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# SOME GRÜSS TYPE INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. In this paper a Grüss type inequalities for Riemann-Stieltjes integral are proved. Applications to the approximation problem of the Riemann-Stieltjes are also pointed out.

## 1. Introduction

In 1935 G. Grüss proved the following famous inequality regarding the integral of the product of two functions and the product of the integrals

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x-\left(\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right)\cdot\left(\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right)\right|\\ \leq\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma)\end{aligned}\tag{1.1}$$

provided that $f$ and $g$ are two integrable functions on $[a,b]$ satisfying the condition $\phi\leq f(x)\leq\Phi$ and $\gamma\leq g(x)\leq\Gamma$ for all $x\in[a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller one.

In [**15**] Dragomir and Fedotov have established the following functional

$$\mathcal{D}(f;u):=\int_a^b f(x)\mathrm{d}u(x)-\frac{u\,(b)-u(a)}{b-a}\int_a^b f(t)\mathrm{d}t\tag{1.2}$$

provided that the Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$ and the Riemann integral $\int_a^b f(t)\mathrm{d}t$ exist.

In the same paper [**15**] the authors proved the following inequality.

**Theorem 1.** *Let $f,u\colon[a,b]\to\mathbb{R}$ be such that $u$ is of bounded variation on $[a,b]$ and $f$ is Lipschitzian with the constant $K>0$. Then we have*

$$|\mathcal{D}(f;u)|\leq\frac{1}{2}K(b-a)\bigvee_a^b(u)\tag{1.3}$$

*The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.*

Also, in [**7**], Dragomir obtained the following inequality

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**Theorem 2.** *Let $f, u\colon [a,b] \to \mathbb{R}$ be such that $u$ is Lipschitzian on $[a,b]$, i.e.,*

$$|u(y) - u(x)| \leq L|x - y| \qquad \forall x, y \in [a,b], \quad (L > 0)$$

*and $f$ is Riemann integrable on $[a,b]$.*

*If $m, M \in \mathbb{R}$ are such that $m \leq f(x) \leq M$ for any $x \in [a,b]$, then the inequality*

$$|\mathcal{D}(f;u)| \leq \frac{1}{2}L(M-m)(b-a) \tag{1.4}$$

*holds. The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.*

For other recent inequalities for the Riemann-Stieltjes integral see [**1**]–[**16**] and the references therein.

The aim of this paper is to obtain several new bounds for $\mathcal{D}(f;u)$. More specifically, the integrand $f$ is assumed to be monotonic nondecreasing on both $[a,x]$ and $[x,b]$, and the integrator $u$ is to be of bounded variation, Lipschitzian and monotonic on $[a,b]$.

## 2. The case of bounded variation integrators

**Theorem 3.** *Let $x \in [a,b]$. Let $u\colon [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. Assume that $f$ is monotonic nondecreasing on both $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \leq [f(b) - f(a)] \cdot \bigvee_a^b(u). \tag{2.1}$$

*Proof.* It is well-known that for a continuous function $p\colon [a,b] \to \mathbb{R}$ and a function $\nu\colon [a,b] \to \mathbb{R}$ of bounded variation, one has the inequality

$$\left| \int_a^b p(t) \mathrm{d}\nu(t) \right| \leq \sup_{t \in [a,b]} |p(t)| \bigvee_a^b(\nu).$$

Therefore, as $u$ is of bounded variation on $[a,b]$, we have

$$\begin{aligned} |\mathcal{D}(f;u)| &= \left| \int_a^b \left[ f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right] \mathrm{d}u(x) \right| \\ &\leq \sup_{x \in [a,b]} \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t \right| \cdot \bigvee_a^b(u) \\ &= \frac{1}{b-a} \sup_{x \in [a,b]} \left| \int_a^b [f(x) - f(t)]\, \mathrm{d}t \right| \cdot \bigvee_a^b(u) \\ &= \frac{1}{b-a} \sup_{x \in [a,b]} \int_a^b |f(x) - f(t)|\, \mathrm{d}t \cdot \bigvee_a^b(u). \end{aligned} \tag{2.2}$$

As $f$ is monotonic nondecreasing on $[a,x]$ and monotonic nondecreasing on $[x,b]$, we get

$$\begin{aligned}\int_a^b |f(x)-f(t)|\,\mathrm{d}t &\le \int_a^x |f(x)-f(t)|\,\mathrm{d}t + \int_x^b |f(x)-f(t)|\,\mathrm{d}t \\ &= (x-a)f(x) - \int_a^x f(t)\mathrm{d}t + \int_x^b f(t)\mathrm{d}t - (b-x)f(x) \\ &= (2x-a-b)f(x) + \int_x^b f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t.\end{aligned}$$

Utilizing the monotonicity property of $f$ on both intervals, we have

$$\int_x^b f(t)\mathrm{d}t \le (b-x)\,f\,(b) \qquad \text{and} \qquad \int_a^x f(t)\mathrm{d}t \ge (x-a)f(a)$$

which imply that

$$\int_a^b |f(x)-f(t)|\,\mathrm{d}t \le (2x-a-b)f(x) + (b-x))f(b) - (x-a)f(a).$$

Taking 'sup' for both sides, we get

$$\begin{aligned}\sup_{x\in[a,b]} \int_a^b |f(x)-f(t)|\mathrm{d}t &\le \sup_{x\in[a,b]} \{(2x-a-b)f(x) + (b-x)f(b) - (x-a)f(a)\} \\ &= (b-a)\,[f(b)-f(a)]\,. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3), we get

$$\begin{aligned}|\mathcal{D}(f;u)| &\le \frac{1}{b-a} \sup_{x\in[a,b]} \int_a^b |f(x)-f(t)|\,\mathrm{d}t \cdot \bigvee_a^b (u) \\ &\le [f(b)-f(a)] \cdot \bigvee_a^b (u),\end{aligned}$$

and the theorem is proved. □

**Corollary 1.** *Let $f$ be as in Theorem 3. Let $u \in C^{(1)}[a,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le [f(b)-f(a)] \cdot \|u'\|_{1,[a,b]} \tag{2.4}$$

*where $\|\cdot\|_1$ is the $L_1$ norm, namely $\|u'\|_{1,[a,b]} := \int_a^b |u'(t)|\,\mathrm{d}t$.*

**Corollary 2.** *Let $f$ be as in Theorem 3. Let $u\colon [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constant $L > 0$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le L(b-a)[f(b)-f(a)]. \tag{2.5}$$

**Corollary 3.** *Let $f$ be as in Theorem 3. Let $u\colon [a,b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le [f(b)-f(a)] \cdot |u\,(b)-u(a)|\,. \tag{2.6}$$

## 3. The case of Lipschitzian integrators

**Theorem 4.** *Let $x \in [a,b]$. Let $u, f\colon [a,b] \to \mathbb{R}$ be such that $u$ is $L$-Lipschitzian on $[a,b]$ and $f$ is monotonic nondecreasing on both $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le L\left[\frac{1}{2}(b-a)(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right]. \tag{3.1}$$

*Proof.* It is well-known that for a Riemann integrable function $p\colon [a,b] \to \mathbb{R}$ and $L$-Lipschitzian function $\nu\colon [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)d\nu(t)\right| \le L\int_a^b |p(t)|\,\mathrm{d}t.$$

Therefore, as $u$ is $L$-Lipschitzian on $[a,b]$, we have

$$\begin{aligned}|\mathcal{D}(f;u)| &= \left|\int_a^b \left[f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right]\mathrm{d}u(x)\right| \\ &\le L\int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\mathrm{d}x = \frac{L}{b-a}\int_a^b\left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right|\mathrm{d}x. \end{aligned}\tag{3.2}$$

As $f$ is monotonic nondecreasing on $[a,x]$ and monotonic nondecreasing on $[x,b]$, we get

$$\begin{aligned}\left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| &\le \int_a^b |f(x)-f(t)|\,\mathrm{d}t \\ &= \int_a^x |f(x)-f(t)|\,\mathrm{d}t + \int_x^b |f(x)-f(t)|\,\mathrm{d}t \\ &= (x-a)f(x) - \int_a^x f(t)\mathrm{d}t + \int_x^b f(t)\mathrm{d}t - (b-x)\,f(x) \\ &= (2x-a-b)\,f(x) + \int_x^b f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t.\end{aligned}$$

Utilizing the monotonicity property of $f$ on both intervals, we have

$$\int_x^b f(t)\mathrm{d}t \le (b-x)\,f(b) \qquad \text{and} \qquad \int_a^x f(t)\mathrm{d}t \ge (x-a)f(a),$$

which imply that

$$\int_a^b |f(x)-f(t)|\,\mathrm{d}t \le (2x-a-b)\,f(x) + (b-x)\,f(b) - (x-a)f(a). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\begin{aligned}
|\mathcal{D}(f;u)| &\le \frac{L}{b-a}\int_a^b \left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| \mathrm{d}x \\
&\le \frac{L}{b-a}\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}x \\
&= \frac{1}{2}L(b-a)\,[f(b)-f(a)] + \frac{L}{(b-a)}\int_a^b (2x-a-b)\,f(x)\mathrm{d}x \\
&\le \frac{1}{2}L(b-a)\,[f(b)-f(a)] + \frac{L}{(b-a)}\cdot \max_{x\in[a,b]}\{2x-a-b\}\cdot \int_a^b f(x)\mathrm{d}x \\
&= L\left[\frac{1}{2}(b-a)\,(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right]
\end{aligned}$$

and the theorem is proved. □

## 4. The case of monotonic integrators

**Theorem 5.** *Let $x \in [a,b]$. Let $u, f\colon [a,b] \to \mathbb{R}$ be a continuous mappings on $[a,b]$. Assume that $u$ is monotonic nondecreasing mapping on $[a,b]$ and $f\colon [a,b] \to \mathbb{R}$ is monotonic nondecreasing on both intervals $[a,x]$ and $[x,b]$. Then we have the inequality*

$$|\mathcal{D}(f;u)| \le 2u(b)\cdot\left[f(b) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]. \tag{4.1}$$

*Proof.* It is well-known that for a monotonic non-decreasing function $\nu\colon [a,b] \to \mathbb{R}$ and continuous function $p\colon [a,b] \to \mathbb{R}$, one has the inequality

$$\left|\int_a^b p(t)d\nu(t)\right| \le \int_a^b |p(t)|\,d\nu(t).$$

Therefore, as $u$ is monotonic non-decreasing on $[a,b]$, we have

$$\begin{aligned}
|\mathcal{D}(f;u)| &= \left|\int_a^b \left[f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right]\mathrm{d}u(x)\right| \le \int_a^b \left|f(x) - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right|\mathrm{d}u(x) \\
&= \frac{1}{b-a}\int_a^b \left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| \mathrm{d}u(x). \qquad (4.2)
\end{aligned}$$

As $f$ is monotonic nondecreasing on $[a,x]$ and monotonic nondecreasing on $[x,b]$, we get

$$\begin{aligned}\left|\int_a^b [f(x)-f(t)]\,\mathrm{d}t\right| &\le \int_a^b |f(x)-f(t)|\,\mathrm{d}t\\ &\le \int_a^x |f(x)-f(t)|\,\mathrm{d}t + \int_x^b |f(x)-f(t)|\,\mathrm{d}t\\ &= (x-a)f(x) - \int_a^x f(t)\mathrm{d}t + \int_x^b f(t)\mathrm{d}t - (b-x)\,f(x)\\ &= (2x-a-b)\,f(x) + \int_x^b f(t)\mathrm{d}t - \int_a^x f(t)\mathrm{d}t.\end{aligned}$$

Utilizing the monotonicity property of $f$ on both intervals, we have

$$\int_x^b f(t)\mathrm{d}t \le (b-x)\,f(b) \qquad \text{and} \qquad \int_a^x f(t)\mathrm{d}t \ge (x-a)f(a)$$

which imply that

$$\int_a^b |f(x)-f(t)|\,\mathrm{d}t \le (2x-a-b)\,f(x) + (b-x)\,f(b) - (x-a)f(a). \tag{4.3}$$

Using (4.2) and (4.3), we get

$$\begin{aligned}&|\mathcal{D}(f;u)|\\ &\le \frac{1}{b-a}\int_a^b [(2x-a-b)\,f(x) + (b-x)\,f(b) - (x-a)f(a)]\,\mathrm{d}u(x)\end{aligned} \tag{4.4}$$

Now, using Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^b (2x-a-b)\,f(x)\mathrm{d}u(x) &= (b-a)\,[f(b)u(b)+f(a)u(a)]\\ &\quad - 2\int_a^b u(x)f(x)\mathrm{d}x - \int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x)\\ \int_a^b (b-x)\,f(b)\mathrm{d}u(x) &= f(b)\int_a^b (b-x)\,\mathrm{d}u(x)\\ &= -(b-a)u(a)f(b) + f(b)\int_a^b u(x)\mathrm{d}x\end{aligned}$$

and

$$\int_a^b (x-a)f(a)\mathrm{d}u(x) = f(a)\int_a^b (x-a)\mathrm{d}u(x) = (b-a)u(b)f(a) - f(a)\int_a^b u(x)\mathrm{d}x.$$

Therefore, by (4.4), we get

$$
\begin{aligned}
&\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}u(x) \\
&= (b-a)\,[f(b)u(b)+f(a)u(a)]-2\int_a^b u(x)f(x)\mathrm{d}x \\
&\quad -\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x)-(b-a)u(a)f(b)+f(b)\int_a^b u(x)\mathrm{d}x \\
&\quad -(b-a)u(b)f(a)+f(a)\int_a^b u(x)\mathrm{d}x \\
&= (b-a)\,(f(b)-f(a))\,(u(b)-u(a))+(f(a)+f(b))\int_a^b u(x)\mathrm{d}x \\
&\quad -2\int_a^b u(x)f(x)\mathrm{d}x-\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x).
\end{aligned} \tag{4.5}
$$

Now, by the monotonicity property of $u$, we have $\int_a^b u(x)\mathrm{d}x \le (b-a)\,u(b)$,

$$\int_a^b u(x)f(x)\mathrm{d}x \ge u\,(a)\int_a^b f(x)\mathrm{d}x$$

and

$$\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x) \ge (a-b)\,u(a)\int_a^b \mathrm{d}f(x) = (a-b)\,u(a)\cdot(f(b)-f(a))$$

which by (4.5) give

$$
\begin{aligned}
&\int_a^b [(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)]\,\mathrm{d}u(x) \\
&= (b-a)\,(f(b)-f(a))\,(u(b)-u(a))+(f(a)+f(b))\int_a^b u(x)\mathrm{d}x \\
&\quad -2\int_a^b u(x)f(x)\mathrm{d}x-\int_a^b (2x-a-b)\,u(x)\mathrm{d}f(x) \\
&\le (b-a)\,(f(b)-f(a))\,(u(b)-u(a))+(b-a)\,(f(a)+f(b))\,u(b) \\
&\quad -2u\,(a)\int_a^b f(x)\mathrm{d}x-(a-b)\,u(a)\cdot(f(b)-f(a)) \\
&= (b-a)\,[(f(b)-f(a))\,u(b)+(f(a)+f(b))\,u(b)]-2u\,(a)\int_a^b f(x)\mathrm{d}x \\
&= 2(b-a)f(b)u(b)-2u\,(a)\int_a^b f(x)\mathrm{d}x.
\end{aligned}
$$

Therefore, by (4.4) we get

$$\begin{aligned}
|\mathcal{D}(f;u)| &\leq \frac{1}{b-a}\int_a^b \left[(2x-a-b)\,f(x)+(b-x)\,f(b)-(x-a)f(a)\right]\mathrm{d}u(x)\\
&\leq 2f(b)u(b)-2\frac{u\,(a)}{b-a}\int_a^b f(x)\mathrm{d}x.
\end{aligned}$$

Now, using the properties of 'max' function and the monotonicity of $u$, we get

$$\begin{aligned}
|\mathcal{D}(f;u)| &\leq 2f(b)u(b)-2\frac{u\,(a)}{b-a}\int_a^b f(x)\mathrm{d}x\\
&\leq 2\cdot\max\left\{u(a),u(b)\right\}\cdot\left[f(b)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]\\
&= 2u(b)\cdot\left[f(b)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right]
\end{aligned}$$

which proves the inequality (4.1). □

## 5. A Numerical quadrature formula for the Riemann-Stieltjes integral

In this section, an approximation for the Riemann-Stieltjes integral $\int_a^b f(x)\mathrm{d}u(x)$, is given in terms of the Riemann integral $\int_a^b f(t)\mathrm{d}t$.

**Theorem 6.** *Let $f,u$ be as in Theorem 3 and consider*

$$I_h := \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$$

*a partition of $[a,b]$. Denote $h_i = x_{i+1}-x_i$, $i = 1,2,\cdots n-1$. Then we have*

$$\int_a^b f(x)\mathrm{d}u(x) = A_n\left(f,u,I_h\right)+R_n\left(f,u,I_h\right), \tag{5.1}$$

*where*

$$A_n\left(f,u,I_h\right)=\sum_{i=0}^{n-1}\frac{u\left(x_{i+1}\right)-u\left(x_i\right)}{h_i}\times\int_{x_i}^{x_{i+1}} f(t)\mathrm{d}t \tag{5.2}$$

*and the Remainder $R_n\left(f,u,I_h\right)$ satisfies the estimation*

$$\left|R_n\left(f,u,I_h\right)\right|\leq\left[f(b)-f(a)\right]\cdot\bigvee_a^b(u). \tag{5.3}$$

*Proof.* Applying Theorem 3 on the intervals $[x_i,x_{i+1}]$, $i = 1,2,\cdots n-1$, we get

$$\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}u(x)-\frac{u\left(x_{i+1}\right)-u\left(x_i\right)}{h_i}\int_{x_i}^{x_{i+1}} f(t)\mathrm{d}t\right|\leq\left[f\left(x_{i+1}\right)-f\left(x_i\right)\right]\bigvee_{x_i}^{x_{i+1}}(u).$$

Summing the above inequality over $i$ from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned}\left|\int_a^b f(x)\mathrm{d}u(x) - A_n(f,u,I_h)\right| &\le \sum_{i=0}^{n-1}\left[f(x_{i+1}) - f(x_i)\right]\bigvee_{x_i}^{x_{i+1}}(u)\\ &\le \max_{i=\overline{0,n-1}}\{f(x_{i+1}) - f(x_i)\}\cdot\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(u)\\ &= [f(b) - f(a)]\cdot\bigvee_a^b(u),\end{aligned}$$

and the theorem is proved. □

**Theorem 7.** *Let $f, u$ be as in Theorem 4. Let $I_h$ be as above. Then we have*

$$\int_a^b f(x)\mathrm{d}u(x) = A_n(f,u,I_h) + R_n(f,u,I_h), \tag{5.4}$$

*where $A_n(f,u,I_h)$ is defined in* (5.2) *and the Remainder $R_n(f,u,I_h)$ satisfies the estimation*

$$|R_n(f,u,I_h)| \le L\left[\frac{1}{2}\nu(h)(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right], \tag{5.5}$$

*where $\nu(h) = \max_{i=\overline{0,n-1}}\{h_i\}$.*

*Proof.* Applying Theorem 4 on the intervals $[x_i, x_{i+1}]$, $i = 1, 2, \cdots n-1$, we get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(x)\mathrm{d}u(x) - \frac{u(x_{i+1}) - u(x_i)}{h_i}\int_{x_i}^{x_{i+1}} f(t)\mathrm{d}t\right|\\ &\qquad\le L\left[\frac{h_i}{2}(f(x_{i+1}) - f(x_i)) + \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x\right].\end{aligned}$$

Summing the above inequality over $i$ from 0 to $n-1$ and using the generalized triangle inequality, we deduce that

$$\begin{aligned}&\left|\int_a^b f(x)\mathrm{d}u(x) - A_n(f,u,I_h)\right|\\ &\quad\le L\sum_{i=0}^{n-1}\left[\frac{h_i}{2}(f(x_{i+1}) - f(x_i)) + \int_{x_i}^{x_{i+1}} f(x)\mathrm{d}x\right]\\ &\quad\le L\left[\frac{1}{2}\max_{i=\overline{0,n-1}}\{h_i\}\cdot\sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i)) + \int_a^b f(x)\mathrm{d}x\right]\\ &\quad\le L\left[\frac{1}{2}\nu(h)(f(b)-f(a)) + \int_a^b f(x)\mathrm{d}x\right],\end{aligned}$$

and the theorem is proved. □

**Remark 1.** Similarly, one may apply Theorem 5 to approximate $\int_a^b f(x)\mathrm{d}u(x)$ in terms of $\int_a^b f(t)\mathrm{d}t$. We shall omit the details to the interested reader.

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