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ON THE GEODESIC TORSION OF A TANGENTIAL INTERSECTION CURVE OF TWO SURFACES IN $\mathbb{R}^3$

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Abstract. In this paper, we find the unit tangent vector and the geodesic torsion of the tangential intersection curve of two surfaces in all three types of surface-surface intersection problems (parametric-parametric, implicit-implicit and parametric-implicit) in three-dimensional Euclidean space.

1. Introduction

We know that the curvatures of a curve can be calculated easily if the curve is given by its parametric equation. But the curvature calculations become harder when the curve is given as an intersection of two surfaces in three-dimensional Euclidean space.

In differential geometry the surfaces are generally given by their parametric or implicit equations. For that reason, the surface-surface intersection (SSI) problems can be three types: parametric-parametric, implicit-implicit, parametric-implicit. The SSI is called transversal or tangential if the normal vectors of the surfaces are linearly independent or linearly dependent, respectively at the intersecting points. In transversal intersection problems, the tangent vector of the intersection curve can be found easily by the vector product of the normal vectors of the surfaces. Because of this, there are many studies related to the transversal intersection problems in literature on differential geometry. Also there are some studies about tangential intersection curve and its properties. Some of these studies are mentioned below.

Willmore [**1**] describes how to obtain the Frenet apparatus of the transversal intersection curve of two implicit surfaces in Euclidean 3-space. Using the implicit function theorem, Hartmann [2] obtains formulas for computing the curvature κ of the transversal intersection curve for all three types of SSI problems. Ye and Maekawa [**3**] present algorithms for computing the differential geometry properties of intersection curves of two surfaces and give algorithms to evaluate the higher-order derivatives for transversal as well as tangential intersections for all three types of intersection problems. Wu, Aléssio and Costa [**4**], using only the normal

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vectors of two regular surfaces, present an algorithm to compute the local geometric properties of the transversal intersection curve. Goldman [**5**], using the classical curvature formulas in differential geometry, provides formulas for computing the curvatures of intersection curve of two implicit surfaces. Using the implicit function theorem, Aléssio [**6**] gives a method to compute the Frenet vectors and also the curvature and the torsion of the intersection curve of two implicit surfaces. Aléssio [**7**] presents algorithms for computing the differential geometry properties of intersection curves of three implicit surfaces in $\mathbb{R}^4$, using the implicit function theorem and generalizing the method of Ye and Maekawa. Düldül [**8**] gives a method for computing the Frenet vectors and the curvatures of the transversal intersection curve of three parametric hypersurfaces in four-dimensional Euclidean space. In our recent study [**9**], we give the geodesic curvature and the geodesic torsion of the intersection curve of two transversally intersecting surfaces in Euclidean 3-space. Aléssio [**10**] presents formulas on geodesic torsion, geodesic curvature and normal curvature of the intersection curve of $n-1$ implicit hypersurfaces in $\mathbb{R}^n$.

In this study, first we find the unit tangent vector of the tangential intersection curve of two surfaces in all three types of SSI problems. Then we calculate the geodesic torsion of the intersection curve and give examples related to the subject.

2. Preliminaries

Consider a unit-speed curve $\alpha\colon I \to \mathbb{R}^3$, parametrized by arclength function s. Let $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ be the moving Frenet frame along α, where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ denote the tangent, the principal normal and the binormal vector fields, respectively. The vector $\mathbf{t}' = \alpha''(s)$ is called the curvature vector and the length of this vector denotes the curvature $\kappa(s)$ of the curve α.

Let $\{\mathbf{t}(s), \mathbf{V}(s), \mathbf{N}(s)\}$ be the moving Darboux frame on the curve α, where $\mathbf{N}(s)$ is the surface normal restricted to α and $\mathbf{V}(s) = \mathbf{N}(s) \times \mathbf{t}(s)$. Then, we have

$$\begin{aligned} \mathbf{t}' &= \kappa_g \mathbf{V} + \kappa_n \mathbf{N} \\ \mathbf{V}' &= -\kappa_g \mathbf{t} + \tau_g \mathbf{N} \\ \mathbf{N}' &= -\kappa_n \mathbf{t} - \tau_g \mathbf{V} \end{aligned} \tag{1}$$

where κ_n, κ_g and τ_g are the normal curvature of the surface in the direction of $\mathbf{t}$, the geodesic curvature and the geodesic torsion of the curve α, respectively, [**11**]. Thus from (1), the normal curvature, the geodesic curvature and the geodesic torsion of the curve α are

$$\kappa_n = \langle \mathbf{t}', \mathbf{N} \rangle, \qquad \kappa_g = \langle \mathbf{t}', \mathbf{V} \rangle, \qquad \tau_g = \langle \mathbf{V}', \mathbf{N} \rangle,$$

where $\langle , \rangle$ denotes the scalar product.

We know that the geodesic curvature and the geodesic torsion of the transversal intersection curve of the surfaces A and B with the parametric equations $\mathbf{X}(u, v)$

and $\mathbf{Y}(p,q)$, respectively, with respect to the surface A are given by

$$\begin{aligned}\kappa_g^A = &\frac{1}{\sqrt{EG-F^2}}\left\{\left[\left(F_u - \frac{E_v}{2}\right)\langle \mathbf{X}_u, \mathbf{t}\rangle - \frac{E_u}{2}\langle \mathbf{X}_v, \mathbf{t}\rangle\right](u')^2\right. \\ &+ \left(G_u\langle \mathbf{X}_u, \mathbf{t}\rangle - E_v\langle \mathbf{X}_v, \mathbf{t}\rangle\right)u'v' \\ &\left.+\left[\frac{G_v}{2}\langle \mathbf{X}_u, \mathbf{t}\rangle - \left(F_v - \frac{G_u}{2}\right)\langle \mathbf{X}_v, \mathbf{t}\rangle\right](v')^2\right\} \\ &+ \sqrt{EG-F^2}(u'v'' - v'u'')\end{aligned} \tag{2}$$

and

$$\begin{aligned}\tau_g^A = &\frac{1}{\sqrt{EG-F^2}}\left\{(EM-FL)(u')^2 + (EN-GL)u'v'\right. \\ &\left.+ (FN-GM)(v')^2\right\}\end{aligned} \tag{3}$$

in which u' and v' can be found by [**3**]

$$\begin{aligned}u' &= \frac{1}{EG-F^2}\left(G\langle \mathbf{t}, \mathbf{X}_u\rangle - F\langle \mathbf{t}, \mathbf{X}_v\rangle\right) \\ v' &= \frac{1}{EG-F^2}\left(E\langle \mathbf{t}, \mathbf{X}_v\rangle - F\langle \mathbf{t}, \mathbf{X}_u\rangle\right)\end{aligned} \tag{4}$$

where E, F, G and L, M, N, respectively, are the first and the second fundamental form coefficients of the surface A (Eqs. (2) and (3) can be found in classic books on differential geometry). The values u'' and v'' in Eq. (2) can be computed from the linear equation system [**9**]

$$\begin{aligned}&\langle \mathbf{X}_u, \mathbf{N}^B\rangle u'' + \langle \mathbf{X}_v, \mathbf{N}^B\rangle v'' = \langle \mathbf{\Lambda}, \mathbf{N}^B\rangle \\ &\langle \mathbf{X}_u, \mathbf{t}\rangle u'' + \langle \mathbf{X}_v, \mathbf{t}\rangle v'' = -\langle \mathbf{X}_{uu}, \mathbf{t}\rangle (u')^2 - 2\langle \mathbf{X}_{uv}, \mathbf{t}\rangle u'v' - \langle \mathbf{X}_{vv}, \mathbf{t}\rangle (v')^2\end{aligned}$$

where $\mathbf{\Lambda} = \mathbf{Y}_{pp}(p')^2 + 2\mathbf{Y}_{pq}p'q' + \mathbf{Y}_{qq}(q')^2 - \mathbf{X}_{uu}(u')^2 - 2\mathbf{X}_{uv}u'v' - \mathbf{X}_{vv}(v')^2$.

$$\begin{aligned}p' &= \frac{1}{eg-f^2}\left(g\langle \mathbf{t}, \mathbf{Y}_p\rangle - f\langle \mathbf{t}, \mathbf{Y}_q\rangle\right) \\ q' &= \frac{1}{eg-f^2}\left(e\langle \mathbf{t}, \mathbf{Y}_q\rangle - f\langle \mathbf{t}, \mathbf{Y}_p\rangle\right)\end{aligned} \tag{5}$$

and e, f, g and l, m, n, respectively, denote the first and the second fundamental form coefficients of the surface B.

Also, the geodesic curvature of the transversal intersection curve of the surfaces A and B with respect to the surface A is

$$\kappa_g^A = \frac{1}{\|\nabla f\|}\{(y'z'' - y''z')f_x + (z'x'' - z''x')f_y + (x'y'' - x''y')f_z\}, \tag{6}$$

where $\mathbf{t} = (x', y', z')$, $\mathbf{t}' = (x'', y'', z'')$ and $f(x,y,z) = 0$ denotes the implicit equation of A [**12**].

2.1. Tangential intersection curve of parametric-parametric surfaces

Let A and B be two regular surfaces given by parametric equations $\mathbf{X}(u, v)$ and $\mathbf{Y}(p, q)$, respectively. Let us assume that these surfaces intersect tangentially along the intersection curve $\alpha(s)$. Then, the unit normal vectors of the surfaces A and B are given by

$$\mathbf{N}^A = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}, \qquad \mathbf{N}^B = \frac{\mathbf{Y}_p \times \mathbf{Y}_q}{\|\mathbf{Y}_p \times \mathbf{Y}_q\|}.$$

Since the surfaces intersect tangentially, the normals $\mathbf{N}^A$ and $\mathbf{N}^B$ are parallel at all points of α. It can be assumed that $\mathbf{N}^A = \mathbf{N}^B = \mathbf{N}$ by orienting the surfaces properly. In this case, we can not find the unit tangent vector $\mathbf{t}$ of the intersection curve by the vector product of the normal vectors. Therefore, we have to find new methods to compute the geometric properties of the intersection curve α.

Since $\mathbf{V}^A = \mathbf{N}^A \times \mathbf{t}$ and $\mathbf{V}^B = \mathbf{N}^B \times \mathbf{t}$, let us denote $\mathbf{V}^A = \mathbf{V}^B = \mathbf{V}$. Thus from (1), the geodesic torsions of the intersection curve α with respect to the surfaces A and B are

$$\tau_g^A = \tau_g^B = \langle \mathbf{V}', \mathbf{N} \rangle.$$

Also, we may write $\alpha(s) = \mathbf{X}(u(s), v(s)) = \mathbf{Y}(p(s), q(s))$ which yield

$$\mathbf{t} = \alpha'(s) = \mathbf{X}_u u' + \mathbf{X}_v v' = \mathbf{Y}_p p' + \mathbf{Y}_q q'. \tag{7}$$

If we take the vector product of both hand sides of (7) with $\mathbf{Y}_p$ and $\mathbf{Y}_q$, and then take the dot product of both hand sides of these equations with $\mathbf{N}$, we have

$$\begin{aligned} p' &= b_{11} u' + b_{12} v' \\ q' &= b_{21} u' + b_{22} v', \end{aligned} \tag{8}$$

where

$$b_{11} = \frac{\det(\mathbf{X}_u, \mathbf{Y}_q, \mathbf{N})}{\sqrt{eg - f^2}}, \qquad b_{12} = \frac{\det(\mathbf{X}_v, \mathbf{Y}_q, \mathbf{N})}{\sqrt{eg - f^2}},$$
$$b_{21} = \frac{\det(\mathbf{Y}_p, \mathbf{X}_u, \mathbf{N})}{\sqrt{eg - f^2}}, \qquad b_{22} = \frac{\det(\mathbf{Y}_p, \mathbf{X}_v, \mathbf{N})}{\sqrt{eg - f^2}}.$$

Thus from (3), we have

$$D_1 (u')^2 + D_2 u' v' + D_3 (v')^2 = d_1 (p')^2 + d_2 p' q' + d_3 (q')^2, \tag{9}$$

where

$$D_1 = \frac{EM - FL}{\sqrt{EG - F^2}}, \qquad D_2 = \frac{EN - GL}{\sqrt{EG - F^2}}, \qquad D_3 = \frac{FN - GM}{\sqrt{EG - F^2}},$$
$$d_1 = \frac{em - fl}{\sqrt{eg - f^2}}, \qquad d_2 = \frac{en - gl}{\sqrt{eg - f^2}}, \qquad d_3 = \frac{fn - gm}{\sqrt{eg - f^2}}.$$

Substituting (8) into (9), we have

$$c_1 (u')^2 + c_2 u' v' + c_3 (v')^2 = 0, \tag{10}$$

where

$$c_1 = d_1 b_{11}^2 + d_2 b_{11} b_{21} + d_3 b_{21}^2 - D_1,$$
$$c_2 = 2d_1 b_{11} b_{12} + d_2(b_{11} b_{22} + b_{12} b_{21}) + 2d_3 b_{21} b_{22} - D_2,$$
$$c_3 = d_1 b_{12}^2 + d_2 b_{12} b_{22} + d_3 b_{22}^2 - D_3.$$

If we denote $\rho = \frac{u'}{v'}$ when $c_1 \neq 0$, or $\nu = \frac{v'}{u'}$ when $c_1 = 0$ and $c_3 \neq 0$, Eq. (10) becomes

$$c_1 \rho^2 + c_2 \rho + c_3 = 0$$

or

$$c_3 \nu^2 + c_2 \nu = 0.$$

Let $\Delta = c_2^2 - 4c_1 c_3$. If $\Delta > 0$, then solving the above equations according to ρ or ν, two different values are found. For these values of ρ and ν, let us consider the vectors

$$\mathbf{r}_i = \frac{\rho_i \mathbf{X}_u + \mathbf{X}_v}{\|\rho_i \mathbf{X}_u + \mathbf{X}_v\|} \quad \text{or} \quad \mathbf{r}_i = \frac{\mathbf{X}_u + \nu_i \mathbf{X}_v}{\|\mathbf{X}_u + \nu_i \mathbf{X}_v\|}, \quad i = 1, 2. \tag{11}$$

We need to determine the vector which denotes the tangent vector $\mathbf{r}_1$ and/or $\mathbf{r}_2$ at the intersection point P.

Let R_1 denotes the plane determined by the common surface normal $\mathbf{N}$ and the vector $\mathbf{r}_1$ at P. R_1 has the parametric equation $\mathbf{Z}(r, w)$. Since the normals of the plane R_1 and the surface A are perpendicular, the intersection of these surfaces is the transversal intersection at P. Let us denote the normal vector of the plane R_1 by $\mathbf{N}_1 = \mathbf{N} \times \mathbf{r}_1$. Then, the unit tangent vector of the transversal intersection curve of the surface A and the plane R_1 is determined by

$$\mathbf{t}_1 = \frac{\mathbf{N} \times \mathbf{N}_1}{||\mathbf{N} \times \mathbf{N}_1||}.$$

From (2), the geodesic curvature $\kappa_{g_1}^A$ of this intersection curve with respect to R_1 is

$$\kappa_{g_1}^A = \sqrt{E_1 G_1 - F_1^2}(r' w'' - r'' w'), \tag{12}$$

where $E_1 = \langle \mathbf{Z}_r, \mathbf{Z}_r \rangle$, $F_1 = \langle \mathbf{Z}_r, \mathbf{Z}_w \rangle$, $G_1 = \langle \mathbf{Z}_w, \mathbf{Z}_w \rangle$ and

$$\begin{aligned} r' &= \frac{1}{E_1 G_1 - F_1^2} \left(G_1 \langle \mathbf{t}_1, \mathbf{Z}_r \rangle - F_1 \langle \mathbf{t}_1, \mathbf{Z}_w \rangle \right), \\ w' &= \frac{1}{E_1 G_1 - F_1^2} \left(E_1 \langle \mathbf{t}_1, \mathbf{Z}_w \rangle - F_1 \langle \mathbf{t}_1, \mathbf{Z}_r \rangle \right). \end{aligned} \tag{13}$$

The unit tangent vector of the transversal intersection curve of A and R_1 is

$$\mathbf{t}_1 = \mathbf{Z}_r r' + \mathbf{Z}_w w' = \mathbf{X}_u u' + \mathbf{X}_v v',$$

where u' and v' can be calculated by taking $\mathbf{t}_1$ instead of $\mathbf{t}$ in Eq. (4). Since $\mathbf{Z}_{rr} = \mathbf{Z}_{rw} = \mathbf{Z}_{ww} = (0, 0, 0)$,

$$\mathbf{t}_1' = \mathbf{Z}_r r'' + \mathbf{Z}_w w'' = \mathbf{X}_u u'' + \mathbf{X}_v v'' + \mathbf{\Lambda}_1^A, \tag{14}$$

where $\mathbf{\Lambda}_1^A = \mathbf{X}_{uu}(u')^2 + 2\mathbf{X}_{uv}u'v' + \mathbf{X}_{vv}(v')^2$. By taking the dot product of both hand sides of (14) with $\mathbf{N}$, we get

$$\langle \mathbf{Z}_r, \mathbf{N}\rangle r'' + \langle \mathbf{Z}_w, \mathbf{N}\rangle w'' = \langle \mathbf{\Lambda}_1^A, \mathbf{N}\rangle. \tag{15}$$

Since $\mathbf{t}_1'$ is perpendicular to $\mathbf{t}_1$,

$$\langle \mathbf{Z}_r, \mathbf{t}_1\rangle r'' + \langle \mathbf{Z}_w, \mathbf{t}_1\rangle w'' = 0 \tag{16}$$

is also obtained. (15) and (16) constitute a linear system with respect to r'' and w'' which has nonvanishing coefficients determinant, i.e., $\delta = -\|\mathbf{Z}_r \times \mathbf{Z}_w\| \cdot \|\mathbf{N} \times \mathbf{N}_1\| \neq 0$. Thus, r'' and w'' can be computed by solving this linear system. So, from Eq. (12), $\kappa_{g_1}^A$ is calculated.

On the other hand, the unit tangent vector of the transversal intersection curve of the surface B and the plane R_1 is also $\mathbf{t}_1$. Then, the geodesic curvature of this intersection curve with respect to R_1 is

$$\kappa_{g_1}^B = \sqrt{E_1G_1 - F_1^2}(r'w'' - r''w'), \tag{17}$$

where r' and w' are calculated by Eq. (13). Let us find r'' and w''. The unit tangent vector of the transversal intersection curve of B and R_1 is

$$\mathbf{t}_1 = \mathbf{Z}_r r' + \mathbf{Z}_w w' = \mathbf{Y}_p p' + \mathbf{Y}_q q',$$

where p' and q' can be computed by taking $\mathbf{t}_1$ instead of $\mathbf{t}$ in Eq. (5). Also,

$$\mathbf{t}_1' = \mathbf{Z}_r r'' + \mathbf{Z}_w w'' = \mathbf{Y}_p p'' + \mathbf{Y}_q q'' + \mathbf{\Lambda}_1^B, \tag{18}$$

where $\mathbf{\Lambda}_1^B = \mathbf{Y}_{pp}(p')^2 + 2\mathbf{Y}_{pq}p'q' + \mathbf{Y}_{qq}(q')^2$. If we solve Eq. (16) and the equation obtained by taking the dot product of both hand sides of (18) with $\mathbf{N}$, we find the unknowns r'' and w''. Thus, $\kappa_{g_1}^B$ is calculated from Eq. (17).

Similarly, if we denote the plane determined by the common surface normal $\mathbf{N}$ and the vector $\mathbf{r}_2$ at P by R_2, we can calculate the geodesic curvatures $\kappa_{g_2}^A$ and $\kappa_{g_2}^B$ (with respect to R_2) of the intersection curve of the plane R_2 with A and R_2 with B, respectively.

We have the following cases for $\Delta > 0$:

1) If $\kappa_{g_1}^A = \kappa_{g_1}^B$, then the transversal intersection curve of both R_1 with A and R_1 with B is the same curve around the point P, i.e., $\mathbf{t} = \mathbf{r}_1$. If $\kappa_{g_2}^A = \kappa_{g_2}^B$, then the transversal intersection curve of both R_2 with A and R_2 with B is the same curve around the point P, i.e., $\mathbf{t} = \mathbf{r}_2$. Hence, P is a branch point.
2) If $\kappa_{g_1}^A = \kappa_{g_1}^B$ and $\kappa_{g_2}^A \neq \kappa_{g_2}^B$ (or $\kappa_{g_1}^A \neq \kappa_{g_1}^B$ and $\kappa_{g_2}^A = \kappa_{g_2}^B$), then the intersection curve is unique, i.e., $\mathbf{t} = \mathbf{r}_1$ (or $\mathbf{t} = \mathbf{r}_2$).
3) If $\kappa_{g_1}^A \neq \kappa_{g_1}^B$ and $\kappa_{g_2}^A \neq \kappa_{g_2}^B$, then P is an isolated contact point.

We have the following cases for $\Delta = 0$:

1) If $c_1 = c_2 = c_3 = 0$, then P is an isolated contact point when $\kappa_{g_1}^A \neq \kappa_{g_1}^B$, or the surfaces have at least second order contact at P when $\kappa_{g_1}^A = \kappa_{g_1}^B$ obtained by taking any tangent vector $\mathbf{r}_1$.

2) If $c_1^2 + c_2^2 + c_3^2 \neq 0$, then $\mathbf{r}_1 = \mathbf{r}_2$. In this case, $\mathbf{t} = \mathbf{r}_1$ when $\kappa_{g_1}^A = \kappa_{g_1}^B$ or P is an isolated contact point when $\kappa_{g_1}^A \neq \kappa_{g_1}^B$.

If $\Delta < 0$, then P is an isolated contact point.

Thus, using the unit tangent vector $\mathbf{t}$ of the tangential intersection curve of the surfaces A and B, u' and v' can be calculated from Eq. (4). Substituting these values into (3), the geodesic torsion of the intersection curve with respect to the surfaces A and B at P is obtained.

Example 1. Let A and B be two surfaces given by the parametric equations, respectively,

$$\mathbf{X}(u,v) = \Big(3\cos u - \cos u \cos v + \frac{1}{\sqrt{10}} \sin u \sin v, 3\sin u - \sin u \cos v - \frac{1}{\sqrt{10}} \cos u \sin v, u + \frac{3}{\sqrt{10}} \sin v\Big)$$

and

$$\mathbf{Y}(p,q) = (2\cos p, 2\sin p, q),$$

where $0 \leq u, v, p, q \leq 2\pi$ (Figure 1). Let us find the unit tangent vector and the geodesic torsions with respect to the surfaces A and B of the intersection curve at the point $P = \mathbf{X}(0,0) = \mathbf{Y}(0,0) = (2,0,0)$.

The partial derivatives of the surface A are $\mathbf{X}_u = (0,2,1)$, $\mathbf{X}_v = (0, -\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$, $\mathbf{X}_{uu} = (-2,0,0)$, $\mathbf{X}_{uv} = (\frac{1}{\sqrt{10}}, 0, 0)$ and $\mathbf{X}_{vv} = (1,0,0)$ at P. Thus we find the unit normal vector and the first and the second fundamental form coefficients of A at P as $\mathbf{N}^A = (1,0,0)$, $E = 5$, $F = \frac{1}{\sqrt{10}}$, $G = 1$, $L = -2$, $M = \frac{1}{\sqrt{10}}$, $N = 1$.

Similarly, for the surface B at the point P, we get $\mathbf{N}^B = (1,0,0)$, $\mathbf{Y}_p = (0,2,0)$, $\mathbf{Y}_q = (0,0,1)$, $\mathbf{Y}_{pp} = (-2,0,0)$, $\mathbf{Y}_{pq} = \mathbf{Y}_{qq} = (0,0,0)$, $e = 4$, $g = 1$, $l = -2$, $f = m = n = 0$.

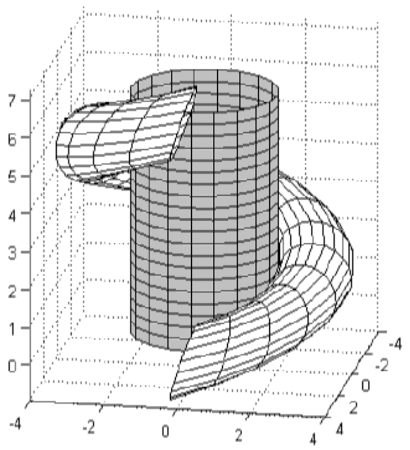

Figure 1. The tangential intersection of the cylinder and the canal surface.

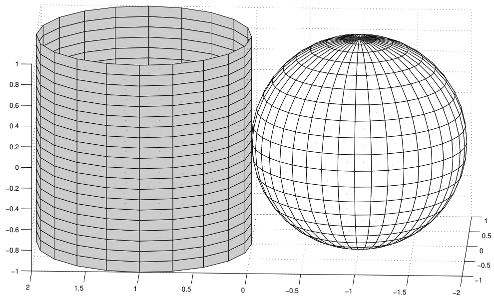

Figure 2. Tangential intersection of a cylinder and sphere.

Also, we have $D_1 = d_2 = 1$, $D_2 = \sqrt{10}$, $D_3 = d_1 = d_3 = 0$ and $b_{11} = b_{21} = 1$, $b_{12} = -\frac{1}{2\sqrt{10}}$, $b_{22} = \frac{3}{\sqrt{10}}$. Therefore, we obtain $5\sqrt{10}\nu + \nu^2 = 0$, i.e., $\Delta = 250 > 0$. By solving this equation, the values $\nu_1 = 0$ and $\nu_2 = -5\sqrt{10}$ are found. So, from (11), we obtain $\mathbf{r}_1 = (0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ and $\mathbf{r}_2 = (0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$.

Let us denote the common unit normal vectors of the surfaces A and B by $\mathbf{N}$. Since the normal vector of R_1 determined by $\mathbf{N}$ and $\mathbf{r}_1$ is $\mathbf{N}_1 = (0, -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$, R_1 has the parametric equation $\mathbf{Z}(r, w) = (r, 2w, w)$. Then, $\mathbf{Z}_r = (1, 0, 0)$, $\mathbf{Z}_w = (0, 2, 1)$, $E_1 = 1$, $F_1 = 0$, $G_1 = 5$, $\mathbf{t}_1 = (0, -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}})$, $r' = 0$, $w' = -\frac{1}{\sqrt{5}}$, $r'' = -\frac{2}{5}$, $w'' = 0$. So, we have $\kappa_{g_1}^A = -\frac{2}{5}$. Similarly, we get $\kappa_{g_1}^B = -\frac{2}{5}$. On the other hand, we find $\kappa_{g_2}^A = \frac{238}{245}$, $\kappa_{g_2}^B = \frac{-1}{10}$. Since $\kappa_{g_1}^A = \kappa_{g_1}^B$ and $\kappa_{g_2}^A \neq \kappa_{g_2}^B$, the vector $\mathbf{r}_1$ is the tangent vector of the tangential intersection curve of the surfaces A and B at P, i.e., $\mathbf{t} = (0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. Also, we find $u' = \frac{1}{\sqrt{5}}$, $v' = 0$ and $p' = q' = \frac{1}{\sqrt{5}}$. Thus, we obtain the geodesic torsions $\tau_g^A = \tau_g^B = \frac{1}{5}$ of the tangential intersection curve at the point P.

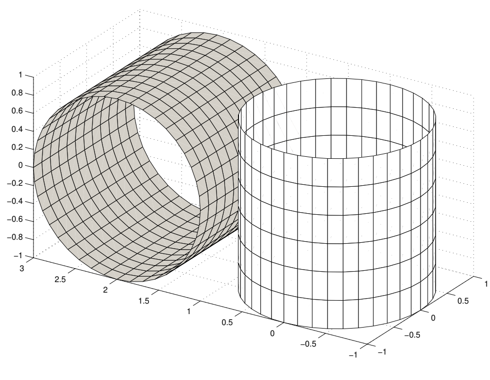

Figure 3. Tangential intersection of two cylinders.

Example 2. Let us consider the parametric surfaces A and B, respectively, with

$$\mathbf{X}(u,v) = (\cos u \cos v, -1 + \sin u \cos v, \sin v), \quad \mathbf{Y}(p,q) = (\cos q, 1 + \sin q, p),$$

where $-\pi < u < \pi$, $-\frac{\pi}{2} < v < \frac{\pi}{2}$, $-1 < p < 1, -\pi < q < \pi$.

These surfaces intersect tangentially at the origin. We have $c_1 = 0$, $c_2 = -1$, $c_3 = 0$, i.e. $\Delta > 0$. Applying the explained method for $\mathbf{r}_1 = (0,0,1)$ and $\mathbf{r}_2 = (-1,0,0)$, we find $\kappa^A_{g_1} = -1$, $\kappa^B_{g_1} = 0$, $\kappa^A_{g_2} = -1$, $\kappa^B_{g_2} = 1$. Since $\kappa^A_{g_1} \neq \kappa^B_{g_1}$ and $\kappa^A_{g_2} \neq \kappa^B_{g_2}$, P is an isolated contact point (Figure 2).

Example 3. The surfaces $A \ldots \mathbf{X}(u,v) = (\cos u, \sin u, v)$ and $B \ldots \mathbf{Y}(p,q) = (p, 2 + \cos q, \sin q)$ $(0 < u,q < 2\pi,\ -1 < v,p < 1)$ intersect tangentially at the point $P = (0,1,0)$. We obtain $\Delta = 0$ with $c_1 = c_2 = c_3 = 0$. Thus, by taking $\mathbf{r}_1 = (-1,0,0)$, we have $\kappa^A_{g_1} \neq \kappa^B_{g_1}$. Hence, P is an isolated contact point (Figure 3).

Example 4. Let us consider the parametric surfaces A and B respectively, with

$$\mathbf{X}(u,v) = (u,v,v^4), \qquad \mathbf{Y}(p,q) = (p,q,0), \quad -1 < u,v,p,q < 1,$$

which are intersect tangentially at origin. For these surfaces we find $\Delta = 0$ with $c_1 = c_2 = c_3 = 0$. By taking $\mathbf{r}_1 = (1,0,0)$ we have $\kappa^A_{g_1} = \kappa^B_{g_1}$. Thus, the surfaces have at least second order contact at origin (Figure 4).

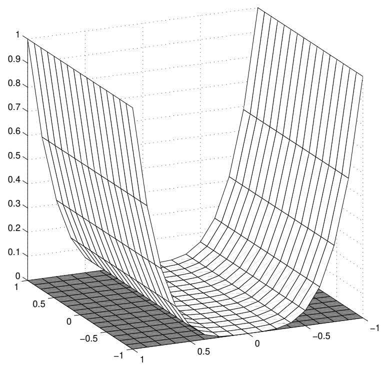

Figure 4. Tangential intersection with higher order contact.

2.2. Tangential intersection curve of implicit-implicit surfaces

Let A and B be two regular tangentially intersecting surfaces with implicit equations $f(x,y,z) = 0$ and $g(x,y,z) = 0$, respectively. Since $\nabla f = (f_x, f_y, f_z) \neq 0$ and $\nabla g = (g_x, g_y, g_z) \neq 0$, the normal vectors of the surfaces are

$$\mathbf{N}^A = \frac{\nabla f}{\|\nabla f\|}, \qquad \mathbf{N}^B = \frac{\nabla g}{\|\nabla g\|}.$$

By orienting the surfaces properly, we can assume $\mathbf{N}^A = \mathbf{N}^B = \mathbf{N}$ along the intersection curve α. Let us denote the unit tangent vector of α with $\alpha'(s) = \mathbf{t} =$

(x', y', z'). Since $\tau_g^A = \langle (\mathbf{V}^A)', \mathbf{N}^A \rangle$ and $\mathbf{V}^A = \mathbf{N}^A \times \mathbf{t}$, we have

$$\tau_g^A = \frac{1}{\|\nabla f\|}\{(a_3 f_y - a_2 f_z)x' + (a_1 f_z - a_3 f_x)y' + (a_2 f_x - a_1 f_y)z'\}, \tag{19}$$

where $(\mathbf{N}^A)' = (a_1, a_2, a_3)$ and

$$\begin{aligned} a_i = & \frac{1}{\|\nabla f\|}\left(f_{x_i x_i} x_i' + f_{x_i x_j} x_j' + f_{x_i x_k} x_k'\right) \\ & - \frac{1}{\|\nabla f\|^3}\Big[f_{x_i}^2\left(f_{x_i x_i} x_i' + f_{x_i x_j} x_j' + f_{x_i x_k} x_k'\right) \\ & + f_{x_i} f_{x_j}\left(f_{x_j x_i} x_i' + f_{x_j x_j} x_j' + f_{x_j x_k} x_k'\right) \\ & + f_{x_i} f_{x_k}\left(f_{x_k x_i} x_i' + f_{x_k x_j} x_j' + f_{x_k x_k} x_k'\right)\Big] \end{aligned}$$

with $x_1 = x$, $x_2 = y$, $x_3 = z$ $(i, j, k = 1, 2, 3$ cyclic).

Similarly, for the geodesic torsion of the intersection curve with respect to the surface B, we find

$$\tau_g^B = \frac{1}{\|\nabla g\|}\{(b_3 g_y - b_2 g_z)x' + (b_1 g_z - b_3 g_x)y' + (b_2 g_x - b_1 g_y)z'\}, \tag{20}$$

where $(\mathbf{N}^B)' = (b_1, b_2, b_3)$ and

$$\begin{aligned} b_i = & \frac{1}{\|\nabla g\|}\left(g_{x_i x_i} x_i' + g_{x_i x_j} x_j' + g_{x_i x_k} x_k'\right) \\ & - \frac{1}{\|\nabla g\|^3}\Big[g_{x_i}^2\left(g_{x_i x_i} x_i' + g_{x_i x_j} x_j' + g_{x_i x_k} x_k'\right) \\ & + g_{x_i} g_{x_j}\left(g_{x_j x_i} x_i' + g_{x_j x_j} x_j' + g_{x_j x_k} x_k'\right) \\ & + g_{x_i} g_{x_k}\left(g_{x_k x_i} x_i' + g_{x_k x_j} x_j' + g_{x_k x_k} x_k'\right)\Big] \end{aligned}$$

with $x_1 = x$, $x_2 = y$, $x_3 = z$ $(i, j, k = 1, 2, 3$ cyclic).

Since the surfaces A and B intersect tangentially along the intersection curve, $\tau_g^A = \tau_g^B$ is valid. Then, from Eq. (19) and (20), we obtain

$$\lambda_1 x' + \lambda_2 y' + \lambda_3 z' = 0, \tag{21}$$

where

$$\begin{aligned} \lambda_1 &= \frac{a_3 f_y - a_2 f_z}{||\nabla f||} + \frac{b_2 g_z - b_3 g_y}{||\nabla g||}, \\ \lambda_2 &= \frac{a_1 f_z - a_3 f_x}{||\nabla f||} + \frac{b_3 g_x - b_1 g_z}{||\nabla g||}, \\ \lambda_3 &= \frac{a_2 f_x - a_1 f_y}{||\nabla f||} + \frac{b_1 g_y - b_2 g_x}{||\nabla g||}. \end{aligned}$$

Also, since the tangent vector $\mathbf{t}$ is perpendicular to the gradient vector ∇f, we have

$$f_x x' + f_y y' + f_z z' = 0. \tag{22}$$

Eq. (21) and Eq. (22) constitute a linear system with unknowns x', y' and z'. Since at least one of the f_x, f_y and f_z is non-zero, we assume f_z is non-zero. Then we get $z' = -\frac{f_x x' + f_y y'}{f_z}$ from Eq. (22). Substituting this value of z' into (21), we find

$$\mu_1 x' + \mu_2 y' = 0, \tag{23}$$

where $\mu_1 = \lambda_1 f_z - \lambda_3 f_x$ and $\mu_2 = \lambda_2 f_z - \lambda_3 f_y$. Since x', y' and z' are components of the unit tangent vector, x' and y' both can not be zero. If we denote $\rho = \frac{x'}{y'}$ when $y' \neq 0$, or $\nu = \frac{y'}{x'}$ when $x' \neq 0$, and solve (23) for ρ or ν, then

$$\mathbf{r}_1 = \frac{(\rho y', y', -\frac{\rho f_x + f_y}{f_z} y')}{\|(\rho y', y', -\frac{\rho f_x + f_y}{f_z} y')\|} \qquad \text{or} \qquad \mathbf{r}_2 = \frac{(x', \nu x', -\frac{f_x + \nu f_y}{f_z} x')}{\|(x', \nu x', -\frac{f_x + \nu f_y}{f_z} x')\|}$$

are found. Now, let us determine the vector which corresponds to the tangent vector at the point P. If we denote the plane determined by $\mathbf{N}$ and $\mathbf{r}_1$ with R_1, then R_1 has the implicit equation $h(x, y, z) = 0$. The intersection of R_1 and A is the transversal intersection. Thus, the unit tangent vector of this intersection curve is

$$\mathbf{t}_1 = \frac{\mathbf{N} \times \mathbf{N}_1}{||\mathbf{N} \times \mathbf{N}_1||} = (x_1', y_1', z_1'),$$

where the vector $\mathbf{N}_1 = \mathbf{N} \times \mathbf{r}_1$ is the normal vector of the plane R_1. Then the geodesic curvature $\kappa_{g_1}^A$ of the transversal intersection curve with respect to R_1 is found from Eq. (6) as

$$\kappa_{g_1}^A = \frac{1}{||\nabla h||}\{(y_1' z_1'' - y_1'' z_1')h_x + (x_1'' z_1' - x_1' z_1'')h_y + (x_1' y_1'' - x_1'' y_1')h_z\}, \tag{24}$$

where $\mathbf{t}_1' = (x_1'', y_1'', z_1'')$. If the linear equation system consisting of the equations

$$\begin{aligned} x_1' x_1'' + y_1' y_1'' + z_1' z_1'' &= 0, \\ h_x x_1'' + h_y y_1'' + h_z z_1'' &= 0, \\ f_x x_1'' + f_y y_1'' + f_z z_1'' &= -\{f_{xx}(x_1')^2 + f_{yy}(y_1')^2 + f_{zz}(z_1')^2 \\ &\quad +2(f_{xy} x_1' y_1' + f_{xz} x_1' z_1' + f_{yz} y_1' z_1')\} \end{aligned}$$

is solved, the unknowns x_1'', y_1'' and z_1'' can be found. Substituting these values into Eq. (24) yield the geodesic curvature $\kappa_{g_1}^A$. Similarly, the geodesic curvature $\kappa_{g_1}^B$ of the transversal intersection curve of the surface B and the plane R_1 can be found.

By using the previous method given in paramteric-parametric intersection, we determine the tangent vector at P of the tangential intersection curve of the surfaces A and B. Then the geodesic torsion τ_g^A (or τ_g^B) of the intersection curve with respect to A (or B) is calculated by Eq. (19) (or Eq. (20)).

Example 5. The implicit surface A is given by $f(x, y, z) = (\sqrt{x^2 + y^2} - 2)^2 + (z - 1)^2 - 1 = 0$ and the implicit surface B is given by $g(x, y, z) = z - 2 = 0$ (Figure 5).

We have $\nabla f = (0, 0, 2)$ and $\nabla g = (0, 0, 1)$ at the point $P = (0, 2, 2)$ on the intersection curve of the surfaces A and B. At the intersection point we have

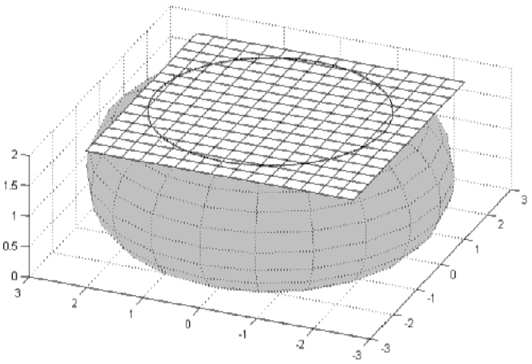

Figure 5. The tangential intersection of the torus and the plane.

$\|\nabla f\| = 2$, $\mathbf{N}^A = (0,0,1)$, $(\nabla f)' = (0, 2y', 2z')$, $(\mathbf{N}^A)' = (0, y', 0)$ for the surface A and $\|\nabla g\| = 1$, $\mathbf{N}^B = (0,0,1)$, $(\nabla g)' = (\mathbf{N}^B)' = (0,0,0)$ for the surface B. Also, the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ are calculated as $\mathbf{r}_1 = (0,1,0)$ and $\mathbf{r}_2 = (1,0,0)$, and the geodesic curvatures are found as $\kappa_{g_1}^A = -1$, $\kappa_{g_1}^B = 0$, $\kappa_{g_2}^A = 0$, $\kappa_{g_2}^B = 0$. Since $\kappa_{g_1}^A \neq \kappa_{g_1}^B$ and $\kappa_{g_2}^A = \kappa_{g_2}^B$, the unit tangent vector of the tangential intersection curve of the surfaces A and B at P is the vector $\mathbf{r}_2$, i.e., $\mathbf{t} = (1,0,0)$. Then the geodesic torsions τ_g^A and τ_g^B are calculated as zero at P.

2.3. Tangential intersection curve of parametric-implicit surfaces

Let A be a regular surface given by the parametric equation $\mathbf{X}(u,v)$ and B be a regular surface given by the implicit equation $g(x,y,z) = 0$. The unit normal vectors of the surfaces A and B on the intersection curve α are given by

$$\mathbf{N}^A = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}, \qquad \mathbf{N}^B = \frac{\nabla g}{\|\nabla g\|}.$$

Let us denote the common surface normal by $\mathbf{N} = \mathbf{N}^A = \mathbf{N}^B$. The unit tangent vector of the curve α is

$$\mathbf{t} = \mathbf{X}_u u' + \mathbf{X}_v v' = (x', y', z'). \tag{25}$$

We know the geodesic torsions of α with respect to the surfaces A and B, respectively, as

$$\tau_g^A = D_1(u')^2 + D_2 u'v' + D_3(v')^2 \tag{26}$$

and

$$\tau_g^B = E_1 x' + E_2 y' + E_3 z', \tag{27}$$

where $E_1 = \frac{b_3 g_y - b_2 g_z}{\|\nabla g\|}$, $E_2 = \frac{b_1 g_z - b_3 g_x}{\|\nabla g\|}$, $E_3 = \frac{b_2 g_x - b_1 g_y}{\|\nabla g\|}$. Since the surfaces A and B intersect tangentially along the curve α, τ_g^A is equal to τ_g^B, and so

$$D_1(u')^2 + D_2 u'v' + D_3(v')^2 - E_1 x' - E_2 y' - E_3 z' = 0. \tag{28}$$

If we substitute the values of x', y', z' in terms of u' and v' into Eq. (28), we obtain a quadratic equation similar to (10). Solving this quadratic equation and applying the same method, the unit tangent vector of the intersection curve at P is found. Also, substituting u' and v' into Eq. (26) or x', y', z' into Eq. (27), the geodesic torsions of α are obtained.

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