Acta Math. Univ. Comenianae
Vol. LXXXII, 2 (2013), pp. 191–200

PROPERTIES OF THE INTERVAL GRAPH OF A BOOLEAN FUNCTION

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ABSTRACT. In the present paper we describe relations between the interval graph of a Boolean function, its abbreviated disjunctive normal form and its minimal disjunctive normal forms. The inteval graph of a Boolean function f has vertices corresponding to the maximal intervals of f and any two vertices are joined with an edge if the corresponding maximal intervals have nonempty intersection.

1. Introduction

A Boolean function can be represented by several types of graphs. Among them, the greatest attention has been devoted to the study of the graph $G(f)$ induced by the vertices of the n-cube, on which the Boolean function f takes the value 1. This geometric representation was introduced by Yablonskiy in [**1**]. The concept of the interval graph of a Boolean function was defined by Sapozhenko in [**3**]. The interval graph is a graph associated with a Boolean function f such that the vertices correspond to maximal intervals of f and two vertices are joined with an edge if the intersection of the corresponding intervals is nonempty. The parameters such as the size and the number of connected components, the radius and the diameter of these graphs are closely related to local algorithms of construction of a minimal disjunctive normal form of a Boolean function (briefly d.n.f.), described by Zhuravlev; for exact definitions see [**5**]. Toman [**6**] employed a method of good and bad vertices of a Boolean function to give an upper bound for the vertex degree of the interval graph for almost all Boolean functions. This method was applied by Toman, Olejar and Stanek in [**9**] where they gave asymtotic upper and lower bounds for the average vertex degree in the interval graph of a Boolean function. Toman and Daubner in [**7**] obtained asymptotic estimate of a vertex degree in the interval graph of a Boolean function. In a recent paper [**8**], they also obtained asymptotic estimates for the size of the neighbourhood of a constant order in the interval graph of a Boolean function.

In the present paper, we prove that if the interval graph of a Boolean function is a complete graph, then the abbreviated d.n.f. of this function is also a minimal

Received June 26, 2012; revised October 16, 2012.
2010 *Mathematics Subject Classification.* Primary 60C05, 05C80, 68R10.
Key words and phrases. Boolean function; interval graph; disjunctive normal form.
Supported by VEGA grant No 1/1005/12.

d.n.f.. In addition, we describe a construction of a Boolean function such that the abbreviated d.n.f. of this function is also a minimal d.n.f., and the interval graph may be an arbitrary simple finite graph. We also study a relationship between the number of vertices and edges in the interval graph and the dimension of a corresponding Boolean function. We also present examples of pairs of Boolean functions with isomorphic interval graphs where one of the functions has a minimal d.n.f. identical with its abbreviated d.n.f. while the other has not. All the necessary definitions and notations are formulated later.

2. Preliminaries and Notation

We use the standard notation from Boolean function theory. An $n-$ary *Boolean function* is a function f: $\{0,1\}^n \to \{0,1\}$. The symbol P^n denotes the set of all $n-$ary Boolean functions. Boolean variables and their negations are called *literals*. A literal of a variable x is denoted by x^α, where $\alpha \in \{0,1\}$, and we set

$$x^\alpha = \begin{cases} \bar{x} & \text{if } \alpha = 0, \\ x & \text{if } \alpha = 1. \end{cases}$$

A conjunction $K = x_{i_1}^{\alpha_{i_1}} \wedge \ldots \wedge x_{i_r}^{\alpha_{i_r}}$ of literals of different variables is called an *elementary conjunction*. The number of literals (r) in K is called *rank* of K. A special case is the conjunction of rank 0, it is called empty and its value is set to 1.

A formula $D = K_1 \vee \ldots \vee K_s$, the disjunction of distinct elementary conjunction, is called a *disjunctive normal form*. The parameter s (the number of elementary conjunctions in D) is called the *length* of D. A d.n.f. with $s = 0$ is called empty and its value is set to 0. A d.n.f. D represents a Boolean function f if the truth tables of f and D coincide. Let us consider the class of all d.n.f.that represent an n-ary Boolean function f. A d.n.f. with minimal number of literals in this class is called a *minimal d.n.f.* of f and the one with minimal length in this class is called a *shortest d.n.f.* of f.

We also use a geometric representation of Boolean functions. The Boolean n-cube is the graph B^n with 2^n vertices $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where $\alpha_i \in \{0,1\}$, in which those pairs of vertices that differ in exactly one coordinate are joined with an edge. For an n-ary Boolean function f let, N_f denote the subset $\{\tilde{\alpha}; f(\tilde{\alpha}) = 1\}$ and N_f^- denote the subset $\{\tilde{\alpha}; f(\tilde{\alpha}) = 0\}$ of all vertices $\tilde{\alpha}$. Notice that there is a one-to-one correspondence between the sets N_f and Boolean functions f. The subgraph of the Boolean n-cube induced by the set N_f is called the graph of f and is denoted by $G(f)$.

The set of vertices $N_i \subseteq \{0,1\}^n$ corresponding to an elementary conjunction K_i of rank r is called the *interval* of rank r. Notice that to every elementary conjunction $K = x_{i_1}^{\alpha_{i_1}} \wedge \ldots \wedge x_{i_r}^{\alpha_{i_r}}$ there corresponds an interval of rank r consisting of all vertices $(\beta_1, \ldots, \beta_n)$ of B^n such that $\beta_{i_j} = \alpha_{i_j}$ for $j = 1, \ldots, r$ and values of other vertex coordinates are arbitrary. In the present paper, we often work with intervals coresponding to elementary conjunctions.

In the geometric model, every interval of rank r represents an $(n-r)$-dimensional subcube of B^n. Therefore we call the interval of rank r also the $(n-r)$-dimensional interval. An interval N is called the *maximal interval* of Boolean function f if $N \subseteq N_f$ and there is no interval $N' \subseteq N_f$ such that $N \subseteq N'$. A d.n.f. which consists of all elementary conjunctions corresponding to maximal intervals is called the *abbreviated d.n.f.* and it is denoted by $D_A(f)$.

Now we can define the *interval graph* $\Gamma(f)$ as the graph associated with a Boolean function f as follows: its vertices correspond to maximal intervals of f and the vertices corresponding to intervals N_i and N_j are joined with an edge in $\Gamma(f)$ if $K_i \wedge K_j$ is nonempty.

For an arbitrary Boolean function f and each of its d.n.f.s $K_1 \vee \ldots \vee K_s$, we have

$$N_f = \bigcup_{j=1}^{s} N_j.$$

In other words, every d.n.f. of a Boolean function f corresponds to a covering of N_f by intervals $N_1, \ldots, N_s$ such that $N_j \subseteq N_f$. Conversely, every covering of N_f by intervals $N_1, \ldots, N_s$ contained in N_f corresponds to some d.n.f. of f. Using the geometric interpretation of d.n.f.s, we can express the irreducibility of d.n.f.. The d.n.f. D of a Boolean function f cannot be simplified if every interval N_j of the covering corresponding to D contains at least one vertex belonging to just this one interval of the covering. Such a d.n.f. is called an *irredundant d.n.f.*.

Let r_j denote the order of the interval N_j. Then the number of literals in d.n.f. is $r = \sum_{j=1}^{s} r_j$ and the construction of a minimal d.n.f. in the geometric model can be formulated as a problem of constructing a covering of N_f by intervals $N_j \subseteq N_f$ with minimal r. On the other hand, the construction of a covering corresponding to the shortest d.n.f. requires to minimize the number of intervals in a covering of N_f.

The set of all conjuctions K_j from $K_1, \ldots, K_s$ corresponding to intervals for which

$$N_j \nsubseteq \bigcup_{\substack{i=1 \\ i \neq j}}^{s} N_i.$$

is called the *core* of d.n.f. $D = \bigvee_{j=1}^{s} K_j$ of a Boolean function f. It is denoted by $\gamma(D(f))$.

3. Complete interval graph and a minimal disjunctive normal form

In this section, we study Boolean functions whose interval graph is a complete graph. To avoid trivial cases, we omit interval graphs consisting from two or fewer vertices, where it is obvious that the corresponding abreviated d.n.f. is also a minimal d.n.f..

Theorem 3.1. *If f is a Boolean function such that $\Gamma(f)$ is complete, then the intersection of the maximal intervals of f is nonempty.*

Proof. Let N_1 and N_2 be arbitrary maximal intervals of f. Assume that the dimension of N_1 is m, that is,

$$K_1 = x_{i_1}^{\alpha_{i_1}} \wedge \ldots \wedge x_{i_{n-m}}^{\alpha_{i_{n-m}}}$$

and the dimension of N_2 is k, that is,

$$K_2 = x_{j_1}^{\alpha_{j_1}} \wedge \ldots \wedge x_{j_{n-k}}^{\alpha_{j_{n-k}}}.$$

We have two possibilities for these intervals.

1. $N_1 \cap N_2 = \emptyset$. This occurs when $i_l = j_s$ for some l and s and $\alpha_{i_l} \neq \alpha_{j_s}$ for at least one fixed coordinate of the intervals N_1, N_2.
2. Otherwise, $N_1 \cap N_2 = N_I$ for some conjunction I. It is clear that $I = K_1 \wedge K_2$. Therefore the number of fixed coordinates of I is $(n-m)+(n-k)-r$, where r is the number of positions of fixed coordinates on which the intervals N_1, N_2 coincide.

Let $N_1 \cap \cdots \cap N_s$ be maximal intervals of f. In the complete interval graph $\Gamma(f)$, any two vertices are joined with an edge, therefore, any two maximal intervals have a common intersection. In other words, for any two maximal intervals, there do not exist fixed coordinates in which these two intervals differ. Therefore, $N_I = N_1 \cap \ldots \cap N_s$. Thus $I = K_1 \wedge \ldots \wedge K_s$. □

Theorem 3.2. *Let f be a Boolean function of n variables and let $D_M(f)$ be a minimal d.n.f. of f. If $\Gamma(f)$ is complete, then $D_A(f) = D_M(f)$.*

Proof. Assume that $D_A(f) = K_1 \vee K_2 \vee \ldots \vee K_s$. This d.n.f. corresponds to the covering $N_1 \cup \ldots \cup N_s$ by the maximal intervals. To be sure that the abbreviated d.n.f. is identical with a irredundant d.n.f., each K_i, $1 \leq i \leq s$, has to be from $\gamma(D_A(f))$. By the assumption of the theorem, $\Gamma(f)$ is complete. From Theorem 3.1 it follows that all maximal intervals have a common intersection $I = N_1 \cap \ldots \cap N_s$.

To prove the result, we now proceed by contradiction. Let us assume that the abbreviated d.n.f. contains a conjuction K_i which does not belong to the core of d.n.f. $\gamma(D_A(f))$. Assume that the dimension of corresponding maximal interval N_i is m. It follows that

$$N_i \subseteq \bigcup_{\substack{j=1 \\ j \neq i}}^{s} N_j.$$

In other words, for each vertex $\tilde{\delta}$ of the maximal interval N_i, we have:

- if $\tilde{\delta} \in N_i \cap I$, then $\tilde{\delta}$ is contained in each maximal interval of $D_A(f)$,
- if $\tilde{\delta} \in N_i \backslash I$ (this set cannot be empty), then there exists at least one maximal interval different from N_i, let us denote it N_j for which $I \subset N_j \wedge \tilde{\delta} \in N_j$.

Let us choose a vertex $\tilde{\alpha}$ such that $\tilde{\alpha} \in I \cap N_i$ and a vertex $\tilde{\beta}$ different from $\tilde{\alpha}$ exactly in m coordinates and such that $\tilde{\beta} \in N_i \backslash I$. The dimension of N_i is m, therefore, such two vertices exist. Hence, there exists a maximal interval N_j which contains both $\tilde{\alpha}$ and $\tilde{\beta}$. As $\tilde{\alpha}$ and $\tilde{\beta}$ generate an m-dimensional subcube, the dimension of N_j is at least m. If it equals m, then $N_j = N_i$. If the dimension of N_j is greater than m, then $N_i \subset N_j$. But this contradicts the fact that N_i is the maximal interval.

We have shown that all maximal intervals belong to the core. Therefore we can not omit any interval from abbreviated d.n.f. while constructing irredundant d.n.f.. Otherwise we would violate irreducibility of d.n.f.. Considering that $D_A(f)$ contains all maximal intervals, we have proved that there does not exist another irredundant d.n.f.. Because minimal d.n.f. is also the unique irredundant, it holds $D_A(f) = D_M(f)$. □

Now we discuss the converse of Theorem 3.2. Let $D_M(f)$ be a minimal d.n.f. of a Boolean function f. Suppose that $D_A(f) = D_M(f)$, then $\Gamma(f)$ need not be complete. In the example below, we show that there exists a Boolean function f such that $D_A(f) = D_M(f)$ but the graph $\Gamma(f)$ is not complete. Thus the converse oof Theorem 3.2 is false.

Example 3.1. The Boolean function is described in Figure 1 left by bold lines. The interval graph of this function is shown in Figure 1 right. The abbreviated d.n.f. of the function f is $D_A(f) = \bar{x}_1 \vee \bar{x}_2\bar{x}_3 \vee x_2x_3$ and it is easy to see that it is also minimal.

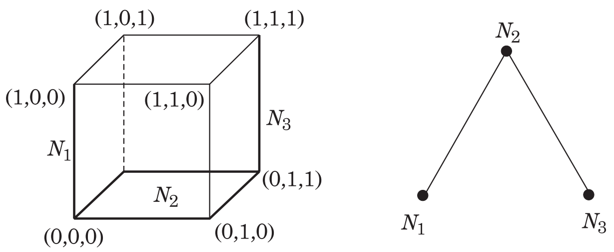


Figure 1. Geometric representation of a Boolean function $f(x_1, x_2, x_3)$, covering with maximal intervals and corresponding interval graph $\Gamma(f)$.

4. The interval graph and a minimal disjunctive normal form

In this section we construct a Boolean function for an arbitrary simple graph G such that the interval graph of this function is isomorphic with G. We also consider the number of variables that such a function needs to have.

Theorem 4.1. *Let G be a graph of order n. There exists a Boolean function $f(x_1, \dots, x_n)$ whose abbreviated d.n.f. is also its minimal d.n.f. and $\Gamma(f) \cong G$.*

Proof. We will prove the Theorem by induction with respect to n.

The base case. We prove the Theorem for graph $G = (V, E)$ consisting of n vertices for $n = 1$ and $n = 2$. This case is described in Figure 2.

We divide the base case into 3 subcases, the number of vertices being the primary criterion and the number of edges being a secondary criterion.

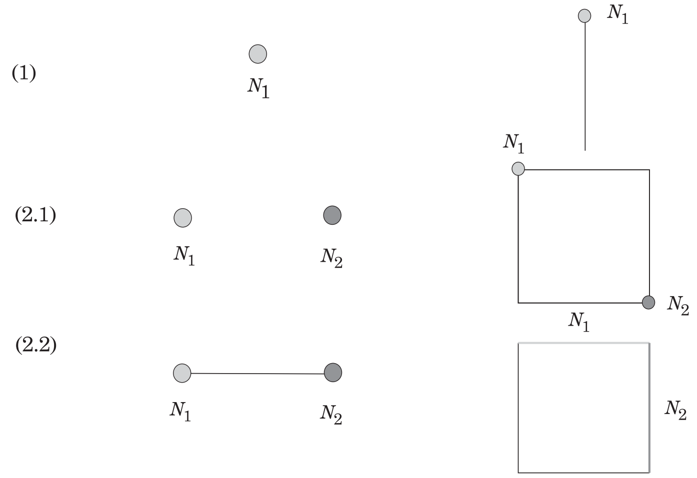


Figure 2. Graph $\Gamma(f) \cong G$ of the function f and corresponding maximal intervals for $n = 1$ and $n = 2$.

(1) $|V| = 1$ and $|E| = 0$. The satisfying Boolean function is $f(x_1) = x_1$.
(2.1) $|V| = 2$ and $|E| = 0$. The Boolean function is $f(x_1, x_2) = \bar{x}_1\bar{x}_2 \vee x_1x_2$.
(2.2) $|V| = 2$ and $|E| = 1$. The Boolean function is $f(x_1, x_2) = x_1 \vee \bar{x}_2$.

It is clear that in these 3 subcases the following holds $D_A(f) = D_M(f)$.

Induction step. Let $G = (V, E)$ be an arbitrary graph with $n + 1$ vertices. We want to show that there exists a Boolean function $f(x_1, \ldots, x_{n+1})$ such that $\Gamma(f) \cong G$ and the abbreviated d.n.f. is also its minimal d.n.f.. Let v be a vertex of G with minimum degree. Let $G' = G - v$. By the induction hypothesis there exists a Boolean function $f'(x_1, \ldots, x_n)$ with $\Gamma(f') \cong G'$ such that the abbreviated d.n.f. is also its minimal d.n.f..

We enlarge the n-cube with maximal intervals corresponding to vertices $\Gamma(f')$ by one. We create the copy and add 2^n edge. An interval of f' will be called active if it corresponds to a neighbour of v. Otherwise it will be called passive. We denote the interval corresponding to v as N_{n+1}.

An edge between vertices implies that the corresponding maximal intervals have a common intersection. We need at least one vertex belonging to the added interval for each active interval. To make sure that these intervals are from the core they contain at least one point which is not contained in any other interval. In our construction maximal intervals have the same dimension as their degree in the interval graph.

We enlarge the original maximal intervals in the direction towards the copy. We will construct maximal interval corresponding to the new vertex in the copy and its dimension will be equal to its degree. Vertices corresponding to passive maximal intervals are not affected.

If the degree of v is 0, we will find the vertex belonging to N_f^- in the original object. Such vertex has to be found there, because the degree of a vertex in G' is not more than $n-1$ and thus the dimension of the corresponding maximal interval in the n-cube is not more than $n-1$. To keep dimensions and number of these intervals and to avoid a situation that we would be able to join two intervals into one bigger, it is necessary to separate these n intervals with vertices belonging to N_f^-. We find one such vertex and move in the direction towards the copy. This vertex we have been looking for will be N_{n+1}. We add all other points in the copy to the set N_f^-.

If degree of v is k, $1 \leq k \leq n$, we need to increase the dimension of k active intervals. Let us denote them $N_{i_1}, \ldots, N_{i_k}$. So we increase the dimensions of all intervals from the set $\{N_{i_1}, \ldots, N_{i_k}\}$ by one towards the copy. It is clear that we do not affect passive intervals in original object and the intervals with a common intersection in the original object will have a common intersection also in the copy.

Now we find the maximal interval N_{n+1} with the dimension of k which contains at least one point from each of the intervals $N_{i_1}, \ldots, N_{i_k}$ and also one vertex which is not covered by any other interval. We can place this k-dimensional interval in such a way that in the direction towards the original object there are only points from N_f^- or from the common intersection of k incident maximal intervals, because 0s separates the maximal intervals in the original object. We add all other points in the copy to the set N_f^-.

This completes the construction of the required function $f(x_1, \ldots, x_{n+1})$. From the manner of construction it follows that all maximal intervals of the abbreviated d.n.f. belong to the core. We have proved, that $D_A(f)$ is also the irredundant d.n.f.. Because minimal d.n.f. is also unique irredundant, it holds $D_A(f) = D_M(f)$. $\square$

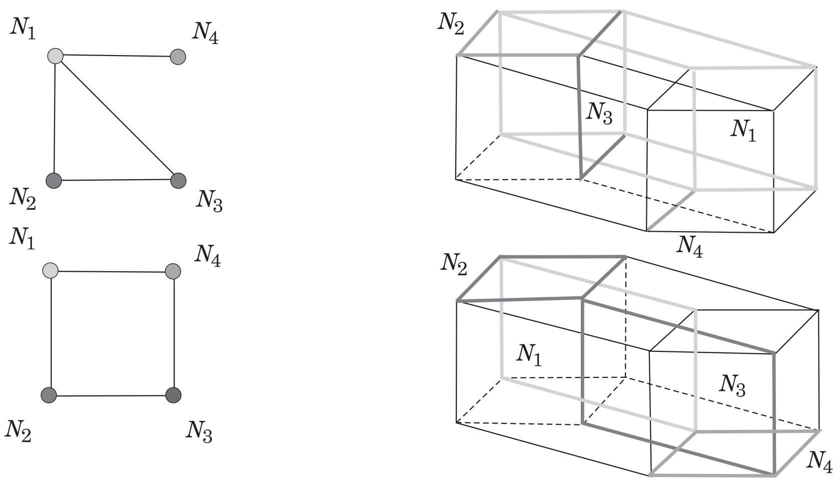


Figure 3. Graph $G = (V, E)$ with $|V| = 4$ and $|E| = 4$.

In Figure 3, there is the graph $G = (V, E)$ with $|V| = 4$ and $|E| = 4$. We illustrate how we construct the maximal intervals of searched Boolean function in 4-dimensional cube such that $\Gamma(f) \cong G$ and the Theorem 4.1 holds. Vertex which we add is denoted as N_4.

In the following part, we consider the number of variables of a Boolean function.

For a graph G of order m, let $n(G)$ be the minimal number n of variables of a Boolean function $f(x_1, \ldots, x_n)$ such that $\Gamma(f) \cong G$. Set

$$\underline{n}(m) = \min\{n(G);\ \ G \text{ of order } m\}$$
$$\overline{n}(m) = \max\{n(G);\ \ G \text{ of order } m\}$$

Theorem 4.2. *For every positive integer m, one has $\underline{n}(m) = \lceil \log m \rceil + 1$ and $\overline{n}(m) = m$.*

Proof. We first prove that $\underline{n}(m) = \lceil \log m \rceil + 1$. The dimension of maximal intervals depends on the degree of the corresponding vertex in the interval graph.

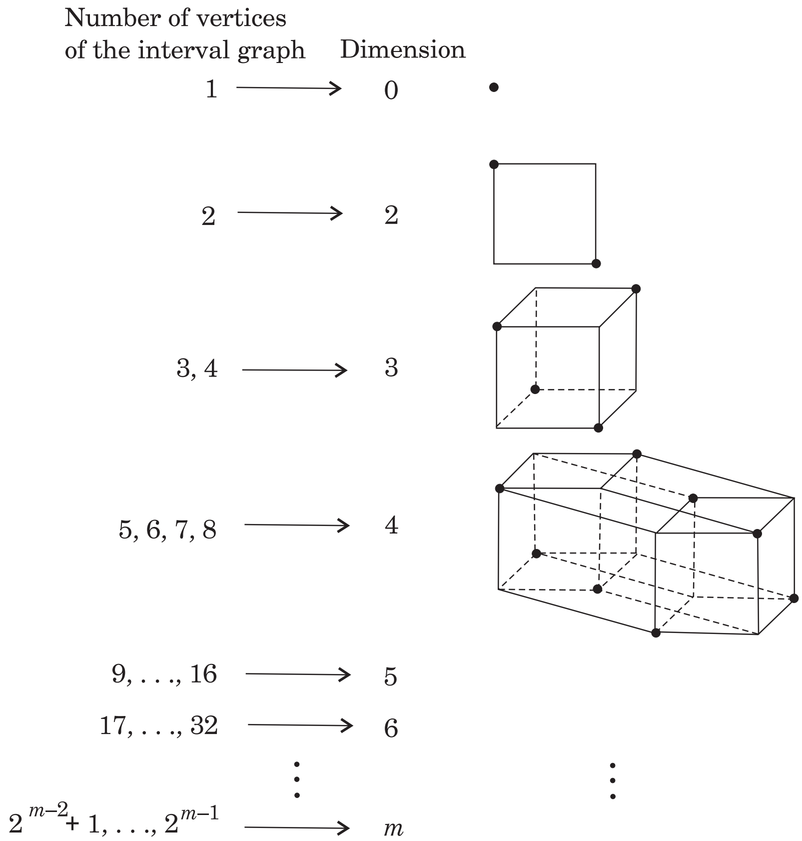
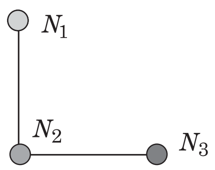


Figure 4. Number of 0-dimensional maximal intervals in the cubes of particular dimensions.

Bigger degree of the vertex means bigger dimension of corresponding maximal interval. The minimal dimension of a Boolean cube can be obtained for the interval graph with no edges. The maximal intervals corresponding to the vertices can be 0-dimensional intervals. To avoid that two 0-dimensional intervals could be joined into 1-dimensional, for each vertex $\tilde{\alpha}$ at the distance 1 from each maximal interval $f(\tilde{\alpha}) = 0$ holds. Any two maximal intervals differ in at least two coordinates. It follows that we can place $2^m/2$ 0-dimensional intervals into the m-cube. From the manner of construction it follows that $\overline{n}(m) = m$. □

In this part we show examples of isomorphic interval graphs such that for one graph the abbreviated d.n.f. is a minimal at the same time and for the other one is not. The Boolean functions are described in both figures by bold lines on the right side and the interval graphs of this functions are shown left.

Example 4.1. The abbreviated and at the same time minimal d.n.f. of the function f from Figure 5 is $D(f) = N_1 \vee N_2 \vee N_3$. The abbreviated d.n.f. of the function f from Figure 6 is $D(f) = N_1 \vee N_2 \vee N_3$ and its minimal d.n.f. is $D(f) = N_1 \vee N_3$.



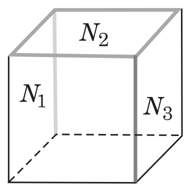
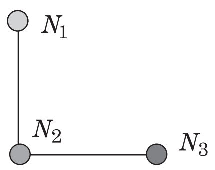


Figure 5. Interval graph and corresponding maximal intervals of a Boolean function $f(x_1, x_2, x_3)$.



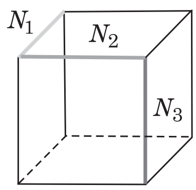


Figure 6. Interval graph and corresponding maximal intervals of a Boolean function $f(x_1, x_2, x_3)$.

5. Conclusion

In the present paper we have proved that if the interval graph of a Boolean function is a complete graph, then the abbreviated d.n.f. of this function is also a minimal d.n.f.. In addition, we have described a construction of a Boolean function such that the abbreviated d.n.f. of this function is also a minimal d.n.f. for an arbitrary simple finite graphs. We have also studied the relationship between the number of vertices and edges in the interval graph and the dimension of the corresponding Boolean function.

It would be interesting to study graph whose vertices are d.n.f.s and whose edges are pairs of d.n.f.s that differ in exactly one conjunction. The study of a simplified interval graph, for example, one without the vertices (maximal intervals) corresponding to the core conjuctions or conjuctions belonging to every irredundant d.n.f. would also be interesting. It would also be interesting to study the behaviour of the interval graph under suitable transformations of Boolean functions.

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