible and full basic triples.

Corollary 2.12. *Let $(E, A, \langle\ ,\ \rangle_A)$ be a basic triple such that E is faithful as an A-module and E is admissible as a $*$-tring. Then there exists a unique homomorphism $\psi : E_0^r \to A$ such that $\langle x, y\rangle_r = \langle x, y\rangle_A$, $\forall x, y \in E$. The homomorphism ψ is injective, and it is an isomorphism if $(E, A, \langle\ ,\ \rangle_A)$ is full.*

Proof. Let $(E, E_0^r, \langle\ ,\ \rangle_r)$ be the full and admissible basic triple provided by Theorem 2.10. The identity map on E is an injective homomorphism of $*$-trings, so by 2.8 there exists a unique homomorphism $\psi : E_0^r \to A$ such that $\langle x, y\rangle_r = \langle x, y\rangle_A$, $\forall x, y \in E$, which is injective because E is faithful as E_0^r-module. It is clear that ψ is also surjective when the given basic triple is full. $\qquad\square$

Corollary 2.13. *Let $(E, A, \langle\ ,\ \rangle_A)$ be a full basic triple such that E is faithful as a A-module. Then A is admissible if E is admissible.*

Proof. Just note that if E is admissible, then $E_0^r \cong A$ by 2.12, and E_0^r is admissible according to 2.10. $\square$

Corollary 2.14. *Let F be an ideal of the admissible $*$-tring E, $(E, E_0^r, \langle\ ,\ \rangle_E)$ and $(F, F_0^r, \langle\ ,\ \rangle_F)$ the full and admissible basic triples associated, respectively, with E and F (given by Theorem 2.10). If $A := span\{\langle x, y\rangle_E : x, y \in F\}$, then A is a $*$-ideal of E_0^r, and the basic triples $(F, F_0^r, \langle\ ,\ \rangle_F)$ and $(F, A, \langle\ ,\ \rangle_E)$ are isomorphic.*

Proof. The triple $(F, A, \langle\ ,\ \rangle_r)$ is admissible and full, with F as induced $*$-tring. Then F is a faithful A-module by 2.7. According to 2.12, there exists a unique map $\psi : F_0^r \to A$ such that (id, ψ) is a homomorphism from $(F, F_0^r, \langle\ ,\ \rangle_F)$ to $(F, A, \langle\ ,\ \rangle_E)$, and ψ is an isomorphism o $*$-algebras. It follows that (id, ψ^{-1}) is the inverse homomorphism of (id, ψ). $\square$

From now on if F is an ideal in the admissible $*$-tring E, we will think of F_0^r as a $*$-ideal of E_0^r via the identification provided by 2.14:

$$F_0^r \cong span\{\langle x, y\rangle_E : x, y \in F\}. \tag{2.2}$$

For the next result recall that an ideal F of the $*$-tring E is hermetic if and only if E/F is admissible.

Proposition 2.15. *Let $\pi : E \to F$ be a homomorphism between the admissible $*$-trings E and F, such that $\ker \pi$ is hermetic. If $I_{\ker \pi} := \{a \in E_0^r : Ea \subseteq \ker \pi\}$, then:*

$$(\ker \pi)_0^r \subseteq \ker \pi_0^r \subseteq I_{\ker \pi}$$

Proof. Taking into account (2.2) above and the second part of 2.10, the first inclusion is clear. The second inclusion follows from the admissibility of $E/\ker \pi$ and (2.1) in Lemma 2.8. $\square$

Remark 2.16. Suppose F is an hermetic ideal of the admissible $*$-tring E. Let $q : E \to E/F$ be the quotient map, $I_F := \{a \in E_0^r : Ea \subseteq F\}$, $p : E_0^r \to E_0^r/I_F$ the canonical projection and $\overline{q_0^r} : E_0^r/I_F \to (E/F)_0^r$ the isomorphism induced by q_0^r, so the following diagram commutes:

$$
\begin{array}{ccc}
E_0^r & \xrightarrow{\quad q_0^r \quad} & (E/F)_0^r \\
& {\scriptstyle p} \searrow \quad \nearrow {\scriptstyle \overline{q_0^r}} & \\
& E_0^r/I_F &
\end{array}
$$

Then:

$$\overline{q_0^r}(p(\langle x, y\rangle_E)) = q_0^r(\langle x, y\rangle_E)) = \langle q(x), q(y)\rangle_{E/F}, \quad \forall x, y \in E.$$

Therefore the pair $((E/F)_0^r.\langle, \rangle_{E/F})$ associated with E/F in Theorem 2.10 may be replaced by the pair $(E_0^r/I_F, [,]_{E/F})$., where $[q(x), q(y)]_{E/F} = p(\langle x, y\rangle_E)$, $\forall x, y \in E$ and the action of E_0^r/I_F on E/F_γ is given by $q(x)p(a) = q(xa)$, $\forall x \in E$, $a \in A$.

Proposition 2.17. *Let $\pi : E \to F$ be a homomorphism between admissible $*$-trings. Then:*

    (1) π is injective if and only if $\pi_0^r : E_0^r \to F_0^r$ is injective.
    (2) If π is onto, or an isomorphism, then so is $\pi_0^r : E_0^r \to F_0^r$.

Proof. Since the second statement is clear we prove only the first one. Now if π_0^r is injective and $x \in E$, the admissibility of E and F implies that:

$$\pi(x) = 0 \iff \langle \pi(x), \pi(x) \rangle_r = 0 \iff \pi_0^r(\langle x, x \rangle_r) = 0 \iff x = 0,$$

so π is injective as well. On the other hand the injectivity of π implies that of π_0^r by 2.8. $\qquad\square$

3. Correspondence between C^*-seminorms.

3.1. C^*-seminorms.

Definition 3.1. A C^*-seminorm on a $*$-tring (E, μ) is a seminorm such that:

    (1) $\|\mu(x, y, z)\| \leq \|x\| \, \|y\| \, \|z\|$, $\forall x, y, z \in E$.
    (2) $\|\mu(x, x, x)\| = \|x\|^3$, $\forall x \in E$.

If $\|\cdot\|$ is a norm, we call it a C^*-norm, and we say that $(E, \|\cdot\|)$ is a pre-C^*-ternary ring. If $(E, \|\cdot\|)$ is also a Banach space, we say that it is a C^*-ternary ring, or just a C^*-tring.

If E is a $*$-tring, the set of C^*-seminorms on E will be denoted by $\mathcal{SN}(E)$, and $\mathcal{N}(E)$ will denote the set of C^*-norms on E. The set $\mathcal{SN}(E)$ is partially ordered by: $\gamma_1 \leq \gamma_2$ if $\gamma_1(x) \leq \gamma_2(x)$, $\forall x \in E$.

Definition 3.2. A $*$-tring E will be called C^*-closable, or just closable, in case $\mathcal{N}(E) \neq \emptyset$. Similar terminology will be used for $*$-algebras.

Observe that any C^*-closable $*$-tring is admissible.

In the next proposition, whose easy proof is left to the reader, we record some basic facts about $*$-trings.

Proposition 3.3. *Let E be a $*$-tring. Then:*

    (1) $N_\gamma := \{x \in E : \gamma(x) = 0\}$ is an hermetic ideal of E, for all $\gamma \in \mathcal{SN}(E)$.
    (2) The intersection of hermetic subrings is also hermetic.
    (3) The quotient E/N is admissible, where $N := \cap\{N_\gamma : \gamma \in \mathcal{SN}(E)\}$ and N_γ is as in 1.
    (4) If $\mathcal{SN}(E)$ separates points of E, then E is admissible.
    (5) If $\mathcal{SN}(E)$ separates points of E and is bounded, then E is C^*-closable.

If H and K are Hilbert spaces and $B(H, K)$ denotes the corresponding space of bounded linear maps, a subspace E of $B(H, K)$ closed under the ternary product $(R, S, T) \mapsto RS^*T \in E$, $\forall R, S, T \in E$, is a $*$-tring with that product. In case E is also closed it is called a ternary ring of operators (TRO). Note that if (E, μ) is a C^*-tring, then $(E, -\mu)$ also is a C^*-tring, called the opposite of (E, μ) and denoted by E^{op}. The opposite of a TRO is called anti-TRO.

New C^*-ternary rings can be obtained by direct sums: if $(E, \|\cdot\|_E, \mu_E)$ and $(F, \|\cdot\|_F, \mu_F)$ are C^*-trings, then $(E \oplus F, \max\{\|\cdot\|_E, \|\cdot\|_F\}, \mu_E \oplus \mu_F)$ is a C^*-tring. We denote it just by $E \oplus F$.

Suppose that E is a full right Hilbert A-module, and define the ternary product on E: $\mu_E(x, y, z) := x\langle y, z\rangle$. Then (E, μ_E) is a C^*-tring with the norm $\|x\| = \sqrt{\langle x, x\rangle}$. Now, if F is a full right Hilbert B-module, then $E \oplus F^{\mathrm{op}}$ is also a C^*-tring. This is the fundamental example of C^*-tring, as shown by Zettl in [12, 3.2] (see also Corollay 3.10 below).

Zettl also showed that there exist unique sub-C^*-trings E_+ and E_- of E such that $E = E_+ \bigoplus E_-$, and E_+ is isomorphic to a TRO, while E_- is isomorphic to an anti-TRO (see [12]). The decomposition above is called the *fundamental decomposition* of E.

Definition 3.4. We say that a C^*-tring E is positive (negative) if $E = E_+$ (respectively: if $E = E_-$).

If E is a C^*-tring, we define $E_p := E_+ \oplus E_-^{\mathrm{op}}$. Then E_p is a positive C^*-tring.

Let E^* be the reverse $*$-tring of (the $*$-tring) E. It is clear that a norm on E is a C^*-norm if and only if is a C^*-norm on E^*. Moreover, E is a (positive) C^*-tring if and only if so is E^*.

3.2. From pre-C^*-trings to pre-C^*-algebras.

In what follows we will examine an intermediate situation between the $*$-algebraic context of 2.10 and the C^*-context originally considered by Zettl.

If α is a seminorm on the vector space X, then $N_\alpha := \{x \in X : \alpha(x) = 0\}$ is a closed subspace of X, so X/N_α is a normed space with the norm $\bar\alpha$ induced by α: $\bar\alpha(x + N_\alpha) = \alpha(x)$. The completion $(X_\alpha, \bar\alpha)$ of $(X/N_\alpha, \bar\alpha)$ will be referred to as the *Hausdorff completion* of the seminormed space (X, α), and the map $x \mapsto x + N_\alpha$ will be called the canonical map.

In case γ is a C^*-seminorm on the ternary ring E, then E/N_γ is a pre-C^*-tring with the induced norm $\bar\gamma$. Thus the corresponding Hausdorff completion E_γ of E is a C^*-tring.

Proposition 3.5. *Suppose E is an admissible $*$-tring and $\gamma \in \mathcal{SN}(E)$. Let $\gamma^r : E_0^r \to [0, \infty)$ be the operator seminorm on E_0^r, that is:*

$$\gamma^r(a) := \sup\{\gamma(xa) : \gamma(x) \le 1\}. \tag{3.1}$$

Then $\gamma^r \in \mathcal{SN}(E_0^r)$, and $\gamma^r \in \mathcal{N}(E_0^r) \iff \gamma \in \mathcal{N}(E)$. Moreover the following relations hold:

$$\gamma(xa) \le \gamma(x)\gamma^r(a), \forall x \in E, a \in E_0^r \tag{3.2}$$

$$\gamma^r(\langle x, y\rangle_r) \le \gamma(x)\gamma(y), \forall x, y \in E \tag{3.3}$$

$$\gamma(x)^2 = \gamma^r(\langle x, x\rangle_r), \forall x \in E \tag{3.4}$$

Proof. Given $a = \sum_{i=1}^{n} \langle x_i, y_i \rangle \in E_0^r$ the linear map $x \mapsto xa$ is bounded because $\gamma(xa) \leq \gamma(x) \sum_{i=1}^{n} \gamma(x_i)\gamma(y_i)$. Then (3.2) and (3.3) follow immediately and Definition 3.1 implies (3.4). With $a \in E_r^0$ as before and $x \in E$ we have

$$(xa, xa, xa) = \sum_{i=1}^{n}((x, x_i, y_i), xa, xa) = \sum_{i=1}^{n}(x, (xa, y_i, x_i), xa),$$

so

$$\gamma(xa)^3 = \gamma(x, xaa^*, xa) \leq \gamma^r(aa^*)\,\gamma^r(a)\,\gamma(x)^3,$$

from where it follows that $\gamma^r(a)^2 \leq \gamma^r(aa^*) \leq \gamma^r(a)\gamma^r(a^*)$. From the computations above is clear that $\gamma^r \in \mathcal{N}(E_0^r) \iff \gamma \in \mathcal{N}(E)$. In particular E_0^r is a C^*-closable algebra whenever E is a C^*-closable tring. $\qquad \square$

Definition 3.6. Suppose $(E, A, \langle\,,\,\rangle_A)$ is a basic triple such that (E, γ) is a C^*-tring and a Banach module over the C^*-algebra (A, α), and that $\langle\,,\,\rangle_A : E \times E \to A$ is continuous. Then the triple is said to be a C^*-basic triple. We say that it is full if the ideal span$\{\langle x, y \rangle_A : x, y \in E\}$ of A is dense in A.

The next two results will be useful for studying the relation between a C^*-basic triple $(E, A, \langle\,,\,\rangle_A)$ and the basic triple $(E, E_0^r, \langle\,,\,\rangle_r)$. What we will show first, in 3.9, is that $(E, E_0^r, \langle\,,\,\rangle_r)$ can be embedded in $(E, A, \langle\,,\,\rangle_A)$.

Proposition 3.7. *Let A be a Banach $*$-algebra and I a $*$-ideal of A, not necessarily closed. Then any C^*-seminorm on I can be extended to a C^*-seminorm on A. If I is dense, such extension is unique.*

Proof. Consider $\alpha \in \mathcal{SN}(I)$, $\alpha \neq 0$. Let I_α be the Hausdorff completion of (I, α), $p : I \to I_\alpha$ the canonical map, and let $\pi : I_\alpha \to B(H)$ be a faithful representation. Now, according to [4, VI-19.11], the representation $\pi p : I \to B(H)$ can be extended to a representation ρ of A. Then $a \mapsto \|\rho(a)\|$ defines a C^*-seminorm on A that extends α. Note that the continuity of ρ implies the continuity of α, from which the uniqueness of the extension follows in case I is dense in A. $\qquad \square$

Corollary 3.8. *Let I be a $*$-ideal of the C^*-algebra A. Then the unique C^*-norm one can define in I is the restriction to I of the norm of A.*

Proposition 3.9. *Let $(E, A, \langle\,,\,\rangle_A)$ be a full C^*-basic triple, and γ and α the corresponding norms of E and A. Then (A, α) is the completion of (E_0^r, γ^r), and $\langle\,,\,\rangle_A$ is the continuous extension of $\langle\,,\,\rangle_r$.*

Proof. Note that E is admissible for it is a C^*-tring. On the other hand E is a faithful A-module: if $a \in A$ is such that $xa = 0 \,\forall x \in E$, then $\langle x, y \rangle_A a = 0 \,\forall x, y \in E$, so it follows that $ba = 0$ for every b in the dense ideal span$\{\langle x, y \rangle_A : x, y \in E\}$ of A, which implies $a = 0$. Thus there exists, by 2.7, a unique homomorphism $\psi : E_0^r \to A$ such that $\psi(\langle x, y \rangle_r) = \langle x, y \rangle_A$, $\forall x, y \in E$. Besides ψ is injective and $\psi(E_0^r) = $ span$\{\langle x, y \rangle_A : x, y \in E\}$ (thus we may suppose E_0^r is a dense ideal of A). Now 3.8 implies γ_0^r is the restriction of α to $\psi(E_0^r)$ and, since the latter is dense in A, we conclude that A is the completion of E_0^r. $\qquad \square$

As a consequence we obtain the following result, due to H. Zettl:

Corollary 3.10 (*cf.* [12, Proposition 3.2]). *Let (E, γ) be a C^*-tring and E^r the completion of E_0^r with respect to γ^r. Then $(E, E^r, \langle\,,\,\rangle_r))$ is, up to isomorphism, the unique full C^*- basic triple whose first component is E.*

Proposition 3.11. *Let $\pi : E_1 \to E_2$ be a homomorphism of $*$-trings between the C^*-trings E_1 and E_2. Then there exists a unique homomorphism $\pi^r : E^r \to F^r$ such that $\pi^r(\langle x, y\rangle_E) = \langle \pi(x), \pi(y)\rangle_F$, $\forall x, y \in E$, and $\pi(xa) = \pi(x)\pi^r(a)$ $\forall x \in E$, $a \in E^r$. Consequently π is always contractive, and is isometric if and only if it is injective.*

Proof. It is clear that, if the homomorphism π^r exists, it must be an extension of $\pi_0^r : E_0^r \to F_0^r$. Let $\rho : F^r \to B(H)$ be a faithful representation. Then $\rho\pi_0^r$ is a representation of E_0^r. Now, since $(E, E^r, \langle\,,\,\rangle_r)$ is a C^*-triple, E_0^r is a $*$-ideal in E^r. Therefore $\rho\pi_0^r$ can be uniquely extended to a representation $\bar\rho : E^r \to B(H)$ ([4, VI.19.11]). Since $\rho(F^r)$ is closed and $\bar\rho(E^r)$ is a subset of the closure of $\rho\pi_0^r(E_0^r)$, we have $\bar\rho(E^r) \subseteq \rho(F^r)$. Then take $\pi^r := \rho^{-1}\bar\rho$. Note that $\|\pi(x)\|^2 = \|\pi^r(\langle x, x\rangle)\| \leq \|\langle x, x\rangle\| = \|x\|^2$, with equality if π^r is injective. This shows that π is contractive. Finally, if π is injective, so is π_0^r and, as in the proof of 3.8, this implies that π^r also is injective, thus an isometry. $\quad\square$

Corollary 3.12 (*cf.* [1][Proposition 4.1]). *The assignment*

$$(E \xrightarrow{\pi} F) \longmapsto (E, E^r, \langle\,,\,\rangle_r) \xrightarrow{(\pi, \pi^r)} (F, F^r, \langle\,,\,\rangle_r)$$

defines a functor from the category of C^*-trings to the category of full C^*-basic triples.

It follows from Proposition 3.5 that any C^*-seminorm on E_0^r induced by a C^*-seminorm on E by means of (3.1) must satisfy the Cauchy-Schwarz condition (3.3). So it is natural to restrict our attention to the following subsets of C^*-seminorms on E_0^r:

$$\mathcal{SN}_{cs}(E_0^r) := \{\alpha \in \mathcal{SN}(E_0^r) : \alpha(\langle x, y\rangle_r)^2 \leq \alpha(\langle x, x\rangle_r)\,\alpha(\langle y, y\rangle_r)\}$$

$$\mathcal{N}_{cs}(E_0^r) := \mathcal{SN}_{cs}(E_0^r) \cap \mathcal{N}(E_0^r).$$

In fact it will be convenient to place ourselves in a slightly more general setting:

Definition 3.13. Let $(E, A, \langle\,,\,\rangle)$ be a basic triple. We define

$$\mathcal{SN}_{cs}^{\langle,\rangle}(A) := \{\alpha \in \mathcal{SN}(A) : \alpha(\langle x, y\rangle)^2 \leq \alpha(\langle x, x\rangle)\alpha(\langle y, y\rangle), \forall x, y \in E\}.$$

Proposition 3.14. *Let $(E, A, \langle\,,\,\rangle)$ be a basic triple, and consider E with the $*$-tring structure induced by $\langle\,,\,\rangle$. Given $\alpha \in \mathcal{SN}_{cs}^{\langle,\rangle}(A)$, let $\tilde\alpha : E \to [0, \infty)$ be defined by:*

$$\tilde\alpha(x) := \alpha(\langle x, x\rangle)^{1/2} \tag{3.5}$$

Then

    (1) $\tilde\alpha(xa) \leq \tilde\alpha(x)\alpha(a)$.
    (2) $\tilde\alpha \in \mathcal{SN}(E)$
    (3) If E is a faithful A-module and $\tilde\alpha \in \mathcal{N}(E)$, then $\alpha \in \mathcal{N}_{cs}^{\langle,\rangle}(A)$.
    (4) If $\alpha \in \mathcal{N}_{cs}^{\langle,\rangle}(A)$ and $\langle x, x\rangle = 0$ implies $x = 0$, then $\tilde\alpha \in \mathcal{N}(E)$

Proof. Since the Cauchy-Schwarz inequality (3.3) holds for α, it follows as usual that $\tilde{\alpha}$ satisfies the triangular inequality and, since homogeneity is obvious, $\tilde{\alpha}$ is a seminorm on E. On the other hand, since α is a C^*-seminorm and satisfies (3.3) we have, for all $x, y, z \in E$, $a \in A$:

$$\tilde{\alpha}(xa) = \alpha(a^*\langle x, x\rangle a)^{1/2} \leq \alpha(a)\tilde{\alpha}(x)$$

$$\tilde{\alpha}((x, y, z)) = \tilde{\alpha}(x\langle y, z\rangle) \leq \tilde{\alpha}(x)\alpha(\langle y, z\rangle) \leq \tilde{\alpha}(x)\tilde{\alpha}(y)\tilde{\alpha}(z)$$

$$\tilde{\alpha}((x, x, x)) = \alpha(\langle x, x\rangle^3)^{1/2} = \alpha(\langle x, x\rangle)^{3/2} = \tilde{\alpha}(x)^3,$$

so $\tilde{\alpha}$ is a C^*-seminorm on E. The first of the above inequalities implies that α is a norm whenever $\tilde{\alpha}$ so is and E is a faithful A-module. Finally, if α is a norm, it follows directly from (3.5) that $\tilde{\alpha}$ also is a norm when the condition $\langle x, x\rangle = 0$ implies $x = 0$. $\qquad\square$

Corollary 3.15. *If E is an admissible $*$-tring and $\gamma \in \mathcal{SN}(E)$, $\alpha \in \mathcal{SN}_{cs}(E_0^r)$, then $\widetilde{\gamma^r} = \gamma$ and $\tilde{\alpha}^r \leq \alpha$.*

Proof. The first statement follows immediately from (3.4) and (3.5). As for the second one we have $\tilde{\alpha}^r(a) = \sup\{\tilde{\alpha}(xa) : \tilde{\alpha}(x) \leq 1\} \leq \alpha(a)$ by 1. of 3.14. $\qquad\square$

Corollary 3.16. *Let $(E, A, \langle, \rangle)$ be a full basic triple, and $\alpha \in \mathcal{SN}_{cs}^{\langle,\rangle}(A)$. If $\tilde{\alpha} \in \mathcal{SN}(E)$ is given by (3.5), then $I_{N_{\tilde{\alpha}}} = N_\alpha$, where $I_{N_{\tilde{\alpha}}} := \{a \in A : Ea \subseteq N_{\tilde{\alpha}}\}$.*

Proof. The inclusion $N_\alpha \subseteq I_{N_{\tilde{\alpha}}}$ is clear because $\tilde{\alpha}(xa) \leq \tilde{\alpha}(x)\alpha(a)$, $\forall x \in E$, $a \in A$. Conversely, suppose that $a \in A$ is such that $\tilde{\alpha}(xa) = 0$, $\forall x \in E$. Then $\alpha(a^*\langle x, y\rangle a) = \alpha(\langle xa, ya\rangle) \leq \tilde{\alpha}(xa)\tilde{\alpha}(ya) = 0$, $\forall x, y \in E$. Now, since the triple is full, we can write $aa^* = \sum_j \langle x_j, y_j\rangle$, for certain $x_j, y_j \in E$, so we have:

$$0 \leq \alpha(a)^4 = \alpha(a^*a)^2 = \alpha(a^*aa^*a) = \alpha(a^* \sum_j \langle x_j, y_j\rangle a) \leq \sum_j \alpha(a^*\langle x_j, y_j\rangle a) = 0,$$

hence $a \in N_\alpha$. $\qquad\square$

Proposition 3.17. *Let $(E, A, \langle, \rangle)$ be a full basic triple, and $\alpha \in \mathcal{SN}_{cs}^{\langle,\rangle}(A)$. Let $\gamma := \tilde{\alpha} \in \mathcal{SN}(E)$, $\tilde{\alpha}$ given by (3.5). Then E_γ is a C^*-tring, $(E_\gamma^r, \bar{\gamma}^r) = (A_\alpha, \bar{\alpha})$ and $\tilde{\alpha}^r = \alpha$.*

Proof. Denote by $q : E \to E/N_\gamma \subseteq E_\gamma$ and $p : A \to A/N_\alpha \subseteq A_\alpha$ the corresponding canonical maps. We define $E/N_\gamma \times A/N_\alpha \to E/N_\gamma$ and $[,] : E/N_\gamma \times E/N_\gamma \to A/N_\alpha$ such that $q(x)p(a) := q(xa)$ and $[q(x), q(y)] := p(\langle x, y\rangle)$ respectively. Let us see that these operations are continuous in the norms $\bar{\gamma}$ and $\bar{\alpha}$. The action of A/N_α on E/N_γ is continuous, for if $x, y \in E$ and $a \in A$:

$$\bar{\gamma}(q(x)p(a)) = \bar{\gamma}(q(xa)) = \gamma(xa) \leq \gamma(x)\alpha(a) = \bar{\gamma}(q(x))\bar{\alpha}(p(a))$$

And the sesquilinear map $[,]_{E/N_\gamma}$ also is continuous, because:

$$\bar{\alpha}([q(x), q(y)]_{E/F_\gamma}) = \bar{\alpha}(p(\langle x, y\rangle_E)) = \alpha(\langle x, y\rangle_E) \leq \gamma(x)\gamma(y) = \bar{\gamma}(q(x))\bar{\gamma}(q(y)).$$

Therefore these operations extend to continuous maps $E_\gamma \times A_\alpha \to E_\gamma$ and $[,] : E_\gamma \times E_\gamma \to A_\alpha$, so we obtain a full C^*-basic triple $(E_\gamma, A_\alpha, [,])$. Therefore $(A_\alpha, \alpha) = (E_\gamma^r, \bar{\gamma}^r)$ by 3.9. As for the last assertion, we have to prove that $\gamma^r = \alpha$

or, equivalently, that $\bar{\gamma}^r = \bar{\alpha}$. So it is enough to show that $\gamma^r = \bar{\gamma}^r p$. But, if $a \in A$:

$$\bar{\gamma}^r(p(a)) = \sup\{\bar{\gamma}(q(x)p(a)) : \bar{\gamma}(q(x)) \leq 1\} = \sup\{\bar{\gamma}(q(xa)) : \gamma(x) \leq 1\} = \gamma^r(a).$$

$\square$

Propositions 3.5 and 3.14 allow us to define maps $\Phi_r : \mathcal{SN}(E) \to \mathcal{SN}_{cs}(E_0^r)$ and $\Psi_r : \mathcal{SN}_{cs}(E_0^r) \to \mathcal{SN}(E)$ such that $\Phi_r(\gamma) = \gamma^r$, given by (3.1), and $\Psi_r(\alpha) = \tilde{\alpha}$, given by (3.5). We want to show that in fact Φ_r and Ψ_r are mutually inverse maps that preserve the order.

Theorem 3.18. *Let E be an admissible $*$-tring. Then the maps $\Phi_r : \mathcal{SN}(E) \to \mathcal{SN}_{cs}(E_0^r)$ and $\Psi_r : \mathcal{SN}_{cs}(E_0^r) \to \mathcal{SN}(E)$ are mutually inverse isomorphisms of lattices. Moreover $\Phi_r(\mathcal{N}(E)) = \mathcal{N}_{cs}(E_0^r)$ and $\Psi_r(\mathcal{N}_{cs}(E_0^r)) = \mathcal{N}(E)$.*

Proof. By Corollary 3.15 we have $\Psi_r\Phi_r = Id_{\mathcal{SN}(E)}$, and Proposition 3.17 shows that $\Phi_r\Psi_r = Id_{\mathcal{SN}_{cs}(E_0^r)}$, so the maps Φ_r and Ψ_r are mutually inverse. Besides, it follows from 3.5 that $\Phi_r(\gamma)$ is a norm if and only if so is γ. On the other hand is clear that Ψ_r preserves the order, thus it remains to be shown that Φ_r also preserves the order. To this end consider $\gamma_1 \leq \gamma_2$ in $\mathcal{SN}(E)$. Since $id : (E, \gamma_2) \to (E, \gamma_1)$ is continuous, it induces a homomorphism $\pi : E_{\gamma_2} \to E_{\gamma_1}$, which in turn induces, according with Proposition 3.11, a homomorphism $\pi^r : E_{\gamma_2}^r \to E_{\gamma_1}^r$, which is necessarily contractive. Thus if $a \in E_0^r$, we have:

$$\gamma_1^r(a) = \bar{\gamma_1^r}(\pi^r(a + N_{\gamma_2^r})) \leq \bar{\gamma_2^r}(a + N_{\gamma_1^r}) = \gamma_2^r(a),$$

which shows that $\gamma_1^r \leq \gamma_2^r$. $\square$

All we have done to the right side can be done also to the left side. For example, every admissible $*$-tring E induces a (left) admissible and full basic triple $(E, E_0^l, \langle, \rangle_l)$, we have an isomorphism of posets $\Phi_l : \mathcal{SN}(E) \to \mathcal{SN}_{cs}(E_0^l)$ with inverse $\Psi_l : \mathcal{SN}_{cs}(E_0^l) \to \mathcal{SN}(E)$, given by $\Phi_l(\gamma) = \gamma^l$ and $\Psi(\alpha) = \tilde{\alpha}$, where $\gamma^l(a) := \sup\{\gamma(ax) : \gamma(x) \leq 1\}$ and $\tilde{\alpha}(x) := \alpha(\langle x, x\rangle_l)^{1/2}$, etc. Then we obtain the following consequences:

Corollary 3.19. *Let E be an admissible $*$-tring. Then $\Phi_r\Psi_l : \mathcal{SN}_{cs}(E_0^l) \to \mathcal{SN}_{cs}(E_0^r)$ is an isomorphism of lattices such that $\Phi_r\Psi_l(\mathcal{N}_{cs}(E_0^l)) = \mathcal{N}_{cs}(E_0^r)$. The inverse of $\Phi_r\Psi_l$ is $\Phi_l\Psi_r$.*

As mentioned at the end of 3.1 in [12][Theorem 3.1], Zettl proved that any C^*-tring is of the form $E = E_+ \oplus E_-$, where E_+ and E_-^{op} are isomorphic to a TRO. In fact we have $E_+ := \{x \in E : \langle x, x\rangle_r \text{ is positive}\}$, $E_- := \{x \in E : -\langle x, x\rangle_r \text{ is positive}\}$, and E_+ and E_- are ideals of E such that $\langle E_+, E_-\rangle = 0$. If $E_p := E_+ \oplus E_-^{op}$, we will have that $E_p^r = E^r$ and $E_p^l = E^l$, and now E_p is a Morita–Rieffel equivalence between E^l and E^r. Thus we have:

Corollary 3.20. *Let E be an admissible $*$-tring and $\gamma \in \mathcal{SN}(E)$. Then E_γ^l and E_γ^r are Morita–Rieffel equivalent C^*-algebras.*

In general we will have to deal with algebras that strictly contain E_0^r, but whose C^*-seminorms are essentially the same, as the following results show.

Proposition 3.21. *Let I be a selfadjoint ideal of a $*$-algebra A, and suppose that $\alpha \in \mathcal{SN}(I)$. Let $\alpha' : A \to [0, \infty]$ be given by $\alpha'(a) := \sup\{\alpha(ax) : x \in I, \alpha(x) \leq 1\}$. For every $a \in A$ consider $L_a : I \to I$, such that $L_a(x) = ax$, $\forall x \in I$. Then the following statements are equivalent:*

(1) $\alpha'(a) < \infty$, $\forall a \in A$.
(2) L_a is bounded, $\forall a \in A$.
(3) α can be extended to a C^*-seminorm on A.

Suppose that one of the conditions above holds true. Then:

(a) α' is a C^*-seminorm on A, and $\alpha' \leq \beta$ for every $\beta \in \mathcal{SN}(A)$ that extends α.
(b) If α is a norm, then α' is a norm if and only if $Ann_A(I) = 0$, where $Ann_A(I) := \{a \in A : ax = 0, \forall x \in I\}$.

Proof. Since $\|L_a\| = \alpha'(a)$, we have that conditions 1. and 2. are equivalent. It is also clear that 3. $\Rightarrow$ 1. Suppose now that $\alpha'(a) < \infty$, $\forall a \in A$. Let show that α' is a C^*-seminorm on A that extends α. It is easy to check that $\alpha'(ab) \leq \alpha'(a)\alpha'(b)$, $\forall a, b \in A$. Moreover:

$$\alpha'(a^*a) = \sup\{\alpha(a^*ax) : x \in I, \alpha(x) \leq 1\} \geq \sup\{\alpha(x^*a^*ax) : x \in I, \alpha(x) \leq 1\}$$
$$\geq \sup\{\alpha(ax)^2 : x \in I, \alpha(x) \leq 1\} = \alpha'(a)^2.$$

Therefore $\alpha' \in \mathcal{SN}(A)$. The fact that α' extends α, as well as assertion (a), are consequences of the fact that for every C^*-seminorm β on A one has that $\beta(a) = \sup\{\beta(ab) : \beta(b) \leq 1\}$. Finally, suppose that α is a norm on I. Then $\alpha'(a) = 0 \iff \alpha(ax) = 0$, $\forall x \in I$, that is $\alpha'(a) = 0 \iff a \in Ann_A(I)$. □

Theorem 3.22. *Let $(E, A, \langle\,,\,\rangle)$ be an admissible basic triple, with E a faithful A-module, and admissible as $*$-tring. Suppose that any C^*-seminorm on E_0^r can be extended in a unique way to a C^*-seminorm on A (recall Corollary 2.12). Then the lattices $\mathcal{SN}(E)$ and $\mathcal{SN}_{cs}^{\langle,\rangle}(A)$ are isomorphic. If in addition $Ann_A(E_0^r) = 0$, the posets $\mathcal{N}(E)$ and $\mathcal{N}_{cs}^{\langle,\rangle}(A)$ are isomorphic as well.*

Proof. Since any C^*-seminorm on E_0^r can be uniquely extended to a C^*-seminorm on A, we are allowed to identify $\mathcal{SN}(A)$ and $\mathcal{SN}(E_0^r)$ as lattices, and it is clear that this yields also an identification between $\mathcal{SN}_{cs}^{\langle,\rangle}(A)$ and $\mathcal{SN}_{cs}(E_0^r)$, and the latter is isomorphic to $\mathcal{SN}(E)$ by 3.18. If moreover $Ann_A(E_0^r) = 0$, the same argument applies to $\mathcal{N}(E)$ and $\mathcal{N}_{cs}(A)$. □

In case A is a Banach $*$-algebra, any C^*-seminorm on a $*$-ideal can be extended to a C^*-seminorm defined on the whole algebra. Moreover we have:

Proposition 3.23. *Let A be an admissible Banach $*$-algebra and I a dense $*$-ideal of A, not necessarily closed. Then any C^*-norm on I can be uniquely extended to a C^*-norm on A.*

Proof. Let $\alpha \in \mathcal{N}(I)$. By 3.7 α has a unique extension to a C^*-seminorm on A, and by 3.21 this extension must be α' such that $\alpha'(a) = \sup\{\alpha(ax) : x \in I, \alpha(x) \leq 1\}$. Suppose $a \in Ann_A(I)$. Then $aa^* = 0$, because I is dense in A and $ax = 0$, $\forall x \in I$. Thus $a = 0$ for A is admissible. Then α' is a norm by 3.21. □

Corollary 3.24. *Let $(E, A, \langle\,,\,\rangle_E)$ be an admissible basic triple with A a Banach $*$-algebra and E a faithful A-module. Suppose in addition that E is an admissible $*$-tring such that E_0^r is a dense ideal of A (recall Corollary 2.12). Then the lattices $\mathcal{SN}(E)$ and $\mathcal{SN}_{cs}(A)$ are isomorphic, as well as the partially ordered sets $\mathcal{N}(E)$ and $\mathcal{N}_{cs}(A)$.*

Proof. Just combine Theorem 3.22 with Proposition 3.7 and Proposition 3.23. $\square$

Corollary 3.25. *Let $(E, A, \langle\,,\,\rangle_A)$ and $(E, B, \langle\,,\,\rangle_B)$ be respectively left and right admissible basic triples, with A and B Banach $*$-algebras such that E is an $(A-B)$-bimodule with the given structure, and $\langle x, y\rangle_A z = x\langle y, z\rangle_B$, $\forall x, y, z \in E$. If E is faithful as a left A-module and as a right B-module, and E_0^l and E_0^r are dense in A and B respectively, then there is an isomorphism of lattices between $\mathcal{SN}_{cs}^{\langle,\rangle_A}(A)$ and $\mathcal{SN}_{cs}^{\langle,\rangle_B}(B)$, that restricts to an isomorphism between the posets $\mathcal{N}_{cs}^{\langle,\rangle_A}(A)$ and $\mathcal{N}_{cs}^{\langle,\rangle_B}(B)$.*

3.3. **Positive modules.** In general is not a simple task to decide if a given C^*-seminorm satisfies the Cauchy-Schwarz property with respect to a certain sesquilinear map. However this is always the case for the positive modules we introduce next.

Let α be a C^*-seminorm on the $*$-algebra A, and let $p_\alpha : A \to A_\alpha$ be the canonical map, where A_α is the Hausdorff completion of A. If $\Lambda \subseteq \mathcal{SN}(A)$, then $A_\Lambda^+ := \cap_{\alpha \in \Lambda} p_\alpha^{-1}(A_\alpha^+)$ is a cone. When $\Lambda = \mathcal{SN}(A)$, we write A^+ instead of A_Λ^+. Therefore A^+ is the set of elements of A that are positive in any C^*-Hausdorff completion of A. Of course the map $\Lambda \mapsto A_\Lambda^+$ is order reversing.

Definition 3.26. Given $\Lambda \subseteq \mathcal{SN}(A)$, we say that $a \in A$ is positive in (A, Λ), or that it is Λ-positive, if $a \in A_\Lambda^+$. The elements of A^+ are just called the positive elements of A.

It is clear that A^+ contains the cone $C_A := \{\sum_{i,j=1}^n a_i^* a_j : n \in \mathbb{N}, a_i \in A, i = 1, \ldots n\}$, and that $p_\alpha(C_A)$ is dense in A_α^+, $\forall \alpha \in \mathcal{SN}(A)$. Also note that if $\phi : A \to B$ is a homomorphism between $*$-algebras, then $\phi(A^+) \subseteq B^+$ and $\phi(C_A) \subseteq C_B$.

If $\mathcal{SN}(A)$ is bounded, with $\alpha := \max \mathcal{SN}(A)$, then a is positive in A if and only if a is positive in (A, α). In particular, if A is a Banach $*$-algebra, then $a \in A^+$ if and only if $\iota(a) \in C^*(A)^+$, where $\iota : A \to C^*(A)$ is the natural map of A into its C^*-enveloping algebra $C^*(A)$.

Lemma 3.27. *Let A be C^*-closable. Then $A^+ = \bigcap\{p_\alpha^{-1}(A_\alpha^+) : \alpha \in \mathcal{N}(A)\}$.*

Proof. Clearly we have that $A^+ \subseteq \bigcap\{p_\alpha^{-1}(A_\alpha^+) : \alpha \in \mathcal{N}(A)\}$. Let $\beta \in \mathcal{SN}(A)$. Since the maximum of two C^*-seminorms is again a C^*-seminorm, and since A is C^*-closable, we may pick $\beta' \in \mathcal{N}(A)$ such that $\beta' \geq \beta$. Then the identity map on A induces a homomorphism $\phi : A_{\beta'} \to A_\beta$, determined by $\phi(p_{\beta'}(a)) = p_\beta(a)$, $\forall a \in A$. If $a \in \bigcap\{p_\alpha^{-1}(A_\alpha^+) : \alpha \in \mathcal{N}(A)\}$ then $p_{\beta'}(a) \in A_{\beta'}^+$, and therefore $p_\beta(a) \in A_\beta^+$. This proves the converse inclusion. $\square$

Once we have a cone of positive elements on a $*$-algebra A, we are able to define a notion similar to that of Hilbert module.

Definition 3.28. Let A be a $*$-algebra, E a right A-module, and $\Lambda \subseteq \mathcal{SN}(A)$. We say that a map $\langle \cdot, \cdot \rangle : E \times E \to A$ is a Λ-semi-pre-inner product on E if:

(1) $\langle x, \lambda_1 y + \lambda_2 z \rangle = \lambda_1 \langle x, y \rangle + \lambda_2 \langle x, z \rangle, \ \forall x, y, z \in E, \ \lambda_1, \lambda_2 \in \mathbb{C}$.

(2) $\langle x, ya \rangle = \langle x, y \rangle a, \ \forall x, y \in E, \ a \in A$.

(3) $\langle y, x \rangle = \langle x, y \rangle^*, \ \forall x, y \in E$.

(4) $\langle x, x \rangle$ is Λ-positive, $\forall x \in E$.

The pair $(E, \langle \, , \, \rangle)$ is then called a right positive Λ-module. In case $\Lambda = \mathcal{SN}(A)$ we say that $(E, \langle \, , \, \rangle)$ is a right positive A-module.

Similarly we define left semi-pre-inner-products and left positive modules.

Definition 3.29. An admissible $*$-tring E is right (left) positive if $(E, \langle \, , \, \rangle_r)$ is a positive E_0^r-module (respectively: $(E, \langle \, , \, \rangle_l)$ is a positive E_0^l-module). It is said positive if it is both left and right positive.

Observe that if E is a C^*-tring, which is positive as an admissible $*$-tring, then it is obviously a positive C^*-tring. Conversely, it is readily checked that any positive C^*-tring is a positive admissible $*$-tring.

Proposition 3.30. *Let (A, α) be a C^*-seminormed algebra and $(E, \langle \, , \, \rangle)$ a right positive (A, α)-module. Let $\tilde{\alpha} : E \to [0, \infty)$ be given by $\tilde{\alpha}(x) = \sqrt{\alpha(\langle x, x \rangle)}$, $\forall x \in E$. Consider E as a $*$-tring with $(x, y, z) := x\langle y, z \rangle, \ \forall x, y, z \in E$. Then:*

(1) We have $\alpha(a) \leq \alpha(b)$ whenever a and $b - a$ are positive elements of A.

(2) $\tilde{\alpha}(x)^2 \langle y, y \rangle - \langle x, y \rangle^* \langle x, y \rangle$ is positive in (A, α), and $\alpha(\langle x, y \rangle) \leq \tilde{\alpha}(x)\tilde{\alpha}(y)$, $\forall x, y \in E$ (Cauchy-Schwarz).

(3) $\alpha(\langle x, x \rangle)a^* a - a^* \langle x, x \rangle a \geq 0, \ \forall x \in E, \ a \in A$.

(4) $\tilde{\alpha}(xa) \leq \tilde{\alpha}(x)\alpha(a), \ \forall x \in E, \ a \in A$.

(5) $\tilde{\alpha} \in \mathcal{SN}(E)$.

Proof. Let $p_\alpha : A \to A/I_\alpha =: A_\alpha$ be the natural map, where I_α is the ideal $I_\alpha := \{a \in A : \alpha(a) = 0\}$, and let $\bar{\alpha}$ be the quotient norm on A_α. Now let $F := \operatorname{span}\{xb \in E : x \in E, b \in I_\alpha\}$. Then $EI_\alpha \subseteq F$, so E/F is an A/I_α-module. Moreover, $\langle E, F \rangle \subseteq I_\alpha$ and $\langle F, E \rangle \subseteq I_\alpha$, so we can consider the map $[\, , \,] : E/F \times E/F \to A/I_\alpha$ given by $[q(x), q(y)] = p_\alpha(\langle x, y \rangle)$, which satisfies properties 1.–4. of Definition 3.28 above. If a and $b - a$ are positive in A, then $0 \leq p_\alpha(a) \leq p_\alpha(b)$ in A_α, and therefore $\bar{\alpha}(p_\alpha(a)) \leq \bar{\alpha}(p_\alpha(b))$, that is $\alpha(a) \leq \alpha(b)$. This proves 1. Now, the first part of the second statement follows from the proof of [6, Proposition 1.1], since $p_\alpha(\tilde{\alpha}(x)^2 \langle y, y \rangle - \langle y, x \rangle \langle x, y \rangle) = \bar{\alpha}([q(x), q(x)])[q(y), q(y)] - [q(y), q(x)] [q(x), q(y)]$ in A_α. The second part of 2. follows from the first one and from 1. To see 3. just observe that by applying p_α to the element $\alpha(\langle x, x \rangle)a^* a - a^* \langle x, x \rangle a$ of A we get the positive element $\bar{\alpha}([x, x])p_\alpha(a)^* p_\alpha(a) - p_\alpha(a)^* [x, x]p_\alpha(a)$ of A_α. Assertion 4. easily follows from 1. and 3: by 3. we have $a^* \langle x, x \rangle a \leq \tilde{\alpha}(x)^2 a^* a$, then $\tilde{\alpha}(xa)^2 = \alpha(\langle xa, xa \rangle) = \alpha(a^* \langle x, x \rangle a)$, and by 1. this is less or equal to $\alpha(\tilde{\alpha}(x)^2 a^* a) = \tilde{\alpha}(x)^2 \alpha(a)^2$. It is clear that $\tilde{\alpha}(\lambda x) = |\lambda|\tilde{\alpha}(x), \ \forall x \in E$, $\lambda \in \mathbb{C}$, and from the Cauchy-Schwarz inequality just proved it readily follows that $\tilde{\alpha}$ also satisfies the triangle inequality, so it is a seminorm on E. Now, if $x, y, z \in E$: $\tilde{\alpha}(x\langle y, z \rangle)^2 = \alpha(\langle y, z \rangle^* \langle x, x \rangle \langle y, z \rangle)$. Thus, in the case $x = y = z$:

$$\tilde{\alpha}((x, x, x)) = \alpha(\langle x, x \rangle^3)^{1/2} = \alpha(\langle x, x \rangle^{1/2})^3 = \tilde{\alpha}(x)^3.$$

According to 3. we have $\langle y,z\rangle^*\langle x,x\rangle\langle y,z\rangle \leq \alpha(\langle x,x\rangle)\langle y,z\rangle^*\langle y,z\rangle$ in (A,α). From this fact, together with 4. and the Cauchy-Schwarz inequality we conclude that

$$\tilde{\alpha}(x\langle y,z\rangle)^2 \leq \tilde{\alpha}(x)^2\alpha(\langle y,z\rangle)^2 \leq (\tilde{\alpha}(x)\,\tilde{\alpha}(y)\,\tilde{\alpha}(z))^2$$

so $\tilde{\alpha}$ is a C^*-seminorm on E. $\qquad\square$

Corollary 3.31. *If E is a right positive $*$-tring, then $\mathcal{SN}_{cs}(E_0^r) = \mathcal{SN}(E_0^r)$, and $\mathcal{SN}(E) \cong \mathcal{SN}(E_0^r)$ and $\mathcal{N}(E) \cong \mathcal{N}(E_0^r)$ as ordered sets.*

Proposition 3.32. *Let E be an admissible $*$-tring and $\gamma \in \mathcal{SN}(E)$. If E is a right positive (E_0^r, γ^r)-module, then E is also a left positive (E_0^l, γ^l)-module. Therefore E is right positive if and only if is left positive.*

Proof. Let E_γ be the Hausdorff completion of (E,γ). Since E_γ is a right Hilbert module over E_γ^r, it turns out that E_γ is a positive C^*-tring, and therefore a left Hilbert module over E_γ^l, so E is a left positive (E_0^l, γ^l)-module. $\qquad\square$

Proposition 3.33. *Let B be an admissible Banach $*$-algebra and suppose E is a right closed ideal of B such that $\mathrm{span}\{x^*y : x,y \in E\}$ is dense in B. Let A be the closure in B of $\mathrm{span}\{xy^* : x,y \in E\}$. If xx^* is positive in A, $\forall x \in E$, then the restriction map $\varphi : \mathcal{SN}(B) \to \mathcal{SN}(A)$, $\beta \mapsto \beta|_A$, is a lattice isomorphism such that $\varphi(\mathcal{N}(B)) = \mathcal{N}(A)$, and for each $\beta \in \mathcal{SN}(B)$ the Hausdorff completion B_β of B is Morita–Rieffel equivalent to the Hausdorff completion $A_{\varphi(\beta)}$ of A. In particular, the corresponding enveloping C^*-algebras $C^*(B)$ and $C^*(A)$ of B and A are Morita–Rieffel equivalent C^*-algebras.*

Proof. Let $\langle\,,\,\rangle_B : E \times E \to B$ and $\langle\,,\,\rangle_A : E \times E \to A$ be such that $\langle x,y\rangle_B = x^*y$ and $\langle x,y\rangle_A = xy^*$. Then E is both a positive B-module and a positive A-module. Since B is admissible, so are E and A. Besides E is a faithful B-module, for if $xb = 0$ $\forall x \in E$, then $\sum_j x_j^* y_j b = 0$ $\forall x_j, y_j \in E$, so $b^*b = 0$, and this implies $b = 0$ because B is admissible. Similarly, E is a faithful A-module. It follows by 2.12 that we can identify E_0^r with $\mathrm{span}\{x^*y : x,y \in E\}$ and E_0^l with $\mathrm{span}\{xy^* : x,y \in E\}$. Now the proof ends with an invocation to Corollary 3.25 $\qquad\square$

4. C^*-TERNARY RINGS

As previously mentioned, Zettl found a unique decomposition $E = E_+ \bigoplus E_-$ of any C^*-tring E, E_+ being isomorphic to a TRO and E_- being isomorphic to an anti-TRO (see the discussion preceding Corollary 3.20). Of course, because of the uniqueness of the fundamental decomposition, there is a left version of the stuation above: $E_+ := \{x \in E : \langle x,x\rangle_l \in E_+^l\}$, $E_- := \{x \in E : \langle x,x\rangle_l \in -E_+^l\}$, $\langle E_+, E_-\rangle_l = 0$, $E^l = E_+^l \oplus E_-^l$, and $(E_+, -\langle\cdot,\cdot\rangle_l)$ and $(E_-, -\langle\cdot,\cdot\rangle_l)$ are full left Hilbert E_+^l and E_-^l modules respectively. This way, E is an $(E^l - E^r)$ Banach bimodule that satisfies

$$\langle x,y\rangle_l z = \mu(x,y,z) = x\langle y,z\rangle_r, \ \forall x,y,z \in E.$$

If E is a C^*-tring, we define $E_p := E_+ \oplus E_-^{\mathrm{op}}$. Then E_p is a positive C^*-tring, and $E_p^r = E^r$, $E_p^l = E^l$. Therefore E_p is a $(E^l - E^r)$-imprimitivity bimodule, so in

particular E^l and E^r are Morita–Rieffel equivalent. Note also that if $\phi : E \to F$ is a homomorphism of C^*-trings, then $\phi(E_+) \subseteq F_+$ and $\phi(E_-) \subseteq F_-$, because $\langle \phi(x), \phi(x) \rangle = \phi^r(\langle x, x \rangle)$. Therefore $\phi : E_p \to F_p$ is also a homomorphism of C^*-trings. Thus $E \mapsto E_p$ is a functor.

Let E^* be the reverse $*$-tring of E. It is clear that a norm on E is a C^*-norm if and only if it is a C^*-norm on E^*. Moreover, E is a (positive) C^*-tring if and only if so is E^*, and $E^l = (E^*)^r$, $E^r = (E^*)^l$. Note that E and E^* are essentially the same object as C^*-trings. Thus the properties of E^r and E^l deduced from properties of E will be the same.

Definition 4.1. By a left (right) ideal of the C^*-ternary ring E we mean a closed subspace F of E such that $(E, E, F) \subseteq F$ (respectively: $(F, E, E) \subseteq F$). An ideal of E is both a left and a right ideal of E. We denote by $L(E)$, $R(E)$, and $I(E)$ the families of left, right and twosided ideals of E.

Our definition of ideal, for a closed subspace F of E, is equivalent to the definition which just requires the condition $(E, F, E) \subset F$ to be satisfied. Note that E_+ and E_- are ideals in every C^*-tring E. Moreover, since E_+ and E_- are orthogonal, it easily follows that a closed subspace F of E is an ideal of E if and only if it is an ideal in E_p. Thus the ideal structures of E and of E_p are the same.

If A is a C^*-algebra, we will denote by $I(A)$ and $H(A)$ respectively the families of (closed) twosided ideals and hereditary C^*-subalgebras of A.

As in the algebraic case, if E is a C^*-tring and F is a sub-C^*-tring of E, then the subalgebra $\overline{\mathrm{span}}\langle F, F \rangle_r$ of E^r may be taken to represent the C^*-algebra F^r. With this choice of F^r we have the following result:

Proposition 4.2. *The map $L(E) \to H(E^r)$ given by $F \mapsto F^r$ is a bijection, with inverse given by $A \mapsto EA$. When restricted to $I(E)$, the map $F \mapsto F^r$ is a bijection onto $I(E^r)$. Moreover, all of these maps are lattice isomorphisms.*

Proof. We prove that the map $L(E) \to H(E^r)$ is a bijection. Recalling that we may replace E by E_p (which can be seen as a full right Hilbert E^r-module), the rest of the proof follows from [8, 3.22]. If A is a C^*-subalgebra of E^r: $(E, E, EA) = E\langle E, EA \rangle = E\langle E, E \rangle A = (E, E, E)A \subseteq EA$, so EA is a left ideal in E. Conversely, if F is a left ideal in E:

$$\langle F, F \rangle \langle E, E \rangle \langle F, F \rangle = \langle\, E\langle F, F \rangle\,,\, E\langle F, F \rangle\,\rangle = \langle (E, F, F), (E, F, F) \rangle \subseteq \langle F, F \rangle.$$

Thus, taking the closed linear spans in both sides of the above inclusion we have: $F^r E^r F^r = F^r$, which shows that F^r is hereditary. To see that the correspondences are mutually inverses, note that if F is a C^*-tring, then $F = FF^r$. On the other hand, if A is a hereditary C^*-subalgebra of E^r, then $EA = \overline{\mathrm{span}}\langle EA, EA \rangle_r = \overline{\mathrm{span}}A\langle E, E \rangle_r A = AE^r A = A$. $\qquad\square$

Corollary 4.3. *Let $\pi : E \to F$ be a homomorphism of $*$-trings, where E and F are C^*-trings. Then $(\ker \pi)^r = \ker(\pi^r)$.*

Proof. It is clear that $\ker \pi \supseteq E \ker \pi^r$, so $(\ker \pi)^r \supseteq \ker \pi^r$. On the other hand $(\ker \pi)^r = \overline{\mathrm{span}}\{\langle x, y \rangle_r : x, y \in \ker \pi\} \subseteq \ker \pi^r$. $\qquad\square$

Remark 4.4. By Proposition 2.17 if $\pi : E \to F$ is a surjective homomorphism between C^*-trings, then $\pi_0^r : E_0^r \to F_0^r$ is also surjective, so also is $\pi^r : E^r \to F^r$ for the image of π^r is closed. However the converse is false: consider the Hilbert space inclusion $\mathbb{C} \overset{\iota}{\hookrightarrow} \mathbb{C}^2$; then ι is not onto, although ι^r is the identity on $\mathbb{C}$.

For a proof of the next result the reader is referred to [8, 3.25].

Proposition 4.5. *Let F be an ideal of a C^*-tring E, and consider the quotient E/F with its natural structure of $*$-tring. Then E/F is a C^*-tring with the quotient norm, and $(E/F)^r = E^r/F^r$.*

Corollary 4.6. *Let E and G be C^*-trings, and $\pi : E \to G$ a homomorphism of $*$-trings. Consider $F = \ker(\pi)$, and let $p : E^r \to E^r/F^r$ be the quotient map. Then there exists a unique homomorphism of C^*-algebras $\overline{\pi^r} : E^r/F^r \to G^r$ such that $\overline{\pi^r} p = \pi^r$. The homomorphism $\overline{\pi^r}$ is injective. In particular, if $\pi : E \to E/F$ is the quotient map, where F is an ideal of E, then $\overline{\pi^r} : E^r/F^r \to (E/F)^r$ is a natural isomorphism.*

Proof. Proposition 3.11 provides a unique homomorphism of C^*-algebras $\pi^r : E^r \to G^r$ such that $\langle \pi(x), \pi(y) \rangle = \pi^r(\langle x, y \rangle)$, $\forall x, y \in E$. The existence and uniqueness of $\overline{\pi^r}$, as well as its injetivity, follow now from the quotient universal property, together with the fact that $\ker(\pi^r) = F^r$ by Corollary 4.3. Finally, if F is an ideal of E, by Proposition 4.5 we have that E/F is a C^*-tring, and the projection $\pi : E \to E/F$ is a homomorphism of $*$-trings. $\qquad\square$

Corollary 4.7. *The functor $E \mapsto E^r$, $\pi \mapsto \pi^r$, from the category of C^*-trings into the category of C^*-algebras, is exact. More precisely: if*

$$0 \longrightarrow F_1 \overset{\phi}{\longrightarrow} F_2 \overset{\psi}{\longrightarrow} F_3 \longrightarrow 0$$

is an exact sequence of C^*-trings, then the sequence:

$$0 \longrightarrow F_1^r \overset{\phi^r}{\longrightarrow} F_2^r \overset{\psi^r}{\longrightarrow} F_3^r \longrightarrow 0$$

also is exact.

Corollary 4.8. *If $\pi : E \to F$ is a homomorphism of C^*-trings, then $\pi(E)$ is closed in F. The ideals of a C^*-tring E are exactly the kernels of the homomorphisms defined on E.*

5. Applications

5.1. C^*-algebras associated with Fell bundles. The proof of Theorem 1.1 of [2] relies on the existence of a certain inner product (see Corollary 5.3 below), although no proof is included there of the fact that such inner product is indeed positive. In the following lines we provide such a proof, and we refine the above mentioned result.

Recall that a right ideal $\mathcal{E} = (E_t)_{t \in G}$ of a Fell bundle $\mathcal{B} = (B_t)_{t \in G}$ is a sub-Banach bundle of $\mathcal{B}$ such that $\mathcal{E}\mathcal{B} \subseteq \mathcal{E}$.

Given a right Hilbert B-module X, let denote by D_X the cone of finite sums $\sum_i \langle x_i, x_i \rangle \subseteq B^+$. It is clear that if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of right Hilbert B-modules and $X := \oplus_\lambda X_\lambda$ (direct sum of Hilbert modules), then $\sum_\lambda D_{X_\lambda} \subseteq D_X$ -with equality if Λ is finite- and $\sum_\lambda D_{X_\lambda}$ is dense in D_X.

Similarly, for the right ideal $\mathcal{E}$ of the Fell bundle $\mathcal{B}$, we define $D_\mathcal{E} := \{\sum_{i=1}^n c_i^* c_i : n \in \mathbb{N}, c_i \in \mathcal{E}, \forall i\} \subseteq B_e^+$. Then we have:

Lemma 5.1. *Let $\mathcal{E} = (E_t)_{t \in G}$ be a right ideal of the Fell bundle $\mathcal{B} = (B_t)_{t \in G}$. Then $\mathrm{span}(\mathcal{E}^*\mathcal{E} \cap B_e)$ is dense in B_e if and only if the cone $D_\mathcal{E}$ satisfies the following property:*

$$\forall b \in B_e, \ \epsilon > 0, \ \text{there exists } d \in D_\mathcal{E} \text{ such that } \|d\| \leq 1 \text{ and } \|b - bd\| < \epsilon. \quad (5.1)$$

Proof. Suppose that $b \in B_e$ is such that for any $\epsilon > 0$ there exists $d \in D_\mathcal{E}$ such that $\|b - bd\| < \epsilon$. Since $D_\mathcal{E} \subseteq \mathrm{span}(\mathcal{E}^*\mathcal{E} \cap B_e)$ and the latter is an ideal in B_e, we conclude that $b \in \overline{\mathrm{span}}(\mathcal{E}^*\mathcal{E} \cap B_e)$. Then $\mathrm{span}(\mathcal{E}^*\mathcal{E} \cap B_e)$ is dense in B_e whenever $D_\mathcal{E}$ satisfies (5.1). Note now that $D_\mathcal{E} = \sum_{t \in G} D_{E_t}$, which is dense in D_E, where $E := \oplus_{t \in G} E_t$. Thus $D_\mathcal{E}$ satisfies (5.1) if and only if that property holds for D_E. Assume that $\mathrm{span}(\mathcal{E}^*\mathcal{E} \cap B_e)$ is dense in B_e. Then E is a full Hilbert module over B_e, and therefore it satisfies (5.1) by [6, (ii) of Lemma 7.2]. $\quad \square$

Lemma 5.2. *Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over the locally compact group G, $\mathcal{A} = (A_t)$ a sub-Fell bundle of $\mathcal{B}$, and $\mathcal{E} = (E_t)$ a right ideal of $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{E}$, $\mathcal{E}\mathcal{E}^* \subseteq \mathcal{A}$ and $\mathrm{span}(\mathcal{E}^*\mathcal{E} \cap B_e)$ is dense in B_e. If $\xi \in L^1(\mathcal{E})$, then $\xi * \xi^*$ can be arbitrarily approximated in $L^1(\mathcal{A})$ by a finite sum $\sum_{j=1}^m \eta_j * \eta_j^*$, where $\eta_j \in L^1(\mathcal{A})$, $\forall j = 1, \ldots, m$.*

Proof. We will suppose that $\xi \in C_c(\mathcal{E})$, which is clearly enough. Since $C_0(\mathcal{E})$ is a nondegenerate right Banach B_e-module, given a positive integer n there exists $b_n \in B_e$ such that $\|\xi - \xi b_n\| < 1/n$ and $0 \leq b_n \leq 1$. Then we can find $c_n \in D_\mathcal{E}$ such that $\|b_n^{1/2} - b_n^{1/2} c_n\| < 1/n$. Set $d_n := b_n^{1/2} c_n b_n^{1/2}$ and note that $d_n \in D_\mathcal{E}$ because $\mathcal{E}$ is a right ideal. The continuity of the operations imply $\|b_n - d_n\| \to 0$ and $\|\xi - \xi d_n\|_1 \to 0$. Thus $\|\xi * \xi^* - \xi d_n * \xi^*\|_1 \to 0$.

Now for every n there exist $u_1, \ldots, u_{m_n} \in \mathcal{E}$ such that $d_n = \sum_{j=1}^{m_n} u_j^* u_j$. Thus $\xi d_n * \xi^* = \sum_{j=1}^{m_n} (\xi u_j^* u_j) * \xi^* = \sum_{j=1}^{m_n} (\xi u_j^*) * (\xi u_j^*)^*$ and, as $\mathcal{E}$ is a right ideal, $\xi u_j^* \in C_c(\mathcal{A})$. This completes the proof. $\quad \square$

Corollary 5.3. *Under the assumptions of Lemma 5.2, let $\| \ \|_\mathcal{A} : L^1(\mathcal{A}) \to [0, \infty)$ be the maximal C^*-norm of $L^1(\mathcal{A})$. Then $L^1(\mathcal{E}) \times L^1(\mathcal{E}) \to L^1(\mathcal{A})$ given by $(\xi, \eta) \mapsto \xi * \eta^*$ is an inner product.*

Corollary 5.4. *Under the assumptions of Lemma 5.2, the map $\varphi : \mathcal{SN}(L^1(\mathcal{B})) \to \mathcal{SN}(L^1(\mathcal{B}))$ given by $\beta \mapsto \beta|_{L^1(\mathcal{A})}$ is an isomorphism of partially ordered sets that sends the maximal and reduced norms on $L^1(\mathcal{B})$ to the maximal and reduced norms on $L^1(\mathcal{A})$ respectively, and such that $\varphi(\mathcal{N}(L^1(\mathcal{B}))) = \mathcal{N}(L^1(\mathcal{A}))$. Moreover, the Hausdorff completions of $L^1(\mathcal{B})$ and $L^1(\mathcal{A})$ with respect to β and $\varphi(\beta)$ respectively are Morita–Rieffel equivalent.*

Proof. We only have to prove the correspondence between the reduced C^*-norms, but this is the content of [2]. $\quad \square$

5.2. **Tensor products of C^*-trings.** In the present section we apply the previous results to the study of tensor products of C^*-trings. Maximal and minimal tensor product for TROs were constructed in [5] using linking algebras, but we define tensor products of C^*-trings E and F using the tensor products of E^r and F^r. The main result is Theorem 5.12.

From now on the algebraic tensor product of the $\mathbb{C}$-vector spaces $E_1, \ldots, E_n$ will be denoted by $E_1 \odot \ldots \odot E_n$, or just by $\odot_{j=1}^n E_j$. Let E_{ij}, F_i be complex vector spaces, $\forall i = 1, \ldots, m$, $j = 1, \ldots, n$, and suppose that $\alpha_i : \prod_{j=1}^n E_{ij} \to F_i$ is a n-linear map, for each $i = 1, \ldots, m$. Then it is clear that there exists a unique n-linear map $\alpha := \alpha_1 \odot \cdots \odot \alpha_m : \prod_{j=1}^n \odot_{i=1}^m E_{ij} \to \odot_{i=1}^m F_i$ such that $\alpha(\odot_{i=1}^m e_{i1}, \ldots, \odot_{i=1}^m e_{in}) = \odot_{i=1}^m \alpha_i(e_{i1}, \ldots, e_{in})$. Using this fact we have the following result, whose straightforward proof is left to the reader.

Proposition 5.5. *If (E, μ), (F, ν) are $*$-trings, then $(E \odot F, \mu \odot \nu)$ is also a $*$-tring. Furthermore, if $(E, A, \langle, \rangle_A)$ and $(F, B, \langle, \rangle_B)$, are full basic triples associated to (E, μ) and (F, ν), respectively, then $(E \odot F, A \odot B, \langle, \rangle_A \odot \langle, \rangle_B)$ is a full basic triple associated to $(E \odot F, \mu \odot \nu)$.*

Definition 5.6. A C^*-tensor product of two $*$-trings $(E, \mu, \|\cdot\|)$ and $(F, \nu, \|\cdot\|)$ is a completion of the corresponding algebraic tensor product $(E \odot F, \mu \odot \nu)$ with respect to a C^*-norm. If γ is such a C^*-norm, we denote by $E \otimes_\gamma F$ the corresponding C^*-tensor product.

Definition 5.7. We say that a C^*-tring E is nuclear if for every C^*-tring F there exists just one C^*-tensor product $E \otimes F$.

We will see next that $\mathcal{SN}(E \odot F) = \mathcal{SN}(E_p \odot F)$, which implies, in particular, that a C^*-tring E is nuclear if and only if E_p is nuclear.

Proposition 5.8. *Let E be a $*$-tring, and F_1, F_2 ideals of E such that $E = F_1 \oplus F_2$. If $\gamma \in \mathcal{SN}(E)$, and $x = y + z$, with $y \in F_1$ and $z \in F_2$, then $\gamma(x) = \max\{\gamma(y), \gamma(z)\}$.*

Proof. Since $\gamma(x) = \sup\{\gamma((x, u, u)) : u \in E, \gamma(u) \leq 1\}$, it follows that $\gamma(x) \geq \gamma(y)$ and $\gamma(x) \geq \gamma(z)$, so $\gamma(x) \geq \max\{\gamma(y), \gamma(z)\}$. To prove the converse inequality, let us first introduce the following notation. For $u \in E$ let $u_0 := z$, $u_n := (u_{n-1}, u_{n-1}, u_{n-1})$ if $n \geq 1$. Then we have that $\gamma(u_n) = \gamma(u_{n-1})^3$, $\forall n \geq 1$, so $\gamma(u_n) = \gamma(u)^{3^n}$, $\forall n \geq 0$. Since $(E, F_1, F_2) = 0$, it follows that $x_n = y_n + z_n$. Thus: $\gamma(x) = \gamma(x_n)^{1/3^n} = \gamma(y_n + z_n)^{1/3^n} \leq (\gamma(y_n) + \gamma(z_n))^{1/3^n} = (\gamma(y)^{3^n} + \gamma(z)^{3^n})^{1/3^n} \xrightarrow{n} \max\{\gamma(y), \gamma(z)\}$, whence $\gamma(x) \leq \max\{\gamma(y), \gamma(z)\}$. $\square$

Corollary 5.9. *Let E and F be C^*-trings. Then $\mathcal{SN}(E \odot F) = \mathcal{SN}(E_p \odot F)$ and $\mathcal{N}(E \odot F) = \mathcal{N}(E_p \odot F)$. Consequently a C^*-tring E is nuclear if and only if E_p is nuclear.*

Our aim is to prove that there is an isomorphism between $\mathcal{N}(E \odot F)$ and $\mathcal{N}(E^r \odot F^r)$. The key step is to show that each C^*-norm on $E_0^r \odot F_0^r$ has unique extension to a C^*-norm on $E^r \odot F^r$.

Lemma 5.10. *Let I and J be $*$-ideals (not necessarily closed) of the C^*-algebras A and B, respectively. Then the map $\Theta : \mathcal{N}(A \odot B) \to \mathcal{N}(I \odot J)$, $\gamma \mapsto \gamma|_{I \odot J}$, is*

an order preserving surjection. If, in addition, I and J are dense in A and B, respectively, then Θ is a bijection.

Proof. Clearly Θ is order preserving. Fix $\delta \in \mathcal{N}(I \odot J)$. Given $a \in A$ and $z = \sum_{j=1}^n x_i \odot y_j \in I \odot J$, define $w := \sum_{j=1}^n (\|a\|^2 - a^*a)^{1/2} x_i \odot y_j \in A \odot B$. In case A is unital it is clear that $w \in I \odot J$. If A is not unital, I is an ideal of the unitization of A, so $w \in I \odot J$ in any case. Then

$$\|a\|^2 z^* z - (\sum_{j=1}^n ax_i \odot y_j)^* (\sum_{j=1}^n ax_i \odot y_j) = w^* w \in (I \otimes_\delta J)^+$$

and $\delta(\sum_{j=1}^n ax_i \odot y_j) \leq \|a\| \delta(\sum_{j=1}^n x_i \odot y_j)$. Similarly, if $b \in B$, we also have $\delta(\sum_{j=1}^n x_i \odot by_j) \leq \|b\| \delta(\sum_{j=1}^n x_i \odot y_j)$. Thus $\delta((a \odot b)z) \leq \|a\| \|b\| \delta(z)$, $\forall a \in A$, $b \in B$ and $z \in I \odot J$. Therefore, according to 3.21, the map $\delta' : A \odot B \to \mathbb{R}$ such that $\delta'(c) := \sup\{\delta(cz) : \delta(z) \leq 1\}$ is a C*-seminorm on $A \odot B$ that extends δ. In case I and J are dense in A and B, respectively, $I \odot J$ is dense in $A \odot B$ with respect to any C*-norm [10, Corollary T.6.2]. Thus Θ is injective. □

Proposition 5.11. *Let E and F be positive C*-trings and consider the admissible full basic triples $(E, E_0^r, \langle\,,\,\rangle_r^E)$ and $(F, F_0^r, \langle\,,\,\rangle_r^F)$ given by Theorem 2.10. Then the full basic triple $(E \odot F, E_0^r \odot F_0^r, \langle\,,\,\rangle_r^E \odot \langle\,,\,\rangle_r^F)$ is admissible. Furthermore, $E \odot F$ is positive and*

$$\mathcal{SN}_{cs}^{\langle\,,\,\rangle_r^E \odot \langle\,,\,\rangle_r^F}(E_0^r \odot F_0^r) = \mathcal{SN}(E_0^r \odot F_0^r)$$

Proof. To simplify our notation we denote $[\,,\,]$ the map $\langle\,,\,\rangle_r^E \odot \langle\,,\,\rangle_r^F$. Note $E_0^r \odot F_0^r$-module is admissible because it is a *-subalgebra of the C*-closable *-algebra $E^r \odot F^r$. We will show that $E \odot F$ is a positive $E_0^r \odot F_0^r$-module. Lemma 5.10 implies there is a maximal C*-norm on $E_0^r \odot F_0^r$, namely the restriction of the maximal C*-norm of $E^r \odot F^r$. The comments preceding Lemma 3.27 imply that, to show $E \odot F$ is positive, it suffices to prove that $[u, u] \geq 0$ in the maximal tensor product $E^r \otimes_{\max} F^r$. Given $u = \sum_{j=1}^n x_j \otimes y_j \in E \odot F$ we have

$$[u, u] = \sum_{j,k=1}^n \langle x_j, x_k \rangle_r^E \odot \langle y_j, y_k \rangle_F^r.$$

Then Lemmas 4.2 and 4.3 of [6] give the desired result.

To show $[u, u] = 0$ implies $u = 0$ we use the linking algebras $\mathbb{L}(E)$ and $\mathbb{L}(F)$ and the linear maps

$$\alpha \colon E \odot F \to \mathbb{L}(E) \odot \mathbb{L}(F), \ x \odot y \mapsto \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) \odot \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right),$$

$$\beta \colon \mathbb{L}(E) \odot \mathbb{L}(F) \to E \odot F, \ \left(\begin{smallmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{smallmatrix}\right) \odot \left(\begin{smallmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{smallmatrix}\right) \mapsto x_{12} \odot y_{12},$$

$$\gamma \colon E \odot F \to \mathbb{L}(E) \odot \mathbb{L}(F), \ a \odot b \mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}\right) \odot \left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right).$$

Then $\alpha(u)^* \alpha(u) = \gamma([u, u]) = 0$, so $\alpha(u) = 0$ and $u = \beta(\alpha(u)) = 0$. □

Theorem 5.12. *Let E and F be C*-ternary rings. Then every set among the partially ordered sets $\mathcal{N}(E^l \odot F^l)$, $\mathcal{N}(E \odot F)$ and $\mathcal{N}(E^r \odot F^r)$ is isomorphic to*

each other. Besides, if $\gamma \in \mathcal{N}(E \odot F)$ and γ^l and γ^r are the corresponding C^*-norms on $\mathcal{N}(E^l \odot F^l)$ and $\mathcal{N}(E^r \odot F^r)$ respectively, then $E \bigotimes_\gamma F$ is a Morita–Rieffel equivalence bimodule between $E^l \bigotimes_{\gamma^l} F^l$ and $E^r \bigotimes_{\gamma^r} F^r$.

Proof. Proposition 5.11 together with Corollary 3.31 imply $\mathcal{N}(E \odot F)$ is isomorphic (as a partially ordered set) to $\mathcal{N}(E \odot F)_0^r$. By 5.5 the posets $\mathcal{N}(E \odot F)_0^r$ and $\mathcal{N}(E_0^r \odot F_0^r)$ are isomorphic, and the latter is isomorphic to $\mathcal{N}(E^r \odot F^r)$ by Lemma 5.10. Thus $\mathcal{N}(E \odot F) \cong \mathcal{N}(E^r \odot F^r)$. Similarly we have $\mathcal{N}(E \odot F) \cong \mathcal{N}(E^l \odot F^l)$. $\qquad\square$

Corollary 5.13. *Let E and F be C^*-trings. Then there exist a maximum C^*-norm $\|\cdot\|_{\max}$ on $E \odot F$, and a minimum C^*-norm $\|\cdot\|_{\min}$ on $E \odot F$, and*

$$\left(E \bigotimes_{\max} F\right)^l = E^l \bigotimes_{\max} F^l, \qquad \left(E \bigotimes_{\max} F\right)^r = E^r \bigotimes_{\max} F^r,$$

$$\left(E \bigotimes_{\min} F\right)^l = E^l \bigotimes_{\min} F^l \qquad \left(E \bigotimes_{\min} F\right)^r = E^r \bigotimes_{\min} F^r.$$

Corollary 5.14 (*cf.* [5, Theorem 6.5]). *The following assertions are equivalent for a C^*-tring E:*

  (1) E *is a nuclear C^*-tring* (5.7).
  (2) E^l *is a nuclear C^*-algebra.*
  (3) E^r *is a nuclear C^*-algebra.*

The equivalence between 2. and 3. in 5.14 is exactly the following well-known result ([3], [11]): *if A and B are two Morita–Rieffel equivalent C^*-algebras then A is nuclear if and only if so is B.*

5.3. **Exact C^*-trings.** To end the section we introduce the notion of exact C^*-tring, extending the notion of exact TRO of [5], and we prove a result similar to Corollary 5.14. The reader is referred to [9] for the theory of exact C^*-algebras.

Suppose that $0 \longrightarrow F_1 \xrightarrow{\phi} F_2 \xrightarrow{\psi} F_3 \longrightarrow 0$ is an exact sequence of C^*-trings, that is, ϕ and ψ are homomorphisms of C^*-trings, ϕ is injective, ψ is surjective, and $\ker \psi = \phi(F_1)$. Let E be a C^*-tring. Then the sequence

$$0 \longrightarrow E \odot F_1 \xrightarrow{id \odot \phi} E \odot F_2 \xrightarrow{id \odot \psi} E \odot F_3 \longrightarrow 0$$

also is exact. We have an inclusion

$$(E \odot F_2)/(E \odot F_1) \hookrightarrow (E \bigotimes_{\min} F_2)/(E \bigotimes_{\min} F_1)$$

and the latter quotient is a C^*-tring. Then there exists a C^*-norm γ on $E \odot F_3$ such that

$$0 \longrightarrow E \bigotimes_{\min} F_1 \xrightarrow{id \otimes \phi} E \bigotimes_{\min} F_2 \xrightarrow{id \otimes \psi} E \bigotimes_\gamma F_3 \longrightarrow 0$$

is exact. Since γ is greater or equal to the minimum norm, the identity map on $E \odot F_3$ extends to a surjective homomorphism $E \bigotimes_\gamma F_3 \to E \bigotimes_{\min} F_3$.

Definition 5.15. We say that a C^*-tring E is exact if for each exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

of C^*-trings we have that

$$0 \longrightarrow E \otimes_{\min} F_1 \longrightarrow E \otimes_{\min} F_2 \longrightarrow E \otimes_{\min} F_3 \longrightarrow 0$$

also is exact.

Proposition 5.16. *Let E and F be C^*-trings, and suppose that G is an ideal of F (Definition 4.1). Then*

$$0 \longrightarrow E \otimes_{\min} G \longrightarrow E \otimes_{\min} F \longrightarrow E \otimes_{\min} (F/G) \longrightarrow 0$$

is exact if and only if the following sequence is exact:

$$0 \longrightarrow E^r \otimes_{\min} G^r \longrightarrow E^r \otimes_{\min} F^r \longrightarrow E^r \otimes_{\min} (F^r/G^r) \longrightarrow 0$$

Proof. Suppose first that the sequence below is exact:

$$0 \longrightarrow E \otimes_{\min} G \longrightarrow E \otimes_{\min} F \longrightarrow E \otimes_{\min} (F/G) \longrightarrow 0$$

By Corollaries 5.13 and 4.7, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (E \otimes_{\min} G)^r & \longrightarrow & (E \otimes_{\min} F)^r & \longrightarrow & (E \otimes_{\min} (F/G))^r & \longrightarrow & 0 \\
& & \cong \downarrow & & \cong \downarrow & & \cong \downarrow & & \\
0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\min} (F/G)^r & \longrightarrow & 0 \\
& & = \downarrow & & = \downarrow & & \cong \downarrow & & \\
0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\min} F^r/G^r & \longrightarrow & 0
\end{array}
$$

Since the upper two rows are exact, the third one also is exact.

To prove the converse, note first that

$$0 \longrightarrow E \otimes_{\min} G \longrightarrow E \otimes_{\min} F \longrightarrow (E \otimes_{\min} F)/(E \otimes_{\min} G) \longrightarrow 0$$

is exact, and $(E \otimes_{\min} F)/(E \otimes_{\min} G)$ is a C^*-completion of the ternary ring $E \odot (F/G)$. Denoting the corresponding C^*-norm by γ, we have a surjective homomorphism $\phi : E \otimes_\gamma (F/G) \to E \otimes_{\min} (F/G)$ which extends the identity on $E \odot (F/G)$. Now, applying the exact functor $E \mapsto E^r$ we obtain the commutative diagram with exact rows that follows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_\gamma F^r/G^r & \longrightarrow & 0 \\
& & = \downarrow & & = \downarrow & & \phi^r \downarrow & & \\
0 & \longrightarrow & E^r \otimes_{\min} G^r & \longrightarrow & E^r \otimes_{\min} F^r & \longrightarrow & E^r \otimes_{\min} F^r/G^r & \longrightarrow & 0
\end{array}
$$

It follows that the homomorphism ϕ^r is an isomorphism. $\qquad\square$

Corollary 5.17 (*cf.* [5, Theorem 6.1]). *A C^*-tring E is exact (5.15) if and only if E^r is an exact C^*-algebra.*

Proof. Immediate from Proposition 5.16 $\qquad\square$

As previously for nuclear C^*-algebras, we easily obtain from 5.17 the following known result([7]): *if A and B are Morita–Rieffel equivalent C^*-algebras, then A is exact if and only if B is exact.*

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