Acta Math. Univ. Comenianae
Vol. LXXXII, 2 (2013), pp. 253–263


# CHARACTERIZATIONS OF TWISTED PRODUCT MANIFOLDS TO BE WARPED PRODUCT MANIFOLDS

S. KAZAN and B. ŞAHIN

Abstract. In this paper, we give characterizations of a twisted product manifold to be a warped product manifold by imposing certain conditions on the Weyl conformal curvature tensor and the Weyl projective tensor. We also find similar results for multiply twisted product manifolds.

## 1. Introduction

The notion of warped product manifolds was introduced by Bishop and O'Neill for constructing negatively curvature manifolds. Later this notion has been extended to a doubly warped product manifold, a doubly twisted product manifold and a multiply warped/twisted product manifold. More precisely, let $(B, g_B)$ and $(F, g_F)$ be semi-Riemannian manifolds of dimensions $r$ and $s$, respectively, and let $\pi\colon B \times F \to B$ and $\sigma\colon B \times F \to F$ be the canonical projections. Also let $b\colon B \times F \to (0, \infty)$ and $f\colon B \times F \to (0, \infty)$ be smooth functions. Then the *doubly twisted product* $M = {}_f B \times_b F$ of $(B, g_B)$ and $(F, g_F)$ with twisting functions $b$ and $f$ is defined to be the product manifold $M = B \times F$ with the metric tensor $g = f^2 g_B \oplus b^2 g_F$ given by $g = f^2 \pi^* g_B + b^2 \sigma^* g_F$. In particular, if $f = 1$, then ${}_1 B \times_b F = B \times_b F$ is called the *twisted product* of $(B, g_B)$ and $(F, g_F)$ with twisting function $b$. Moreover, if $b$ only depends on the points of $B$, then $B \times_b F$ is called the a warped product manifold of $(B, g_B)$ and $(F, g_F)$ with warping function $b$ [**8**].

On the other hand, Hiepko [**9**] gave a characterization of a warped product manifold in terms of distributions defined on the manifolds. Similar characterizations were given by R. Ponge and H. Reckziegel for twisted product semi-Riemannian manifolds in [**11**]. Recently, Fernandez-Lopez, Garcia-Rio, Küpeli and Ünal [**8**] gave a characterization of a twisted product manifold to be a warped product manifold by using the Ricci tensor of the manifold.

The central object of interest in conformal geometry is distinguished tensor which arises as the irreducible traceless part of the Riemannian curvature tensor $R$.

Received October 30, 2012; revised May 2, 2013.
2010 *Mathematics Subject Classification.* Primary 53C50, 53C42.
*Key words and phrases.* twisted product; warped product; Ricci curvature tensor; Weyl curvature tensor.

For $n \geq 4$, the Weyl tensor is

$$C = R - \frac{1}{n-2} L \otimes g$$

where $L$ is standing for the Schouten tensor and $\otimes$ is denoting the Kulkarni-Nomizu product. The Weyl tensor has a distinguished property that is invariant under conformal transformations of the metric $g$. If the Weyl tensor $C$ vanishes, the manifold is conformally flat. It is known that every Riemannian manifold with the Weyl parallel tensor has a constant scalar and such manifolds must be either conformally flat or locally symmetric. However, this result is not valid for semi-Riemannian manifolds (see: [**3**], [**4**], [**5**], [**6**], [**7**]).

In this paper, we investigate what kind of product manifolds occur for twisted product manifolds whose the Weyl tensor $C$ has a special form. We find that a twisted product manifold becomes a warped product manifold by imposing certain properties on the the Weyl conformal curvature tensor and Weyl projective curvature tensor. We also find a similar result for multiply twisted product manifolds.

## 2. Preliminaries

In this section, we review preparatory results for the next section.
Suppose $B$ and $F$ are semi-Riemannian manifolds, and let $f > 0$ be a smooth function on $B$. The *warped product* $M = B \times_f F$ is the product manifold $B \times F$ furnished with the metric tensor

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where $\pi$ and $\sigma$ are the projections of $B \times F$ onto $B$ and $F$, respectively. If $f = 1$, then $M = B \times_f F$ reduces to a semi-Riemannian product manifold [**10**]. This notion has been extended to the several forms. Let $(M_1, g_1)$ and $(M_2, g_2)$ be semi-Riemannian manifolds, $\lambda \colon M_1 \times M_2 \to \mathbb{R}$ a positive, differentiable function, $g$ a semi-Riemannian metric on the manifold $M_1 \times M_2$ and assume that the canonical foliations $L_1$ and $L_2$ intersect perpendicularly everywhere. Then (b) and (c) conditions of [**11**, Proposition 3] imply that $g$ is a metric of

(b) a twisted product manifold $M_1 \times_\lambda M_2$ if and only if $L_1$ is a totally geodesic and $L_2$ is a totally umbilic foliation,

(c) a warped product manifold $M_1 \times_\lambda M_2$ if and only if $L_1$ is a totally geodesic and $L_2$ is a spheric foliation.

On the other hand, a generalisation of the warped product manifold is the multiply warped product manifold. Let $(B, g_B)$ and $(F_i, g_{F_i})$ be semi-Riemannian manifolds and also let $b_i \colon B \to (0, \infty)$ be smooth functions for any $i \in 1, 2, \ldots, m$. The functions $b_i \colon B \times F_i \to (0, \infty)$ are called warping functions for any $i \in 1, 2, \ldots, m$. The multiply warped product manifold is the product manifold $B \times F_1 \times F_2 \times \ldots \times F_m$ with the metric tensor $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \ldots \oplus b_m^2 g_{F_m}$ defined by

$$g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus \ldots \oplus (b_m \circ \pi)^2 \sigma_m^*(g_{F_m}), \tag{2.1}$$

where $\pi\colon B \times F_i \to B$ and $\sigma\colon B \times F_i \to F_i$ are canonical projections. If $m = 1$, then we obtain a singly warped product manifold. If each $b_i \equiv 1$, then we have a product manifold [**12**].

Let $(B, g_B)$ and $(F, g_F)$ be semi-Riemannian manifolds with Levi-Civita connections $\nabla^B$ and $\nabla^F$, respectively, and let both $\nabla$ denote the Levi-Civita connection and the gradient of the doubly twisted product manifold ${}_fB \times_b F$ of $(B, g_B)$ and $(F, g_F)$ with twisting functions $b$ and $f$. Also, let $k = \log(b)$, $l = \log(f)$ and let $\mathfrak{L}(B)$ and $\mathfrak{L}(F)$ be the sets of lifts of vector fields on $B$ and $F$ to $B\times F$, respectively.

For a doubly twisted product manifold, we have the following proposition.

**Proposition 2.1** ([**8**]). *Let $M =_f B \times_b F$ be a doubly twisted product manifold. If $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, then we have*

$$\nabla_X V = V(l)X + X(k)V. \tag{2.2}$$

Define $h^k_B(X, Y) = XY(k) - (\nabla^B_X Y)(k)$ for $X, Y \in \mathfrak{L}(B)$. If $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, then $XV(k) = VX(k)$ and the Hessian form $h^k$ of $k$ on ${}_fB\times_b F$ satisfies

$$h^k(X, V) = XV(k) - X(k)V(l) - X(k)V(k), \tag{2.3}$$

$$h^k(X, Y) = h^k_B(X, Y) - X(l)Y(k) - X(k)Y(l) + g(X, Y)g(\nabla k, \nabla l). \tag{2.4}$$

Now, we define multiply twisted product manifolds, and then we give connections and curvature tensors of multiply twisted product manifolds.

Let $(B, g_B)$ and $(F_i, g_{F_i})$ be $r$ and $s_i$ dimensional semi-Riemannian manifolds, respectively, where $i \in 1, 2, \ldots, m$. If $F = F_1 \times F_2 \times \ldots \times F_m$, then $M = B \times F$ is also an $n$-dimensional semi-Riemannian manifold, where $s = \sum_{i=1}^m s_i$ and $n = r+s$.

**Definition 2.2** ([**14**]). A multiply twisted product manifold $(M, g)$ is a product manifold of $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \ldots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \ldots \oplus b_m^2 g_{F_m}$, where for each $i \in 1, 2, \ldots, m$, $b_i\colon B \times F_i \to (0, \infty)$ is smooth.

Here, $(B, g_B)$ is called the base manifold and $(F_i, g_{F_i})$ is called the fiber manifold and $b_i$ is called the twisted function. Obviously, twisted product manifolds and multiply warped product manifolds are the special cases of multiply twisted product manifolds. Indeed, if $m = 1$, then we obtain a single twisted product manifold. If $b_i \equiv 1$ for all $b_i$, then we have a product manifold.

For a multiply twisted product, we have the following proposition.

**Proposition 2.3** ([**14**]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \ldots \times_{b_m} F_m$ be a multiply twisted product manifold with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \ldots \oplus b_m^2 g_{F_m}$ and let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$, $W \in \mathfrak{L}(F_j)$. Then*

(a) $\nabla_X Y = \nabla^B_X Y$,
(b) $\nabla_X V = \nabla_V X = X(k_i)V$,
(c) $\nabla_V W = 0$ *for* $i \neq j$,
(d) $\nabla_V W = \nabla^{F_i}_V W + V(k_i)W + W(k_i)V - g(V, W)\nabla k_i$ *for* $i = j$

*where* $k_i = log(b_i)$.

Define $h^{k_i}_B(X, Y) = XY(k_i) - (\nabla^B_X Y)(k_i)$, for $X, Y \in \mathfrak{L}(B)$. If $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F_i)$, then $XV(k_i) = VX(k_i)$ and the Hessian form $h^{k_i}$ of $k_i$ on $B\times_{b_i} F_i$

satisfies

$$h^{k_i}(X,V) = XV(k_i) - X(k_i)V(k_i),$$
$$h^{k_i}(X,Y) = h_B^{k_i}(X,Y).$$

For Riemennian tensor and Ricci tensor of a multiply twisted product manifold, we have the following propositions.

**Proposition 2.4** ([**14**]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \ldots \times_{b_m} F_m$ be a multiply twisted product manifold and let $X, Y, Z \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$, $W \in \mathfrak{L}(F_j)$. Then*

(a) $R(X,Y)Z = R^B(X,Y)Z$,
(b) $R(X,Y)V = 0$,
(c) $R(X,W)V = R(W,V)X = R(W,X)V = 0$ *for* $i \neq j$,
(d) $R(X,W)V = [X(k_i)V(k_i) + h^{k_i}(X,V)]W - g(W,V)[X(k_i)\nabla k_i + H^{k_i}(X)]$ *for* $i = j$,

*where $H^{k_i}$ is the Hessian tensor of $k_i$ on $B \times_{b_i} F_i$, i.e., $h^{k_i}(X,Y) = g(H^{k_i}(X),Y)$.*

**Proposition 2.5** ([**14**]). *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \ldots \times_{b_m} F_m$ be a multiply twisted product manifold and let $X, Y \in \mathfrak{L}(B)$, $V, W \in \mathfrak{L}(F_i)$. Then*

$$\mathrm{Ric}\,(X,V) = (s_i - 1)XV(k_i). \tag{2.5}$$

Let $M$ be an $n$-dimensional Riemannian manifold with the metric tensor $g$. If $E_1, E_2, \ldots, E_n$ are local orthonormal vector fields of $M$, then

$$\mathrm{Ric}\,(X,Y) = \sum_{i=1}^{n} g(R(E_i,X)Y,E_i)$$

defines a global tensor field the Ric of type (0,2). Ric tensor field is called the Ricci tensor [**13**].

Let $M$ be an $n$-dimensional Riemannian manifold with metric tensor $g$. Then the *Weyl conformal curvature tensor field* of $M$ is the tensor field $C$ of type $(1,3)$ defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z \\ &\quad + \frac{1}{n-2}\left[\mathrm{Ric}\,(X,Z)Y - \mathrm{Ric}\,(Y,Z)X + g(X,Z)QY - g(Y,Z)QX\right] \\ &\quad - \frac{\tau}{(n-1)(n-2)}\left[g(X,Z)Y - g(Y,Z)X\right] \end{aligned} \tag{2.6}$$

for any vector fields $X, Y$ and $Z$ on $M$, where $\tau$ is scalar curvature [**13**].

Also, the Weyl projective curvature tensor is given by

$$W_P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[\mathrm{Ric}\,(Y,Z)X - \mathrm{Ric}\,(X,Z)Y]. \tag{2.7}$$

Let $R_B$, $R_F$ and $\mathrm{Ric}_B$, $\mathrm{Ric}_F$ be the curvature tensors and Ricci tensors of $(B, g_B)$ and $(F, g_F)$, respectively, and let $R$ and Ric, respectively; be the curvature tensor and the Ricci tensor of ${}_fB \times_b F$, respectively.

For the Riemannian tensor and the Ricci tensor of a doubly twisted product manifold, we have the following proposition.

**Proposition 2.6** ([**8**]). *Let $M =_f B \times_b F$ be a doubly twisted product manifold. If $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, then*

$$R(X,Y)V = h^l(X,V)Y - h^l(Y,V)X + V(l)X(l)Y - V(l)Y(l)X \tag{2.8}$$

$$\mathrm{Ric}\,(X,V) = (1-r)VX(l) + (1-s)XV(k) + (n-2)X(k)V(l). \tag{2.9}$$

## 3. Some characterizations of twisted product manifolds to be warped product manifolds along one of the factors

In this section, we will show that the Weyl conformal tensor is a useful notion for characterizing twisted product manifold. Let $B \times_b F$ be the a twisted product manifold of $(B, g_B)$ and $(F, g_F)$ with twisting function $b$. Then we say that $B \times_b F$ is $mixed\ Weyl\ conformal\text{-}flat$ if $C(X,V) = 0$ for all $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$.

Moreover, $F$ is $Weyl\ conformal\text{-}flat$ along $B$ if $C(X,Y) = 0$, and $B$ is $Weyl\ conformal\text{-}flat$ along $F$ if $C(V,W) = 0$ for all $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$.

**Lemma 3.1.** *Let $M = B \times_b F$ be a twisted product manifold with a twisting function $f$. Then for $X \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, we obtain*

$$R(X,W)V = [X(k)V(k) + h^k(X,V)]W - g(W,V)[X(k)\nabla k + H^k(X)], \tag{3.1}$$

*where $H^k$ is the Hessian tensor of $k$ on $B \times_b F$, i.e., $h^k(X,Y) = g(H^k(X), Y)$.*

*Proof.* Let $X \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then, we have

$$\nabla_V W = \nabla_V W^F + V(k) + W(k)V - g(V,W)\nabla k \tag{3.2}$$

for $V, W \in \mathfrak{L}(F)$ and $k = log(b)$. Hence, using (2.2) and (3.2), we get

$$R(X,W)V = XV(k)W - Xg(W,V)\nabla k - g(W,V)\nabla_X \nabla k + X(k)g(W,V)\nabla k.$$

Thus, using again (2.2), we obtain (3.1). □

For the Weyl tensor of a twisted manifold, we obtain the following proposition.

**Proposition 3.2.** *Let $M = B \times F$ be a twisted product manifold with a twisting function $f$. Then, for $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, we have*

$$C(X,Y)V = \left(\frac{1-s}{n-2}\right)[XV(k)Y - YV(k)X] \tag{3.3}$$

$$C(V,W)X = \left(\frac{r-1}{n-2}\right)[XV(k)W - XW(k)V]. \tag{3.4}$$

*Proof.* Let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Then, we have

$$\begin{aligned} C(X,Y)V = R(X,Y)V & \\ &+ \frac{1}{n-2}[\mathrm{Ric}\,(X,V)Y - \mathrm{Ric}\,(Y,V)X + g(X,V)QY - g(Y,V)QX] \\ &- \frac{\tau}{(n-1)(n-2)}[g(X,V)Y - g(Y,V)X]. \end{aligned}$$

Since $\mathfrak{L}(B)$ and $\mathfrak{L}(F)$ are orthogonal, we get

$$C(X,Y)V = R(X,Y)V + \frac{1}{n-2}[\mathrm{Ric}\,(X,V)Y - \mathrm{Ric}\,(Y,V)X].$$

Then, from (2.9), we have

$$C(X,Y)V = \left(\frac{1-s}{n-2}\right)[XV(k)Y - YV(k)X]$$

which is (3.3).

Now, for $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, from (2.6), we have

$$C(V,W)X = R(V,W)X + \frac{1}{n-2}[\mathrm{Ric}\,(V,X)W - \mathrm{Ric}\,(W,X)V].$$

Applying the first Bianchi identity, we obtain

$$\begin{aligned} C(V,W)X &= R(X,W)V - R(X,V)W \\ &\quad + \frac{1}{n-2}\left[\mathrm{Ric}\,(V,X)W - \mathrm{Ric}\,(W,X)V\right]. \end{aligned} \tag{3.5}$$

Thus, from Lemma 3.1, we get

$$\begin{aligned} C(V,W)X &= [X(k)V(k) + h^k(X,V)]W - g(W,V)[X(k)\nabla k + H^k(X)] \\ &\quad - [X(k)W(k) + h^k(X,W)]V - g(V,W)[X(k)\nabla k + H^k(X)] \\ &\quad + \frac{1-s}{n-2}[XV(k)W - XW(k)V] \\ &= X(k)V(k)W + XV(k) - X(k)V(k)W - X(k)W(k)V \\ &\quad - XW(k)V + X(k)W(k)V \end{aligned}$$

which yields

$$C(V,W)X = \left(\frac{r-1}{n-2}\right)[XV(k)W - XW(k)V]. \tag{3.6}$$

□

In a similar way, we can give the following result for the Weyl conformal curvature tensor of a multiply twisted product manifold.

**Corollary 3.3.** *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times \ldots \times_{b_m} F_m$ be a multiply twisted product manifold with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus \ldots \oplus b_m^2 g_{F_m}$ and let $X, Y \in \mathfrak{L}(B)$, $V \in \mathfrak{L}(F_i)$. Then*

$$\begin{aligned} C(X,Y)V &= \left(\frac{s_i-1}{n-2}\right)[XV(k_i)Y - YV(k_i)X], \\ C(V,W)X &= \left(\frac{n+s_i-3}{n-2}\right)[XV(k_i)W - XW(k_i)V]. \end{aligned} \tag{3.7}$$

Now, for a twisted product manifold, we can give our main theorem:

**Theorem 3.4.** *Let $B \times_b F$ be the twisted product manifold of semi-Riemannian manifolds $(B, g_B)$, with $\dim(B) > 1$ and $(F, g_F)$ with a twisting function $b$ and $\dim F > 1$. Then $B \times_b F$ can be expressed as a warped product manifold, $B \times_\delta F$ of $(B, g_B)$ and $(F, g_{\mathfrak{F}})$ with a twisting function $\delta$ if and only if $B$ is Weyl conformal flat along $F$, where $g_{\mathfrak{F}}$ is a conformal metric tensor to $g_F$.*

*Proof.* Let $X \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$. Then, from Proposition 3.2, we have

$$C(V,W)X = \left(\frac{r-1}{n-2}\right)\left[XV(k)W - XW(k)V\right]. \tag{3.8}$$

If $C(V,W)X = 0$ for all $X \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$, then it follows that $VX(k) = 0$ and $XV(k) = 0$. $VX(k) = 0$ implies that $X(k)$ only depends on the points of $B$, and likewise, $XV(k) = 0$ implies that $V(k)$ only depends on the points of $F$. Thus, $k$ can be expressed as a sum of two functions $\alpha$ and $\beta$ which are defined on $B$ and $F$, respectively, that is, $k(p,q) = \alpha(p)+\beta(q)$ for any $(p,q) \in B \times F$. Hence $b = \exp(\delta)\exp(\gamma)$, that is, $b(p,q) = \delta(p)\gamma(q)$, where $\delta = \exp(\alpha)$ and $\gamma = \exp(\beta)$ for any $(p,q) \in B \times F$. Thus we can write $g = g_B \oplus \delta^2 g_{\mathfrak{F}}$, where $g_{\mathfrak{F}} = \gamma^2 g_F$, that is, the twisted product manifold $B \times_b F$ can be expressed as a warped product manifold $B \times_\delta F$, where the metric tensor of $F$ is $g_{\mathfrak{F}}$ given above. Therefore, the converse is obvious from equation (3.4). □

In a similar way, by using (3.3), we have the following theorem.

**Theorem 3.5.** *Let $B \times_b F$ be the twisted product manifold of semi-Riemannian manifolds $(B, g_B)$ and $(F, g_F)$ with a twisting function $b$ and $dimF > 1$. Then $B \times_b F$ can be expressed as a warped product manifold, $B \times_\delta F$ of $(B, g_B)$ and $(F, g_{\mathfrak{F}})$ with a twisting function $\delta$ if and only if $F$ is Weyl conformal flat along $B$, where $g_{\mathfrak{F}}$ is a conformal metric tensor to $g_F$.*

*Proof.* Let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Then, from Proposition 3.2, we obtain

$$C(X,Y)V = \left(\frac{1-s}{n-2}\right)\left[XV(k)Y - YV(k)X\right]. \tag{3.9}$$

The rest is similar to the previous theorem. □

Theorems 3.4 and 3.5 are generalizations of the results given in [**2**].

Now, we will give the following result about the parallel conformal Weyl tensor of the twisted product manifolds.

**Theorem 3.6.** *Let $M = B \times_b F$ be a twisted product manifold having the parallel conformal Weyl tensor such that $dimB \neq 1$ and $H^k(Y) \neq -Y(k)\nabla k$. Then, the twisted product manifold $B \times_b F$ can be expressed as a warped product.*

*Proof.* Let $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$

$$\begin{aligned}(\nabla_Y C)(V,W,X) = {} & \nabla_Y C(V,W)X - C(Y(k)V,W)X \\ & - C(V,Y(k)W)X - C(V,W)\nabla^B_Y X.\end{aligned}$$

Then, from (3.6), we have

$$\begin{aligned}(\nabla_Y C)(V,W,X) = {} & \left(\frac{r-1}{n-2}\right)\{YXV(k)W + XV(k)\nabla_Y W - YXW(k)V \\ & - XW(k)\nabla_Y V - 2Y(k)XV(k)W + 2Y(k)XW(k)V \\ & - ((\nabla^B_Y X)V(k))W + ((\nabla^B_Y X)W(k))V\}.\end{aligned}$$

Rearranging this expression, we get

$$\begin{aligned}(\nabla_Y C)(V,W,X) = {} & \left(\frac{r-1}{n-2}\right)\{YXV(k)W - XV(k)Y(k)W - YXW(k)V \\ & + XW(k)Y(k)V - ((\nabla^B_Y X)V(k))W \\ & + ((\nabla^B_Y X)W(k))V\}.\end{aligned} \tag{3.10}$$

Since

$$XV(k) = 2X(k)V(k) \qquad \text{and} \qquad V(k) = b^2 g_F(V, \nabla k), \tag{3.11}$$

we have

$$\begin{aligned}YXV(k) &= Y[2X(k)V(k)] = 2[YX(k)V(k) + X(k)YV(k)] \\ &= 2[YX(k)V(k) + X(k)2Y(k)V(k)].\end{aligned} \tag{3.12}$$

Taking into account (3.11), (3.12) in (3.10), we get

$$\begin{aligned}(\nabla_Y C)(V,W,X) = 2\left(\frac{r-1}{n-2}\right)&[V(k)W - W(k)V] \\ &\cdot [h^k(Y,X) + X(k)Y(k)].\end{aligned} \tag{3.13}$$

On the other hand, since $H^k(Y) \neq -Y(k)\nabla k$, the second term in (3.13) is not zero, thus $V(k) = 0$ or $W(k) = 0$, this gives the result. □

In the sequel we show that the condition $H^k(X) = -X(k)\nabla k$ is enough for a twisted product manifold to be a warped product manifold.

**Theorem 3.7.** *Let $M = B \times_b F$ be a twisted product manifold. If $H^k(X) = -X(k)\nabla k$, then $M$ can be written as a warped product manifold.*

*Proof.* Let $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. From hypothesis, we get

$$g(H^k(X), V) \quad = \quad -g(X(k)\nabla k, V) = -X(k)V(k). \tag{3.14}$$

On the other hand, we have

$$g(H^k(X), V) \quad = \quad h^k(X,V) = XV(k) - X(k)V(k). \tag{3.15}$$

Thus, from (3.14) and (3.15), we obtain $XV(k) = 0$, this gives result. □

**Theorem 3.8.** *Let $M = B \times_b F$ be a twisted product manifold such that $\dim F \neq 1$. If $H^k(X) = -X(k)\nabla k$, then $M$ becomes the Ricci-parallel warped product manifold.*

*Proof.* Let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$, then we have

$$\begin{aligned}(\nabla_X \operatorname{Ric})(Y,V) &= \nabla_X \operatorname{Ric}(Y,V) - \operatorname{Ric}(\nabla_X Y, V) - \operatorname{Ric}(Y, \nabla_X V) \\ &= \nabla_X \operatorname{Ric}(Y,V) - \operatorname{Ric}(\nabla^B_X Y, V) - \operatorname{Ric}(Y, X(k)V).\end{aligned} \tag{3.16}$$

Thus, using (2.9), in (3.16), we have

$$
\begin{aligned}
(\nabla_X \operatorname{Ric})(Y,V) &= \nabla_X((1-s)YV(k)) - (1-s)(\nabla^B_X Y)V(k) \\
&\quad - X(k)(1-s)YV(k) \\
&= (1-s)X(YV(k)) - (1-s)(\nabla^B_X Y)V(k) \\
&\quad - X(k)(1-s)YV(k).
\end{aligned} \tag{3.17}
$$

Here, using (3.11) and (3.12), we obtain

$$
\begin{aligned}
(\nabla_X \operatorname{Ric})(Y,V) &= (1-s)\{2[XY(k)V(k) + Y(k)2X(k)V(k)] \\
&\quad - 2(\nabla^B_X Y)(k)V(k) - X(k)2Y(k)V(k)\} \\
&= 2(1-s)[XY(k)V(k) - (\nabla^B_X Y)(k)V(k) + X(k)Y(k)V(k)] \\
&= 2(1-s)[h^k(X,Y)V(k) + X(k)Y(k)V(k)]
\end{aligned} \tag{3.18}
$$

which gives assertion. □

Now, we give characterizations by using the Weyl projective curvature tensor of a twisted product manifold. Let $B \times_b F$ be the twisted product manifold of $(B, g_B)$ and $(F, g_F)$ with a twisting function $b$. Then, we say that $B \times_b F$ is *mixed Weyl projective-flat* if $W_P(X,V) = 0$ for all $X \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Moreover, $F$ is *Weyl projective flat* along $B$ if $W_P(X,Y) = 0$, and $B$ is *Weyl projective flat* along $F$ if $W_P(V,W) = 0$ for all $X, Y \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F)$.

**Theorem 3.9.** *Let $B \times_b F$ be the twisted product manifold of semi-Riemannian manifolds $(B, g_B)$ with $\dim(B) > 1$ and $(F, g_F)$ with a twisting function $b$ and $\dim F > 1$. Then $B \times_b F$ can be expressed as a warped product manifold, $B \times_\delta F$ of $(B, g_B)$ and $(F, g_{\mathfrak{F}})$ with a twisting function $\delta$ if and only if $B$ is Weyl projective flat along $F$, where $g_{\mathfrak{F}}$ is a projective metric tensor to $g_F$.*

*Proof.* Let $X \in \mathfrak{L}(B)$ and $V, U \in \mathfrak{L}(F)$. From (2.7), we have

$$
W_P(V,U)X = \frac{r}{n-1}[XV(k)U - XU(k)V].
$$

Thus, we obtain

$$
W_P(V,U)X = \frac{r(n-2)}{(r-1)(n-1)} C(V,U)X,
$$

where $C$ is the Weyl conformal curvature tensor of the twisted product manifold. Then, the proof is obvious from Theorem 3.4. □

In a similar way, we have the following theorem.

**Theorem 3.10.** *Let $B \times_b F$ be the twisted product manifold of semi-Riemannian manifolds $(B, g_B)$ and $(F, g_F)$ with a twisting function $b$ and $dimF > 1$. Then $B \times_b F$ can be expressed as a warped product manifold, $B \times_\delta F$ of $(B, g_B)$ and $(F, g_{\mathfrak{F}})$ with a twisting function $\delta$ if and only if $F$ is Weyl projective flat along $B$, where $g_{\mathfrak{F}}$ is a projective metric tensor to $g_F$.*

*Proof.* Let $X, Y \in \mathfrak{L}(B)$ and $V \in \mathfrak{L}(F)$. Then, using (2.7), we obtain

$$W_P(X,Y)V = \frac{(n-2)}{(n-1)} C(X,Y)V,$$

where $C$ is the Weyl conformal curvature tensor of the twisted product manifold. Thus, the proof is obvious from Theorem 3.5. □

Finally, we give a similar result for multiply twisted product manifolds.

**Corollary 3.11.** *Let $B \times_{b_i} F_i$ be the multiply twisted product manifold of semi-Riemannian manifolds $(B, g_B)$ with $dim(B) > 1$ and $(F_i, g_{F_i})$ with twisting functions $b_i$ and $dim F_i > 1$. Then $B \times_{b_i} F_i$ can be expressed as a multiply warped product manifold, $B \times_{\delta_i} F_i$ of $(B, g_B)$ and $(F_i, g_{\mathfrak{F}_i})$ with twisting functions $\delta_i$ if and only if $B$ is Weyl conformal-flat along $F_i$, where $g_{\mathfrak{F}_i}$ is a conformal metric tensor to $g_{F_i}$.*

*Proof.* Let $X \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F_i)$. Then, from Corollary 3.3, we have

$$C(V,W)X = \left(\frac{n+s_i-3}{n-2}\right)[XV(k_i)W - XW(k_i)V]. \tag{3.19}$$

If $C(V,W)X = 0$ for all $X \in \mathfrak{L}(B)$ and $V, W \in \mathfrak{L}(F_i)$, then it follows that $VX(k_i) = 0$ and $XV(k_i) = 0$. $VX(k_i) = 0$ implies that $X(k_i)$ only depends on the points of $B$, and likewise, $XV(k_i) = 0$ implies that $V(k_i)$ only depends on the points of $F_i$. Thus, $k_i$ can be expressed as a sum of two functions $\alpha_i$ and $\beta_i$ which are defined on $B$ and $F_i$, respectively, that is, $k_i(p,q) = \alpha_i(p) + \beta_i(q)$ for any $(p,q) \in B \times F_i$. Hence $b_i = \exp(\delta_i)\exp(\gamma_i)$, that is, $b_i(p,q) = \delta_i(p)\gamma_i(q)$, where $\delta_i = \exp(\alpha_i)$ and $\gamma_i = \exp(\beta_i)$ for any $(p,q) \in B \times F_i$. Thus we can write, $g = g_B \oplus \delta_i^2 g_{\mathfrak{F}_i}$, where $g_{\mathfrak{F}_i} = \gamma_i^2 g_{F_i}$, that is, the twisted product manifold $B \times_{b_i} F_i$ can be expressed as a warped product manifold $B \times_{\delta_i} F_i$, where the metric tensor of $F_i$ is $g_{\mathfrak{F}_i}$ given above. Therefore, the converse is obvious from equation (3.7). □

Finally, from Corollary 3.3, we have the following result.

**Corollary 3.12.** *Let $B \times_{b_i} F_i$ be the multiply twisted product manifold of semi-Riemannian manifolds $(B, g_B)$ with $dim(B) > 1$ and $(F_i, g_{F_i})$ with twisting functions $b_i$ and $dim F_i > 1$. Then $B \times_{b_i} F_i$ can be expressed as a multiply warped product manifold, $B \times_{\delta_i} F_i$ of $(B, g_B)$ and $(F_i, g_{\mathfrak{F}_i})$ with twisting functions $\delta_i$ if and only if $B$ is Weyl projective-flat along $F_i$, where $g_{\mathfrak{F}_i}$ is a projective metric tensor to $g_{F_i}$.*

*Proof.* The proof is obvious from Theorem 3.9. □

We note that by using the notions Weyl conformal-flat along $B$ and the Weyl projective-flat along $B$, the result given in Corollary 3.11 and 3.12 can be given for a multiply twisted product manifold.

**Acknowledgment.** The authors are grateful to the referee for his/her valuable comments and suggestions.

## References


**1.** Bishop R. L. and O'Neill B., *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49.

**2.** Brozos-Vázquez M., García-Río E. and Vázquez-Lorezo, R., *Some remarks on lokally conformally flat static space-times*, J. Math. Phys. **46** (2005), 022501.

**3.** Derdzinski A. and Roter W., *Compact pseudo-Riemannian manifolds with parallel Weyl tensor*, Ann. Global Anal. Geom. **37** (2010), 73–90.

**4.** ______, *The local structure of conformally symmetric manifolds*, Bull. Belg. Math. Soc. Simon Stevin **16** (2009), 117–128.

**5.** ______, *On compact manifolds admitting indefinite metrics with parallel Weyl tensor*, J. Geom. Phys. 58 (2008), 1137-1147.

**6.** ______, *Global properties of indefinite metrics with parallel Weyl tensor*, Pure and applied differential geometry PADGE 2007, 6372, Ber. Meth., Shaker Verlag, Aachen, 2007.

**7.** ______, *Projectively flat surfaces, null parallel distributions, and conformally symmetric manifolds*, Tohoku Math. J. **59(2)** (2007), 565–602.

**8.** Fernandez-Lopez M., Garcia-Rio E., Kupeli D. N. and Ünal B., *A curvature condition for a twisted product to be a warped product*, Manuscripta Math. **106** (2001), 213–217.

**9.** Hiepko S., *Eine innere Kennzeichnung der verzerrten Produkte*, Math. Ann. **241** (1979), 209–215.

**10.** O'Neill B., *Semi-Riemannian Geometry With Applications to Relativity*, New York, Academic Press, 1983.

**11.** Ponge R. and Reckziegel H., *Twisted Products in Pseudo-Riemannian Geometry*, Geometriae Dedicata **48**, (1993), 15–25.

**12.** Ünal B., *Multiply warped products*, Journal of Geometry and Physics **34** (2000), 287–301.

**13.** Yano K. and Kon, M., *Structures on Manifolds*, Series in Pure Math., Vol 3, World Sci., 1984.

**14.** Wang Yong., *Multiply twisted products*, arXiv:1207.0199v1 .


S. Kazan, Department of Mathematics, Inonu University, 44280 Malatya, Turkey,
*e-mail*: `sema.bulut@inonu.edu.tr`

B. Şahin, Department of Mathematics, Inonu University, 44280 Malatya, Turkey,
*e-mail*: `bayram.sahin@inonu.edu.tr`