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SOME FIXED POINT THEOREMS FOR ORDERED REICH TYPE CONTRACTIONS IN CONE RECTANGULAR METRIC SPACES

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Abstract. In this paper, we prove some fixed point theorems for ordered Reich type contraction in cone rectangular metric spaces without assuming the normality of cone. Our results generalize and extend some recent results in cone rectangular metric spaces, cone metric spaces and rectangular metric space. Some examples illustrating the results are included.

1. Introduction and preliminaries

K-metric and K-normed spaces were introduced in the mid-20th century (see [**29**]) by using an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Indeed, this idea of replacement of real numbers by an ordered "set" can be seen in [**18, 19**] (see also references therein). Huang and Zhang [**10**] re-introduced such spaces under the name of cone metric spaces, defining convergent and Cauchy sequence in terms of interior points of underlying cone. They proved the basic version of the fixed point theorem with the assumption that the cone is normal. Subsequently several authors (see, e.g. [**1, 6, 9, 11, 12, 15, 20, 21, 25, 28**]) generalized the results of Huang and Zhang. In [**25**] Rezapour and Hamlbarani removed the normality of cone and proved the results of Huang and Zhang in non-normal cone metric spaces.

Branciari [**7**] introduced a class of generalized metric spaces by replacing triangular inequality by similar one which involves four or more points instead of three and improves Banach contraction principle. Azam and Arshad [**4**] proved fixed point result for Kannan type contraction in rectangular metric spaces. After the work of Huang and Zhang [**10**] Azam et al. [**5**] introduced the notion of cone rectangular metric space and proved fixed point result for Banach type contraction in cone rectangular space. Samet and Vetro [**26**] obtained the fixed point results in c-chainable cone rectangular metric spaces.

Ordered normed spaces and cones have applications in applied mathematics, for instance, in using Newton's approximation method [**27**] and in optimization theory

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[**8**]. The existence of fixed point in partially ordered sets was investigated by Ran and Reurings [**23**] and then by Nieto and Lopez [**22**]. Fixed point results in ordered cone metric spaces were obtained by several authors (see, e.g. [**2, 3, 14, 20**]).

The fixed point results for Reich type mappings and fixed point results for ordered contractions not investigated yet even in rectangular metric spaces. In this paper, we extend and generalize the results of Azam et al. [**5**] and Azam and Arshad [**4**] in ordered cone rectangular metric spaces by proving the fixed point theorem for Reich type contractions in the setting of ordered cone rectangular metric space. Results in the present paper are the extension and generalization of fixed point results of Banach, Kannan [**16, 17**] and Reich [**24**] in ordered cone rectangular metric spaces.

We need the following definitions and results consistent with [**8**] and [**10**].

Definition 1 ([**10**])**.** Let E be a real Banach space and P be a subset of E. The set P is called a cone if:

(i) P is closed, nonempty and $P \neq \{\theta\}$, here θ is the zero vector of E;
(ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;
(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering "$\preceq$" with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. While $x \ll y$ if and only if $y - x \in P^0$, where P^0 denotes the interior of P.

Let P be a cone in a real Banach space E, then P is called normal, if there exists a constant $K > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 2 ([**10**])**.** Let X be a nonempty set, E be a real Banach space. Suppose that the mapping $d\colon X \times X \to E$ satisfies

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. In the following we always suppose that E is a real Banach space, P is a solid cone in E, i.e., $P^0 \neq \phi$ and "$\preceq$" is partial ordering with respect to P.

The concept of cone metric space is more general than that of a metric space, because each metric space is a cone metric space with $E = \mathbb{R}$ and $P = [0, +\infty)$.

For examples and basic properties of normal and non-normal cones and cone metric spaces we refer [**10**] and [**25**].

The following remark will be useful in sequel.

Remark 1 ([**13**])**.** Let P be a cone in a real Banach space E and $a, b, c \in P$, then

(a) If $a \preceq b$ and $b \ll c$, then $a \ll c$.

(b) If $a \ll b$ and $b \ll c$, then $a \ll c$.
(c) If $\theta \preceq u \ll c$ for each $c \in P^0$, then $u = \theta$.
(d) If $c \in P^0$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, we have $a_n \ll c$.
(e) If $\theta \preceq a_n \preceq b_n$ for each n and $a_n \to a,\ b_n \to b$, then $a \preceq b$.
(f) If $a \preceq \lambda a$ where $0 < \lambda < 1$, then $a = \theta$.

Definition 3 ([**5**])**.** Let X be a nonempty set. Suppose the mapping $d\colon X \times X \to E$, satisfying

(i) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \preceq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [rectangular property].

Then d is called a cone rectangular metric on X,, and (X, d) is called a cone rectangular metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in (X, d)$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $\lim_n x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$ with $\theta \ll c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$ and $m \in \mathbb{N}$ we have $d(x_n, x_{n+m}) \ll c$. Then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete cone rectangular metric space. If the underlying cone is normal, then (X, d) is called normal cone rectangular metric space.

Example 1. Let $X = \mathbb{N}, E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$.
Define $d\colon X \times X \to E$ as follows:

$$d(x,y) = \begin{cases} (0,0) & \text{if } x = y, \\ (3,9) & \text{if } x \text{ and } y \text{ are in } \{1,2\}, x \neq y, \\ (1,3) & \text{if } x \text{ and } y \text{ both can not be at a time in } \{1,2\}, x \neq y. \end{cases}$$

Now (X, d) is a cone rectangular metric space, but (X, d) is not a cone metric space because it lacks the triangular property

$$(3, 9) = d(1, 2) > d(1, 3) + d(3, 2) = (1, 3) + (1, 3) = (2, 6)$$

as $(3, 9) - (2, 6) = (1, 3) \in P$.

Note that in above example (X, d) is a normal cone rectangular metric space. Following is an example of non-normal cone rectangular metric space.

Example 2. Let $X = \mathbb{N}$, $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and $P = \{x \in E : x(t) \geq 0 \text{ for } t \in [0, 1]\}$. Then this cone is not normal (see [**25**]).

Define $d : X \times X \to E$ as follows

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 3\,\mathrm{e}^t & \text{if } x \text{ and } y \text{ are in } \{1,2\}, x \neq y, \\ \mathrm{e}^t & \text{if } x \text{ and } y \text{ both can not be at a time in } \{1,2\}, x \neq y. \end{cases}$$

Then (X, d) is non-normal cone rectangular metric space, but (X, d) is not a cone metric space because it lacks the triangular property.

Definition 4. If a nonempty set X is equipped with a partial order " $\sqsubseteq$ " and mapping $d\colon X \times X \to E$ such that (X, d) is a cone rectangular metric space, then $(X, \sqsubseteq, d)$ is called an ordered cone rectangular metric space. Let $f\colon X \to X$ be a mapping. The mapping f is called nondecreasing with respect to " $\sqsubseteq$ " if for each $x, y \in X$, $x \sqsubseteq y$ implies $fx \sqsubseteq fy$.

A self map f on $(X, \sqsubseteq, d)$ is called ordered Banach type contraction if for all $x, y \in X$ with $x \sqsubseteq y$, there exists $\lambda \in [0, 1)$ such that

$$d(fx, fy) \preceq \lambda d(x, y). \tag{1}$$

If (1) is satisfied for all $x, y \in X$, then f is called Banach contraction.

f is called ordered Kannan type contraction if for all $x, y \in X$ with $x \sqsubseteq y$, there exists $\lambda \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \preceq \lambda[d(x, fx) + d(y, fy)]. \tag{2}$$

If (2) is satisfied for all $x, y \in X$, then f is called Kannan contraction.

f is called ordered Reich type contraction if for all $x, y \in X$ with $x \sqsubseteq y$, $\lambda, \mu, \delta \in [0, 1)$ such that $\lambda + \mu + \delta < 1$ and

$$d(fx, fy) \preceq \lambda d(x, y) + \mu d(x, fx) + \delta d(y, fy). \tag{3}$$

If (3) is satisfied for all $x, y \in X$, then f is called Reich contraction.

Kannan showed that the conditions (1) and (2) are independent of each other (see [**16, 17**]) and Reich showed that the condition (3) is a proper generalization of (1) and (2) (see [**24**]). Note that Reich type contraction turns into Banach and Kannan type contractions with $\mu = \delta = 0$ and $\lambda = 0, \mu = \delta$, respectively.

Definition 5. Let X be a nonempty set equipped with partial order " $\sqsubseteq$ ". A nonempty subset $\mathcal{A}$ of X is said to be well ordered if every two elements of $\mathcal{A}$ are comparable with respect to " $\sqsubseteq$ ".

Now we can state our main results.

2. Main Results

Theorem 1. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone rectangular metric space and $f\colon X \to X$ be a mapping. Suppose that the following conditions hold*

(I) *f is an ordered Reich type contraction, i.e., it satisfies* (3)*;*
(II) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*
(III) *f is nondecreasing with respect to " $\sqsubseteq$ ";*
(IV) *if $\{x_n\}$ is a nondecreasing sequence in X and converging to some z, then $x_n \sqsubseteq z$.*

Then f has a fixed point. Furthermore, the set of fixed points of f is well ordered if and only if fixed point of f is unique.

Proof. Starting with the given x_0, we can construct the Picard sequence $\{x_n\}$ as follows. As $x_0 \in X$ is such that $x_0 \sqsubseteq fx_0$, suppose $fx_0 = x_1$, then $x_0 \sqsubseteq x_1$. Again as f is nondecreasing with respect to " $\sqsubseteq$ ", we obtain $fx_0 \sqsubseteq fx_1$ suppose

$fx_1 = x_2$. Continuing in this manner, we obtain the nondecreasing sequence so-called Picard sequence $\{x_n\}$ such that

$$x_0 \sqsubseteq x_1 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots \text{ and } x_{n+1} = fx_n \qquad \text{for all } n \geq 0.$$

Note that if $x_{n+1} = x_n$ for any n, then x_n is a fixed point of f. So we assume that $x_{n+1} \neq x_n$ for all $n \geq 0$.

As $x_n \sqsubseteq x_{n+1}$, for any $n \geq 0$, we obtain from (I)

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\preceq \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, fx_{n-1}) + \delta d(x_n, fx_n) \\ &= \lambda d(x_{n-1}, x_n) + \mu d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}) \\ d(x_n, x_{n+1}) &\preceq \frac{\lambda + \mu}{1 - \delta} d(x_{n-1}, x_n), \end{aligned}$$

i.e., $d_n \preceq \alpha d_{n-1}$, where $\alpha = \dfrac{\lambda + \mu}{1 - \delta}$ and $d_n = d(x_n, x_{n+1})$.

Repeating this process, we obtain that

$$d_n \preceq \alpha^n d_0 \qquad \text{for all} \quad n \geq 1. \tag{4}$$

We can also assume that x_0 is not a periodic point. Indeed, if $x_0 = x_n$ for any $n \geq 2$, then from (4) it follows that

$$\begin{aligned} d(x_0, fx_0) &= d(x_n, fx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}), \\ d_0 &= d_n \\ d_0 &\preceq \alpha^n d_0. \end{aligned}$$

As $\alpha = \frac{\lambda+\mu}{1-\delta} < 1$ (since $\lambda + \mu + \delta < 1$), the above inequality shows that $d_0 = \theta$, i.e., $d(x_0, fx_0) = \theta$, so x_0 is a fixed point of f. Thus we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$.

Again, as $x_n \sqsubseteq x_{n+2}$, we obtain from (I) and (4)

$$\begin{aligned} d(x_n, x_{n+2}) &= d(fx_{n-1}, fx_{n+1}) \\ &\preceq \lambda d(x_{n-1}, x_{n+1}) + \mu d(x_{n-1}, fx_{n-1}) + \delta d(x_{n+1}, fx_{n+1}) \\ &= \lambda d(x_{n-1}, x_{n+1}) + \mu d(x_{n-1}, x_n) + \delta d(x_{n+1}, x_{n+2}) \\ &\preceq \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+2}) + d(x_{n+2}, x_{n+1})] \\ &\quad + \mu d(x_{n-1}, x_n) + \delta d(x_{n+1}, x_{n+2}) \\ &= \lambda[d_{n-1} + d(x_n, x_{n+2}) + d_{n+1}] + \mu d_{n-1} + \delta d_{n+1} \\ &= (\lambda + \mu) d_{n-1} + (\lambda + \delta) d_{n+1} + \lambda d(x_n, x_{n+2}) \\ &\preceq (\lambda + \mu)\alpha^{n-1} d_0 + (\lambda + \delta)\alpha^{n+1} d_0 + \lambda d(x_n, x_{n+2}) \\ d(x_n, x_{n+2}) &\preceq \frac{(\lambda + \mu) + (\lambda + \delta)\alpha^2}{1 - \lambda} \alpha^{n-1} d_0 \\ &\preceq \frac{2\lambda + \mu + \delta}{1 - \lambda} \alpha^{n-1} d_0, \end{aligned}$$

so

$$d(x_n, x_{n+2}) \preceq \beta\alpha^{n-1}d_0 \qquad \text{for all} \quad n \geq 1, \tag{5}$$

where $\beta = \dfrac{2\lambda+\mu+\delta}{1-\lambda} \geq 0$.

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases.
If p is odd, say $2m+1$, then using rectangular inequality and (4), we obtain

$$\begin{aligned} d(x_n, x_{n+2m+1}) &\preceq d(x_{n+2m}, x_{n+2m+1}) + d(x_{n+2m-1}, x_{n+2m}) + d(x_n, x_{n+2m-1}) \\ &= d_{n+2m} + d_{n+2m-1} + d(x_{n+2m-1}, x_n) \\ &\preceq d_{n+2m} + d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + \cdots + d_n \\ &\preceq \alpha^{n+2m}d_0 + \alpha^{n+2m-1}d_0 + \alpha^{n+2m-2}d_0 + \cdots + \alpha^n d_0 \\ &= [\alpha^{2m} + \alpha^{2m-1} + \cdots + 1]\alpha^n d_0 \\ &\preceq \frac{\alpha^n}{1-\alpha}d_0, \end{aligned}$$

so

$$d(x_n, x_{n+2m+1}) \preceq \frac{\alpha^n}{1-\alpha}d_0. \tag{6}$$

If p is even, say $2m$, then using rectangular inequality, (4) and (5), we obtain

$$\begin{aligned} d(x_n, x_{n+2m}) &= d(x_{n+2m}, x_n) \\ &\preceq d(x_{n+2m}, x_{n+2m-1}) + d(x_{n+2m-1}, x_{n+2m-2}) + d(x_{n+2m-2}, x_n) \\ &= d_{n+2m-1} + d_{n+2m-2} + d(x_{n+2m-2}, x_n) \\ &\preceq d_{n+2m-1} + d_{n+2m-2} + d_{n+2m-3} + d_{n+2m-4} + \cdots \\ &\quad + d_{n+2} + d(x_{n+2}, x_n) \\ &\preceq \alpha^{n+2m-1}d_0 + \alpha^{n+2m-2}d_0 + \alpha^{n+2m-3}d_0 + \cdots + \alpha^{n+2}d_0 + \beta\alpha^{n-1}d_0 \\ &= [\alpha^{2m-1} + \alpha^{2m-2} + \cdots + \alpha^2]\alpha^n d_0 + \beta\alpha^{n-1}d_0 \\ &\preceq \frac{\alpha^n}{1-\alpha}d_0 + \beta\alpha^{n-1}d_0 \end{aligned}$$

so

$$d(x_n, x_{n+2m}) \preceq \frac{\alpha^n}{1-\alpha}d_0 + \beta\alpha^{n-1}d_0. \tag{7}$$

As $\beta \geq 0$, $0 \leq \alpha < 1$, it follows that $\frac{\alpha^n}{1-\alpha}d_0 \to \theta$, $\beta\alpha^{n-1}d_0 \to \theta$, so by (a) and (d) of Remark 1, for every $c \in E$ with $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+2m}) \ll c, d(x_n, x_{n+2m+1}) \ll c$ for all $n > n_0$. Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, so there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n-1} = u$$

We shall show that u is a fixed point of f. By (IV), we have $x_n \sqsubseteq u$, therefore, it follows from (I) that

$$\begin{aligned} d(fx_{n-1}, fu) &\preceq \lambda d(x_{n-1}, u) + \mu d(x_{n-1}, fx_{n-1}) + \delta d(u, fu) \\ d(x_n, fu) &\preceq \lambda d(x_{n-1}, u) + \mu d(x_{n-1}, x_n) + \delta[d(u, x_{n+1}) \\ &\quad + d(x_{n+1}, x_n) + d(x_n, fu)] \\ (1-\delta)d(x_n, fu) &\preceq \lambda d(x_{n-1}, u) + \delta d(x_{n+1}, u) + \mu d_{n-1} + \delta d_n. \end{aligned}$$

In view of (4) and the fact that $1-\delta \geq 0$, by (d) of Remark 1, there exists $n_1 \in \mathbb{N}$ such that for every $c \in E$ with $\theta \ll c$, $d_{n-1} \ll \frac{(1-\delta)}{4\mu}c$ and $d_n \ll \frac{(1-\delta)}{4\delta}c$ for all $n > n_1$. Also $x_n \to u$, so there exists $n_2 \in \mathbb{N}$ such that for every $c \in E$ with $\theta \ll c$, $d(x_{n-1}, u) \ll \frac{(1-\delta)}{4\lambda}c$ and $d(x_{n+1}, u) \ll \frac{(1-\delta)}{4\delta}c$ for all $n > n_2$.

Thus, we can choose $n_3 \in \mathbb{N}$ such that

$$d(x_n, fu) \ll c \qquad \text{for all } n > n_3. \tag{8}$$

Again by rectangular inequality,

$$\begin{aligned} d(fu, u) &\preceq d(fu, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, u) \\ &= d(fu, x_n) + d_n + d(x_{n+1}, u). \end{aligned}$$

In view of (8),(4) and the fact that $x_n \to u$ by (c) of Remark 1, we conclude that $d(fu, u) = \theta$, i.e., $fu = u$. Thus, u is fixed point of f.

Suppose that the set of fixed points $\mathcal{A}$ (say) of f is well ordered. We shall prove that u is unique fixed point of f.

Let $v \in \mathcal{A}$ be another fixed point of f, i.e., $fv = v$. As $\mathcal{A}$ is well ordered, let, e.g., $u \sqsubseteq v$. From (I) we obtain

$$\begin{aligned} d(u, v) &= d(fu, fv) \\ &\preceq \lambda d(u, v) + \mu d(u, fu) + \lambda d(v, fv) \\ &= \lambda d(u, v) + \mu d(u, u) + \lambda d(v, v) \\ &= \lambda d(u, v). \end{aligned}$$

As $0 \leq \lambda < 1$, it follows from above inequality and (f) of Remark 1 that $u = v$. Thus, fixed point of f is unique. Conversely, if fixed point of f is unique, then $\mathcal{A}$ is a singleton set, therefore well ordered. □

Taking suitable values of λ, μ, δ in theorem 1, one can obtain following fixed point result in ordered cone rectangular metric spaces.

Corollary 1. *Let $(X, \sqsubseteq, d)$ be an ordered complete cone rectangular metric space and $f : X \to X$ be a mapping. Suppose that following conditions hold:*

(I) *f is*
   (a) *an ordered Banach type contraction, or*
   (b) *an ordered Kannan type contraction;*

(II) *there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;*

(III) *f is nondecreasing with respect to $\sqsubseteq$;*

(IV) *if $\{x_n\}$ is a nondecreasing sequence in X and converging to some z, then $x_n \sqsubseteq z$.*

Then f has a fixed point in X. Furthermore, the set of fixed points of f is well ordered if and only if fixed point of f is unique.

Remark 2. The above corollary is a generalization and extension of results of Azam et al. [**5**] and Azam and Arshad [**4**] in ordered cone rectangular metric spaces in view of used contractive conditions and normality of cone.

Following is an example of ordered Reich type contraction which is not a Reich contraction (in the sense of [**24**]) in cone rectangular metric space.

Example 3. Let $X = \{1, 2, 3, 4\}$ and $E = C^1_{\mathbb{R}}[0, 1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$, $P = \{x(t) : x(t) \geq 0 \ \text{ for } \ t \in [0, 1]\}$. Define $d\colon X \times X \to E$ as follows
$d(1, 2) = d(2, 1) = 3\,\mathrm{e}^t$,
$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = \mathrm{e}^t$,
$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 4\,\mathrm{e}^t$,
$d(x, y) = \theta$ if $x = y$.
Then (X, d) is a complete non-normal cone rectangular metric space, but not cone metric space. Define mappings $f\colon X \to X$ and partial order on X as follows:

$$f1 = 1, \quad f2 = 1, \quad f3 = 4, \quad f4 = 2,$$

and $\sqsubseteq = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 4), (1, 4)\}$.

Then it is easy to verify that f is ordered Reich contraction in $(X, \sqsubseteq, d)$ with $\lambda = \delta = \frac{3}{8}, \mu = \frac{1}{5}$. Indeed, we have to check the validity of (3) only for $(x, y) = (1, 2), (2, 4), (1, 4)$.

If $(x, y) = (1, 2)$, then

$$d(f1, f2) = d(1, 1) = \theta,$$

therefore, (3) holds for arbitrary $\lambda, \mu, \delta \in [0, 1)$ such that $\lambda + \mu + \delta < 1$.

If $(x, y) = (2, 4)$, then

$$d(f2, f4) = d(1, 2) = 3\,\mathrm{e}^t$$

and

$$\lambda d(2, 4) + \mu d(2, f2) + \delta d(4, f4) = 4\lambda e^t + 3\mu\,\mathrm{e}^t + 4\delta\,\mathrm{e}^t,$$

therefore, (3) holds for $\lambda = \delta = \frac{3}{8}, \mu = \frac{1}{5}$.

Similarly, (3) holds for $(x, y) = (1, 4)$ with same values of λ, μ, δ. All other conditions of Theorem 1 are satisfied and f has unique fixed point, namely "1".

On the other hand, f is not a Reich type contraction in cone rectangular space (non-ordered), e.g., for $x = 3, y = 1$ $d(f3, f1) = d(4, 1) = 4\,\mathrm{e}^t$ and $\lambda d(3, 1) + \mu d(3, f3) + \delta d(1, f1) = \lambda\,\mathrm{e}^t + 4\mu\,\mathrm{e}^t$ and $\lambda + \mu + \delta < 1$, therefore, (3) can not hold.

Following example illustrates that fixed point in above results may not be unique (when the set of fixed point is not well ordered).

Example 4. Let $X = \{1, 2, 3, 4\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$. Define $d\colon X \times X \to E$ as follows:
$d(1, 2) = d(2, 1) = (3, 6)$,
$d(2, 3) = d(3, 2) = d(1, 3) = d(3, 1) = (1, 2)$,
$d(1, 4) = d(4, 1) = d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = (2, 4)$,
$d(x, y) = \theta$ if $x = y$.

Then (X, d) is a complete cone rectangular metric space but not cone metric space. Define $f : X \to X$ and partial order on X as follows:

$$fx = \begin{cases} x & \text{if } x \in \{1, 3\}, \\ 4 & \text{if } x = 2, \\ 1 & \text{if } x = 4, \end{cases}$$

and $\sqsubseteq = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (1, 4)\}$.

Then it is easy to see that $d(fx, fy) \preceq \lambda d(x, y)$ for all $x, y \in X$ with $x \sqsubseteq y$, is satisfied for $\lambda \in [\frac{2}{3}, 1)$. Thus, f is an ordered Banach contraction on X. All other conditions of Corollary 1 (except the set of fixed points of f is well ordered) are satisfied and f has two fixed points 1 and 3 in X. Note that $(1, 3), (3, 1) \notin \sqsubseteq$.

On the other hand, for $x = 1$, $y = 3$, there is no λ such that $0 \leq \lambda < 1$ and $d(fx, fy) \preceq \lambda d(x, y)$. Therefore, f is not a Banach contraction on X.

In the following theorem the conditions on f, "nondecreasing" and completeness of space, are replaced by another condition.

Theorem 2. *Let $(X, \sqsubseteq, d)$ be an ordered cone rectangular metric space and $f : X \to X$ be a mapping. Suppose that following conditions hold:*

(I) *f is an ordered Reich type contraction, i.e., it satisfies* (3)*;*

(II) *there exists $u \in X$ such that $u \sqsubseteq fu$ and $d(u, fu) \preceq d(x, fx)$ for all $x \in X$.*

Then f has a fixed point. Furthermore, the set of fixed points of f is well ordered if and only if fixed point of f is unique.

Proof. Let $F(x) = d(x, fx)$ for all $x \in X$ and $z = fu$, then $F(u) \preceq F(x)$ for all $x \in X$. If $F(u) = \theta$, then u is a fixed point of f. If $\theta \prec F(u)$, then by assumption (II) $u \sqsubseteq fu$, so $u \sqsubseteq z$ and by (I), we obtain

$$\begin{aligned} F(z) &= d(z, fz) = d(fu, fz) \\ &\preceq \lambda d(u, z) + \mu d(u, fu) + \delta d(z, fz) \\ &= \lambda d(u, fu) + \mu d(u, fu) + \delta d(z, fz) \\ &= \lambda F(u) + \mu F(u) + \delta F(z) \\ F(z) &\preceq \frac{\lambda + \mu}{1 - \delta} F(u) \prec F(u) \quad (\text{as } \lambda + \mu + \delta < 1), \end{aligned}$$

a contradiction. Therefore, we have $F(u) = \theta$, i.e., $fu = u$. Thus, u is a fixed point of f.

The necessary and sufficient condition for uniqueness of fixed point follows from a similar process as used in Theorem 1. □

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