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ON (WEAK) GORENSTEIN GLOBAL DIMENSIONS

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Abstract. In this note, we characterize the (weak) Gorenstein global dimension for arbitrary associative rings. Also, we extend the well-known Hilbert's syzygy Theorem to the weak Gorenstein global dimension, and we study the weak Gorenstein homological dimensions of direct product of rings which gives examples of non-coherent rings with finite Gorenstein dimensions > 0 and infinite classical weak dimension.

1. Introduction

Throughout this paper, R denotes – if not specified otherwise – a non-trivial associative ring and the word R-module means left R-module.

Let R be a ring and let M be an R-module. As usual, we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$, and $\mathrm{fd}_R(M)$ to denote the classical projective dimension, injective dimension and flat dimension of M, respectively. We denote the character module of M by $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We use also $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$, respectively, to denote the classical global and weak dimension of R.

For a two-sided Noetherian ring R, Auslander and Bridger [**2**] introduced the G-dimension, $\mathrm{Gdim}_R(M)$ for every finitely generated R-module M. They showed that there is an inequality $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ for all finite R-modules M and equality holds if $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [**10**], [**11**] introduced the notion of Gorenstein projective dimension (G-projective dimension for short) as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [**13**] introduced the Gorenstein flat dimension. Some references are [**7, 8, 10, 11, 13, 16**].

Recall that a left (resp. right) R-module M is called *Gorenstein projective* if there exists an exact sequence of projective left (resp. right) R-modules

$$\mathbf{P}: \qquad \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

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such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and the operator $\mathrm{Hom}_R(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a left (resp. right) projective R-module. The resolution $\mathbf{P}$ is called a *complete projective resolution*.

The left and right *Gorenstein injective* R-modules are defined dually.

And an R-module M is called left (resp. right) *Gorenstein flat* if there exists an exact sequence of flat left (resp. right) R-modules

$$\mathbf{F}: \qquad \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \cdots$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and the operator $I \otimes_R -$ (resp. $- \otimes_R I$) leaves F exact whenever I is a right (resp. left) injective R-module. The resolution $\mathbf{F}$ is called a *complete flat resolution*.

The Gorenstein projective, injective and flat dimensions are defined in terms of resolutions and denoted by $\mathrm{Gpd}(-)$, $\mathrm{Gid}(-)$ and $\mathrm{Gfd}(-)$, respectively; see [**7, 12, 16**]).

In [**3**], the authors proved the equality

$$\sup\{\mathrm{Gpd}_R(M) \mid M \text{ is a left } R\text{-module}\} = \sup\{\mathrm{Gid}_R(M) \mid M \text{ is a left } R\text{-module}\}.$$

They called the common value of the above quantities the *left Gorenstein global dimension* of R and denoted it by $l.\,\mathrm{Ggldim}(R)$. Similarly, they set

$$l.\,\mathrm{wGgldim}(R) = \{\mathrm{Gfd}_R(M) \mid M \text{ is a left } R\text{-module}\}$$

which they called the *left weak Gorenstein global dimension* of R. Similarly, with the right modules, we can define the right Gorenstein global and weak dimensions; $r.\,\mathrm{Ggldim}(R)$ and $r.\,\mathrm{wGgldim}(R)$. When R is a commutative ring, we drop the unneeded letters r and l.

The Gorenstein global dimension measures how far away a ring R is from being quasi-Frobenius (*i.e.,* Noetherian and self injective rings); see [**3**, Proposition 2.6]). On the other hand, from Faith-Walker Theorem [**18**, Theorem 7.56], a ring is quasi-Frobenius if and only if every right (resp. left) injective module is projective, or equivalently, every right (resp. left) projective module is injective. Hence, from [**3**, Proposition 2.6], we have the following corollary

Corollary 1.1. *The following statements are equivalent:*

(1) $l.\,\mathrm{Ggldim}(R) = 0$.
(2) $r.\,\mathrm{Ggldim}(R) = 0$.
(3) *A left (and right) R-module is projective if and only if it is injective.*

For rings with high $l.\,\mathrm{Ggldim}(-)$, [**3**, Lemma 2.1] gives a nice characterization to $l.\,\mathrm{Ggldim}(R)$ for an arbitrary ring R, provided the finiteness of this dimension as shown by the next proposition.

Proposition 1.2. [**3**, Lemma 2.1] *If $l.\,\mathrm{Ggldim}(R) < \infty$, then the following statements are equivalent:*

1. $l.\,\mathrm{Ggldim}(R) \leq n$.
2. $\mathrm{id}_R(P) \leq n$ *for every (left) R-module P with finite projective dimension.*

There is a similar result of Corollary 1.1 for the weak Gorenstein global dimension as shown by the proposition bellow. Recall that a ring is called right (resp. left) IF-*ring*, if every right (resp. left) injective module is flat, and it is called IF-ring, if it is both right and left IF-ring; see [**9**].

Proposition 1.3. *The following statements are equivalent for any ring R:*

(1) $l.\,\mathrm{wGgldim}(R) = 0$.

(2) *R is an* IF*-ring.*

(3) $r.\,\mathrm{wGgldim}(R) = 0$.

Proof. We prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), while the inverse implications are proved similarly.

(1) $\Rightarrow$ (2) Suppose that $l.\,\mathrm{wGgldim}(R) = 0$. Let I be a right injective R-module. For an arbitrary left R-module M and every $i > 0$, we have $\mathrm{Tor}_R^i(I, M) = 0$ (from the definition of Gorenstein flat modules). Then, I is flat. Moreover, since every left R-module is Gorenstein flat (since $l.\,\mathrm{wGgldim}(R) = 0$), every left R-module can be embedded in a left flat R-module. In particular, every left injective R-module is contained in a flat module. Then, every left injective R-module is a direct summand of a flat module, and then it is flat as desired.

(2) $\Rightarrow$ (3) Let M be a right R-module. Assemble any flat resolution of M with its any injective resolution, we get an exact sequence of right flat R-modules $\mathbf{F}$ (since every right injective module is flat). Also, since every left injective module I is flat, $\mathbf{F} \otimes_R I$ is exact. Hence, $\mathbf{F}$ is a complete flat resolution. This means that M is Gorenstein flat. Consequently, $r.\,\mathrm{wGgldim}(R) = 0$. □

In the next section, we give a generalization of Corollary 1.1 and Proposition 1.3 in the way of Proposition 1.2 for an arbitrary ring with high (weak) Gorenstein global dimension (see Theorems 2.1, and 2.6). In the third section, we extend the well-known Hilbert's syzygy Theorem to the weak Gorenstein global dimension and we study the weak Gorenstein homological dimension of the direct product of rings[1] which gives examples of non-coherent rings with finite Gorenstein dimensions > 0 and infinite classical weak dimensions.

2. Characterizations of (weak) Gorenstein global dimensions

The first main result of this section is the following theorem.

Theorem 2.1. *Let R be a ring and let n be a positive integer. Then, $l.\,\mathrm{Ggldim}(R) \leq n$ if and only if R satisfies the following two conditions:*

(C1) $\mathrm{id}_R(P) \leq n$ *for every (left) projective R-module P.*

(C2) $\mathrm{pd}_R(I) \leq n$ *for every (left) injective R-module I.*

Proof. ($\Rightarrow$) Suppose that $l.\,\mathrm{Ggldim}(R) \leq n$. We claim (C1). Let P be a projective R-module. Since $\mathrm{Gpd}_R(M) \leq n$ for every R-module M, we have

[1] The extension of the Hilbert's syzygy Theorem and the study the weak Gorenstein homological dimensions of direct product of rings to weak Gorenstein dimension was done in [**3**] over a coherent rings. Here, we give a generalization to an arbitrary ring.

$\mathrm{Ext}_R^i(M,P)=0$ for all $i>n$ ([**16**, Theorem 2.20]). Hence, $\mathrm{id}_R(P)\leq n$ as desired.

Now, we claim (C2). Let I be an injective R-module. Since $l.\,\mathrm{Ggldim}(R)=\sup\{\,\mathrm{Gid}_R(M)\mid \text{M is a left R-module}\}$ for an arbitrary R-module M, we have $\mathrm{Ext}_R^i(I,M)=0$ for all $i>n$ ([**16**, Theorem 2.22]). Hence, $\mathrm{pd}_R(I)\leq n$ as desired.

($\Leftarrow$) Suppose that R satisfies (C1) and (C2). We claim that $l.\,\mathrm{Ggldim}(R)\leq n$. Let M be an arbitrary R module and consider an n-step projective resolution of M as follows:

$$0\longrightarrow G\longrightarrow P_n\longrightarrow\cdots\longrightarrow P_1\longrightarrow M\longrightarrow 0.$$

We have to prove that G is Gorenstein projective. Firstly, for every projective R-module P and every $i>0$, we have $\mathrm{Ext}_R^i(G,P)=\mathrm{Ext}_R^{n+i}(M,P)=0$ by condition (C1). So, from [**16**, Proposition 2.3], it suffices to prove that G admits a right co-proper projective resolution (see [**16**, Definition 1.5]). Pick a short exact sequence

$$0\longrightarrow M\longrightarrow I\longrightarrow M'\longrightarrow 0,$$

where I is an injective R-module and consider an n-step projective resolution of M' as follows

$$0\longrightarrow G'\longrightarrow P'_n\longrightarrow\cdots\longrightarrow P'_1\longrightarrow M'\longrightarrow 0.$$

We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q_1 \arrow[r] \arrow[d] & P_n\oplus P'_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1\oplus P'_1 \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \arrow[r] \arrow[d] & P'_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_1 \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathrm{pd}_R(I)\leq n$ (by (C2)), the module Q_1 is clearly projective. On the other hand, we have $\mathrm{Ext}_R^1(G',P)=\mathrm{Ext}_R^{n+1}(M',P)=0$ for every projective module P since (C1). Thus, the functor $\mathrm{Hom}_R(-,P)$ keeps the exactness of the short exact sequence $0\to G\to Q_1\to G'\to 0$. By repeating this procedure, we obtain a right projective resolution of G

$$0\longrightarrow G\longrightarrow Q_1\longrightarrow Q_2\longrightarrow\cdots$$

such that $\mathrm{Hom}_R(-,P)$ leaves this sequence exact whenever P is projective. Hence, G is Gorenstein projective. Consequently, $\mathrm{Gpd}_R(M)\leq n$ and then $l.\,\mathrm{Ggldim}(R)\leq n$, as desired. $\square$

If we denote $l.\mathcal{P}(R)$ (resp. $r.\mathcal{P}(R)$) and $l.\mathcal{I}(R)$ (resp. $r.\mathcal{I}(R)$), respectively, the sets of all left (resp. right) projective and injective R-modules, we have:

$$l.\operatorname{Ggldim}(R) = \sup\{\operatorname{pd}_R(I), \operatorname{id}_R(P) \mid I \in l.\mathcal{I}(R), P \in l.\mathcal{P}(R)\},$$
$$r.\operatorname{Ggldim}(R) = \sup\{\operatorname{pd}_R(I), \operatorname{id}_R(P) \mid I \in r.\mathcal{I}(R), P \in r.\mathcal{P}(R)\}.$$

There is another way to write the above theorem

Corollary 2.2. *Let R be a ring and let n be a positive integer. The following statements are equivalent:*

(1) $l.\operatorname{Ggldim}(R) \leq n$.
(2) *For any R-module M* $\operatorname{pd}_R(M) \leq n \Leftrightarrow \operatorname{id}_R(M) \leq n$.

Proof. (1) $\Rightarrow$ (2) Let M be an R-module such that $\operatorname{pd}_R(M) \leq n$. For such a module, consider a projective resolution as follows

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

From Theorem 2.1, $\operatorname{id}_R(P_i) \leq n$ for each $i = 0, \dots, n$. Hence, $\operatorname{id}_R(M) \leq n$. Similarly, we prove that $\operatorname{pd}_R(M) \leq n$ for every R-module such that $\operatorname{id}_R(M) \leq n$.

(2) $\Rightarrow$ (1) follows directly from Theorem 2.1 since the conditions (C1) and (C2) are clearly satisfied. □

Proposition 2.3. *Let R be a ring with finite Gorenstein global dimension. Then,* (C1) *and* (C2) *of Theorem 2.1 are equivalent, and then the following statements are equivalent:*

(1) $l.\operatorname{Ggldim}(R) \leq n$.
(2) $\operatorname{id}_R(P) \leq n$ *for every projective R-module P.*
(3) $\operatorname{pd}_R(I) \leq n$ *for every injective R-module I.*

Proof. From Theorem 2.1, only the equivalence of (C1) and (C2) needs a proof. However, we will be satisfied to prove the implication (C1) $\Rightarrow$ (C2) whereas the other implication is proved in an analogous fashion. Let M be an arbitrary R-module. For every projective R-module P and any $i > n$, we have $\operatorname{Ext}^i_R(M, P) = 0$ (by (C1)). Then, by [**16**, Theorem 2.20], $\operatorname{Gpd}_R(M) \leq n$. Therefore, we have $l.\operatorname{Ggldim}(R) \leq n$. Accordingly, by Theorem 2.1, (C2) is satisfied, as desired. □

Using the definition of the Gorenstein flat modules, we have immediately the following lemma

Lemma 2.4. *An R-module M is left (resp. right) Gorenstein flat if and only if*

(1) $\operatorname{Tor}^i_R(I, M) = 0$ *(resp.,* $\operatorname{Tor}^i_R(M, I) = 0$*) for every right (resp. left) injective R-module I and every $i > 0$.*
(2) *There exists an exact sequence of left (resp. right) R-modules*

$$0 \longrightarrow M \longrightarrow F_0 \longrightarrow F_1 \longrightarrow \cdots,$$

where each F_i is flat such that the functor $I \otimes_R -$ (resp. $- \otimes_R I$) keeps the exactness of this sequence whenever I is right (resp. left) injective.

Remark 2.5. Using the lemma above and an n-step flat resolution of left (resp. right) R-module M we conclude that if $\mathrm{Gfd}_R(M) \leq n$, then $\mathrm{Tor}_R^i(I, M) = 0$ (resp., $\mathrm{Tor}_R^i(M, I) = 0$) for every right (resp. left) injective R-module I and all $i > n$. The inverse implication is given by Holm ([**16**, Theorem 3.14]) when $\mathrm{Gfd}_R(M) < \infty$ and the ring is right (resp. left) coherent.

Recall that over a ring R, Ding and Chen [**9**] defined and investigated two global dimensions as follows:

$$r.\,\mathrm{IFD}(R) = \sup\{\,\mathrm{fd}_R(I) \mid I \text{ is a right injective } R\text{-module}\}.$$
$$l.\,\mathrm{IFD}(R) = \sup\{\,\mathrm{fd}_R(I) \mid I \text{ is a left injective } R\text{-module}\}.$$

For such dimensions, in [**9**], Ding and Chen gave a several characterizations over arbitrary rings, over coherent rings, and also over commutative rings. Our second main result of this section is given by the theorem bellow.

Theorem 2.6. *Let R be a ring and let n be a positive integer. The following conditions are equivalent:*

(1) $\sup\{\,l.\,\mathrm{wGgldim}(R), r.\,\mathrm{wGgldim}(R)\} \leq n$.
(2) $\mathrm{Gfd}_R(R/E) \leq n$ *for every (left and right) ideal E.*
(3) $\mathrm{fd}_R(I) \leq n$ *for every (left and right) injective R-module I.*

Consequently

$$\begin{aligned}\sup\{\,l.\,\mathrm{wGgldim}(R), r.\,\mathrm{wGgldim}(R)\} &= \sup\{\,\mathrm{fd}_R(I) \mid I \in l.\mathcal{I}(R) \cup r.\mathcal{I}(R)\} \\ &= \sup\{\,l.\,\mathrm{IFD}(R), r.\,\mathrm{IFD}(R)\}.\end{aligned}$$

Proof. (1) $\Rightarrow$ (2) Obvious by the definition of the left and right weak Gorenstein global dimensions.

(2) $\Rightarrow$ (3) Let I be a left injective R-module. Since $\mathrm{Gfd}_R(R/E) \leq n$ for every right ideal E and by Remark 2.5, we get $\mathrm{Tor}_R^i(R/E, I) = 0$ for all $i > n$. Hence, by [**19**, Lemma 9.18], $\mathrm{fd}_R(I) \leq n$. Similarly, we prove that $\mathrm{fd}_R(I) \leq n$ for every right injective R-module.

(3) $\Rightarrow$ (1) Let M be an arbitrary left R-module and consider an n-step projective resolution of M as follows:

$$0 \longrightarrow G \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0.$$

We have to prove that G is a Gorenstein flat R-module. Firstly, for every right injective R-module I and any $i > 0$, we have $\mathrm{Tor}_R^i(I, G) = \mathrm{Tor}_R^{n+i}(I, M) = 0$ since $\mathrm{fd}_R(I) \leq n$ (by hypothesis).

Pick a short exact sequence of left R-modules

$$0 \longrightarrow M \longrightarrow J \longrightarrow M' \longrightarrow 0,$$

where J is an injective R-module and consider an n-step projective resolution of M' as follows

$$0 \longrightarrow G' \longrightarrow P'_n \longrightarrow \cdots \longrightarrow P'_1 \longrightarrow M' \longrightarrow 0$$

We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & P_n \oplus P'_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \oplus P'_1 \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G' \arrow[r] \arrow[d] & P'_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_1 \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathrm{fd}_R(J) \leq n$, the module F_1 is clearly left flat. On the other hand, we have $\mathrm{Tor}_R^1(I, G') = \mathrm{Tor}_R^{n+1}(I, M') = 0$ for every right injective R-module I (since $\mathrm{fd}_R(I) \leq n$). Thus, the functor $I \otimes_R -$ keeps the exactness of the short exact sequence $0 \to G \to F_1 \to G' \to 0$. By repeating this procedure, we obtain a flat resolution of G as follows:

$$0 \longrightarrow G \longrightarrow F_1 \longrightarrow F_2 \longrightarrow \cdots$$

such that $I \otimes_R -$ leaves this sequence exact whenever I is right injective. Hence, from Lemma 2.4, G is left Gorenstein flat. Therefore, $\mathrm{Gfd}_R(M) \leq n$. Consequently, $l.\,\mathrm{wGgldim}(R) \leq n$ as desired.

Similarly, we prove that $r.\,\mathrm{wGgldim}(R) \leq n$. □

It is true that $l.\,\mathrm{wGgldim}(R) \leq n$ implies that $\mathrm{fd}_R(I) \leq n$ for every right injective R-module (by Remark 2.5). However, the inverse implication is not true in the general case as shown by the next example. That explicates the form of Theorem 2.6.

Example 2.7. Let R be a two-sided coherent ring which is right IF, but not left IF (see [**6**, Example 2]). Then, $l.\,\mathrm{wGgldim}(R) = r.\,\mathrm{wGgldim}(R) = \infty$.

Proof. If $l.\,\mathrm{wGgldim}(R) < \infty$, then using [**16**, Theorem 3.14] and since every right injective R-module is flat (since R is right IF ring), we have $\mathrm{Gfd}_R(M) = 0$ for every left R-module M. Then $l.\,\mathrm{wGgldim}(R) = 0$. So, by Proposition 1.3, R is left IF. So, we obtain a contradiction with the hypothesis conditions.

Now, if $r.\,\mathrm{wGgldim}(R) = n < \infty$, then $\mathrm{fd}_R(I) \leq n$ for every left injective R-module I. On the other hand, $\mathrm{fd}_R(I') = 0 \leq n$ for every right injective R-module I' since R is right IF. Then, by Theorem 2.6, $\sup\{\,l.\,\mathrm{wGgldim}(R), r.\,\mathrm{wGgldim}(R)\} \leq n$. Absurd since $l.\,\mathrm{wGgldim}(R) = \infty$. □

Over a right coherent ring, the characterization of $l.\,\mathrm{wGgldim}(R)$ is simpler as shown by the next proposition.

Proposition 2.8. *Let R be a right coherent ring. Then*

$$l.\operatorname{wGgldim}(R) = \sup\{\, l.\operatorname{IFD}(R), r.\operatorname{IFD}(R)\}.$$

Proof. From Theorem 2.6, only the inequality ($\geq$) needs a proof, and we may assume $l.\operatorname{wGgldim}(R) \leq n < \infty$. Clearly, $l.\operatorname{wGgldim}(R) \geq r.\operatorname{IFD}(R)$ since $\operatorname{fd}_R(I) \leq n$ for every right injective module I (by Remark 2.5). So, we have to prove this fact for $l.\operatorname{IFD}(R)$. Let I be a left injective R-module. Since $l.\operatorname{wGgldim}(R) \leq n$, we have $\operatorname{Gfd}_R(I) \leq n$. Then, by [**16**, Lemma 3.17], there exists a short exact sequence $0 \to K \to G \to I \to 0$ where G is left Gorenstein flat and $\operatorname{fd}_R(K) \leq n-1$ (if $n = 0$, this should be interpreted as $K = 0$). Pick a short exact sequence $0 \to G \to F \to G' \to 0$, where F is left flat and G' is left Gorenstein flat. Hence, consider the following push-out diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & G \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, dotted] & G' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r, dotted] \arrow[d] & D \arrow[r] \arrow[d] & G' \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Clearly, $\operatorname{fd}_R(D) \leq n$ and I is isomorphic to a direct summand of D since it is injective. Then, $\operatorname{fd}_R(I) \leq n$. Consequently, $l.\operatorname{wGgldim}(R) \geq l.\operatorname{IFD}(R)$, as desired. $\square$

Similarly, we have the following proposition

Proposition 2.9. *Let R be a left coherent ring. Then,*

$$r.\operatorname{wGgldim}(R) = \sup\{\, l.\operatorname{IFD}(R), r.\operatorname{IFD}(R)\}.$$

Corollary 2.10. *Let R be a ring. The following statements holds:*

(1) *If R is right coherent, then $r.\operatorname{wGgldim}(R) \leq l.\operatorname{wGgldim}(R)$.*
(2) *If R is left coherent, then $l.\operatorname{wGgldim}(R) \leq r.\operatorname{wGgldim}(R)$.*

Consequently, if R is two-sided coherent, then $r.\operatorname{wGgldim}(R) = l.\operatorname{wGgldim}(R)$.

Proof. We suggest to prove (1) whereas the proof of (2) will be similar. If R is right coherent, we have

$$\begin{aligned}
l.\operatorname{wGgldim}(R) &= \sup\{\, l.\operatorname{IFD}(R), r.\operatorname{IFD}(R)\} && \text{(by Proposition 2.8).}\\
&= \sup\{\, r.\operatorname{wGgldim}(R), l.\operatorname{wGgldim}(R)\} && \text{(by Theorem 2.6).}
\end{aligned}$$

Then, we obtain the desired result. $\square$

Remark 2.11. Using Theorem 2.6, Proposition 2.8, Proposition 2.9 and Corollary 2.10, we can find many other characterizations of $l.\,\mathrm{wGgldim}(R)$ and $r.\,\mathrm{wGgldim}(r)$ by using the characterizations of the $l.\,\mathrm{IFD}(R)$ and $r.\,\mathrm{IFD}(R)$. For example, we use [**9**, Theorems 3.5 and 3.8, Proposition 3.17, Corollary 3.18].

The commutative version of Theorem 2.6 is as follows

Theorem 2.12. *Let R be a commutative ring and let n be a positive integer. The following conditions are equivalent:*

(1) $\mathrm{wGgldim}(R) \leq n$,
(2) $\mathrm{Gfd}_R(R/E) \leq n$ *for every ideal E of R,*
(3) $\mathrm{fd}_R(I) \leq n$ *for every injective R-module I.*

Consequently, $\mathrm{wGgldim}(R) = \mathrm{IFD}(R)$.

Proposition 2.13 ([**9**], Theorems 3.5, 3.8, and 3.21)**.** *For any commutative ring, the following conditions are equivalent:*

(1) $\mathrm{wGgldim}(R)(= \mathrm{IFD}(R)) \leq n$,
(2) $\mathrm{fd}_R(M) \leq n$ *for every* FP-*injective module M,*
(3) $\mathrm{fd}_R(M) \leq n$ *for every R-module M with* $\mathrm{FP}\text{-}\mathrm{id}_R(M) < \infty$,
(4) $\mathrm{id}_R(\mathrm{Hom}_R(A, B)) \leq n$ *for every* FP-*injective module A and for every injective module B,*
(5) $\mathrm{fd}_R(\mathrm{Hom}_R(F, B)) \leq n$ *for every flat modules F and every injective module B.*

Moreover, if R is coherent, then $\mathrm{wGgldim}(R) = \mathrm{FP}\text{-}\mathrm{id}_R(R)$.

3. Weak Global Gorenstein dimensions of polynomial rings and direct products of rings

In [**4**, Theorems 2.11 and 3.5], the authors proved that:

(R1) If $\{R_i\}_{i=1}^n$ is a family of coherent commutative rings, then

$$\mathrm{wGgldim}\left(\prod_{i=1}^{n} R_i\right) = \sup\{\mathrm{wGgldim}(R_i) : 1 \leq i \leq n\}.$$

(R2) If the polynomial ring $R[x]$ in one indeterminate x over a commutative ring R is coherent, then

$$\mathrm{wGgldim}(R[x]) = \mathrm{wGgldim}(R) + 1.$$

In the next theorems, we will see that the coherence condition is not necessary in (R1) and (R2).

Theorem 3.1. *For every family of commutative rings $\{R_i\}_{i=1}^n$, we have*

$$\mathrm{wGgldim}\left(\prod_{i=1}^{n} R_i\right) = \sup\{\mathrm{wGgldim}(R_i) : 1 \leq i \leq n\}.$$

Proof. By induction on n, it suffices to prove this result for $n = 2$.

Assume that $\mathrm{wGgldim}(R_1 \times R_2) \leq k$. Let M_i be an R_i-module for $i = 1, 2$. Since each R_i is a projective $R_1 \times R_2$-module, by [**16**, Proposition 3.10], we have $\mathrm{Gfd}_{R_i}(M_i) \leq \mathrm{Gfd}_{R_i \times R_2}(M_1 \times M_2) \leq k$. This follows that $\mathrm{wGgldim}(R_i) \leq k$ for each $i = 1, 2$.

Conversely, suppose that $\sup\{ \mathrm{wGgldim}(R_i) : i = 1, 2\} \leq k$. Let I be an arbitrary injective $R_1 \times R_2$-module. We can see that

$$I \cong \mathrm{Hom}_{R_1 \times R_2}(R_1 \times R_2, I) \cong \mathrm{Hom}_{R_1 \times R_2}(R_1, I) \times \mathrm{Hom}_{R_1 \times R_2}(R_2, I)$$

and $I_i = \mathrm{Hom}_{R_1 \times R_2}(R_i, I)$ is an injective R_i-module for each $i = 1, 2$. Since $\mathrm{wGgldim}(R_i) \leq k$ for each $i = 1, 2$, we get $\mathrm{fd}_{R_i}(I_i) \leq k$ (by Theorem 2.12). Using [**4**, Lemma 3.7], we have $\mathrm{fd}_{R_1 \times R_2}(I_1 \times I_2) = \sup\{ \mathrm{fd}_{R_i}(I_1), 1 \leq i \leq 2\} \leq k$. Consequently, by Theorem 2.12, $\mathrm{wGgldim}(R_1 \times R_2) \leq k$. This completes the proof.

A remark to the other easy proof to this Theorem is that the category of modules over a finite product of rings is equivalent to the product of the categories of modules over each of the rings in the product. □

Theorem 3.2. *Let $R[x]$ be the polynomial ring in one indeterminate x over a commutative ring R. Then*

$$\mathrm{wGgldim}(R[x]) = \mathrm{wGgldim}(R) + 1.$$

To prove this theorem, we need the following lemmas.

Lemma 3.3 ([**14**], Theorem 2.1)**.** *Let R be any ring and let M be an R-module. Then* $\mathrm{fd}_R(M) = \mathrm{id}_R(M^+)$.

Lemma 3.4 ([**17**], Theorem 202)**.** *Let R be any ring (not necessarily commutative). Let x be a central non-zero-divisor in R, and write $R^* = R/(x)$. Let A be a non-zero R^*-module with* $\mathrm{id}_{R^*}(A) = n < \infty$. *Then* $\mathrm{id}_R(A) = n + 1$.

Proof of Theorem 3.2. Firstly, we claim that $\mathrm{wGgldim}(R) \leq \mathrm{wGgldim}(R[x])$. Let I be an arbitrary injective R-module. Clearly, the $R[X]$-module $\mathrm{Hom}_R(R[x], I)$ is injective. Hence, by Theorem 2.12, $\mathrm{fd}_{R[x]}(\mathrm{Hom}_R(R[x], I)) \leq \mathrm{wGgldim}(R[x])$. On the other hand, by [**15**, Theorem 1.3.12],

$$\mathrm{fd}_R(\mathrm{Hom}_R(R[x], I)) \leq \mathrm{fd}_{R[x]}(\mathrm{Hom}_R(R[x], I)),$$

and it is clear that $I \cong \mathrm{Hom}_R(R, I)$ is isomorphic to a direct summand of $\mathrm{Hom}_R(R[x], I)$ (as an R-modules). Hence, $\mathrm{fd}_R(I) \leq \mathrm{wGgldim}(R[x])$. Then

$$\mathrm{wGgldim}(R) = \sup\{ \mathrm{fd}_R(I) \mid \text{I an injective } R\text{-module}\} \leq \mathrm{wGgldim}(R[x]).$$

Secondly, we will prove that $\mathrm{wGgldim}(R[x]) \leq \mathrm{wGgldim}(R) + 1$. We may assume that $\mathrm{wGgldim}(R) = n < \infty$. Otherwise, the result is obvious. Let I be an arbitrary injective $R[x]$-module. From [**15**, Theorem 1.3.16], $\mathrm{fd}_{R[x]}(I) \leq \mathrm{fd}_R(I)+1$. However, I is also an injective R-module since $R[x]$ is a free (then flat) R-module. Then $\mathrm{fd}_{R[x]}(I) \leq \mathrm{fd}_R(I) + 1 \leq \mathrm{wGgldim}(R) + 1$. Hence,

$$\mathrm{wGgldim}(R[x]) = \sup\{\mathrm{fd}_{R[x]}(I) : \text{I an injective R[x]-module}\} \leq \mathrm{wGgldim}(R) + 1.$$

Finally, we have to prove that $\mathrm{wGgldim}(R[x]) \geq \mathrm{wGgldim}(R) + 1$. From the first part of this proof, we may assume that $\mathrm{wGgldim}(R) = n < \infty$. Otherwise, the result is obvious. Let I be an injective R-module such that $\mathrm{fd}_R(I) = n$ (such a module exists by Theorem 2.12). Then by Lemma 3.3, $\mathrm{id}_R(I^+) = n < \infty$. Therefore, by Lemma 3.4, $\mathrm{id}_{R[x]}(I^+) = n + 1$. Again, by Lemma 3.3, $\mathrm{fd}_{R[x]}(I) = n+1$. On the other hand, by Lemma 3.4, $\mathrm{id}_{R[x]}(I) = 1$. Pick an injective resolution of I over $R[x]$ as follows:

$$0 \to I \to I_0 \to I_1 \to 0$$

where I_0 and I_1 are an injective $R[x]$-modules. Then

$$n + 1 = \mathrm{fd}_{R[x]}(I) \leq \sup\{\, \mathrm{fd}_{R[x]}(I_0), \mathrm{fd}_{R[x]}(I_1) - 1\} \leq \mathrm{wGgldim}(R[x]).$$

Therefore, $\mathrm{wGgldim}(R) + 1 \leq \mathrm{wGgldim}(R[x])$ as desired. This finishes our proof. □

Remark 3.5. Let M be an R-module. Using the definition of the character $M^+ = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, we see that the modulation of M^+ over $R[x]$ is the same:

(1) When we consider M^+ as an R-module and then we consider M^+ as an $R[x]$-module.
(2) When we consider M as an $R[x]$-module (by setting $xM = 0$) from the beginning.

Now, we are able to give a class of non-coherent rings $\{\, R_n\}_{n>0}$ with infinite weak global dimensions such that $\mathrm{wGgldim}(R_n) = n$.

Example 3.6. Consider the local Noetherian non semisimple quasi-Frobenius ring $R := K[X]/(X^2)$ where K is a field, and let S be a non-coherent commutative ring with $\mathrm{wdim}(R) = 1$. For each $n > 0$, set $T_n = R[X_1, X_2, \ldots, X_n]$ the polynomial ring over R. Then

(1) $\mathrm{wdim}(T_n \times S) = \infty$,
(2) $\mathrm{wGgldim}(T_n \times S) = n$,
(3) $T_n \times S$ is not coherent.

Proof. (1) Follows from the fact that $\mathrm{wdim}(R) = \infty$.

(2) Clearly, since R is Noetherian and by using [**4**, Theorem 3.5], [**3**, Corollary 1.2 and Proposition 2.6], and [**12**, Theorem 12.3.1], we have $\mathrm{wGgldim}(T_n) = \mathrm{Ggldim}(T_n) = n$ and $\mathrm{wGgldim}(S) = \mathrm{wdim}(S) = 1$. Hence, by Theorem 3.1, $\mathrm{wGgldim}(T_n \times S) = n$ as desired.

(3) Clearly since S is non-coherent, this completes the proof. □

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