Acta Math. Univ. Comenianae
Vol. LXXXI, 1 (2012), pp. 61–69

PEXIDER TYPE QUARTIC OPERATORS AND THEIR NORMS IN X_λ SPACES

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Abstract. In this paper, we introduce linear operators and obtain their exact norms defined on the function spaces X_λ and Z_λ^6. These operators are constructed from the quartic functional equations and their Pexider versions.

1. Introduction

Let X and Y be complex normed spaces. Given $\lambda \geq 0$, denote by X_λ the linear space of all functions $f\colon X \to Y$ with the condition

$$\|f(x)\| \leq M_f \,\mathrm{e}^{\lambda\|x\|}, \qquad \forall x \in X,$$

where $M_f \geq 0$ is a constant depending on f. It is easy to show that the space X_λ is a normed space if it is equipped with the norm

$$\|f\| := \sup_{x\in X}\{\mathrm{e}^{-\lambda\|x\|}\,\|f(x)\|\}.$$

Let us denote by X_λ^n the linear space of all functions $\varphi\colon \underbrace{X \times \cdots \times X}_{n \text{ times}} \to Y$ for which there exists a constant $M_\varphi \geq 0$ with

$$\|\varphi(x_1, \cdots, x_n)\| \leq M_\varphi \,\mathrm{e}^{\lambda \sum\limits_{i=1}^{n}\|x_i\|}, \qquad \forall x_1, \cdots, x_n \in X.$$

It is easy to see that the space X_λ^n with the norm

$$\|\varphi\| := \sup_{x_1,\cdots,x_n\in X}\{\mathrm{e}^{-\lambda \sum\limits_{i=1}^{n}\|x_i\|}\,\|\varphi(x_1, \cdots, x_n)\|\}$$

is a normed space. We denote by Z_λ^m the normed space $\bigoplus_{i=1}^m X_\lambda = \{(f_1, \cdots, f_m)\colon f_1, \cdots, f_m \in X_\lambda\}$ together with the norm

$$\|(f_1, \cdots, f_m)\| := \max\{\|f_1\|, \cdots, \|f_m\|\}.$$

S. Czerwik and K. Dlutek [**1, 2**] investigated some properties of Pexiderized Cauchy, quadratic and Jensen operators on the function space X_λ. These results

Received May 10, 2011.
2010 *Mathematics Subject Classification.* Primary 47A30, 47B38, 39B52.
Key words and phrases. Pexiderized quartic operator; operator norm; X_λ spaces.

have extended in the paper [**7**]. In fact, M. S. Moslehian, T. Riedel and A. Saadatpour [**7**] studied the Pexiderized generalized Jensen and Pexiderized generalized quadratic operators on the function space X_λ and provided more general results regarding their norms. S. M. Jung [**3**] investigated the norm of the cubic operators on the function spaces Z_λ^5. Recently, A. Najati and A. Rahimi [**8**] introduced Euler-Lagrange type cubic operators and gave their exact norms defined on the function spaces X_λ and Z_λ^5.

S. H. Lee, S. M. Im and I. S. Hwang [**5**] considered the following quartic functional equation

$$\begin{aligned} f(x+y)+f(x-y) = {} & 4f\left(\frac{1}{2}x+y\right)+4f\left(\frac{1}{2}x-y\right) \\ & +24f\left(\frac{1}{2}x\right)-6f(y). \end{aligned} \tag{1.1}$$

They obtained the general solution of equation (1.1) and proved the Hyers-Ulam--Rassias stability of this equation. Y. S. Lee and S. Y. Chung [**6**] introduced the following quartic functional equation, which is equivalent to (1.1),

$$\begin{aligned} f(x+y)+f(x-y) = {} & a^2 f\left(\frac{1}{a}x+y\right)+a^2 f\left(\frac{1}{a}x-y\right) \\ & +2a^2(a^2-1)f\left(\frac{1}{a}x\right)-2(a^2-1)f(y) \end{aligned} \tag{1.2}$$

for fixed integers a with $a \neq 0, \pm 1$. Moreover, D. S. Kang [**4**] introduced the following generalized quartic functional equation

$$\begin{aligned} & f\left(\frac{1}{b}x+\frac{1}{a}y\right)+f\left(\frac{1}{b}x-\frac{1}{a}y\right) \\ & = (ab)^2\left[f\left(\frac{1}{ab}x+\frac{1}{ab}y\right)+f\left(\frac{1}{ab}x-\frac{1}{ab}y\right)\right] \\ & \quad +2a^2(a^2-b^2)f\left(\frac{1}{ab}x\right)-2b^2(a^2-b^2)f\left(\frac{1}{ab}y\right) \end{aligned} \tag{1.3}$$

for fixed integers a, b with $a, b \neq 0, a \pm b \neq 0$.

Next, we will introduce linear operators which are constructed from the quartic and the Pexiderization of the quartic function equations (1.2) and (1.3).

Definition 1.1. The operators $Q_1^P, Q_2^P : Z_\lambda^6 \to X_\lambda^2$ are defined by

$$\begin{aligned} Q_1^P(f_1, \cdots, f_6)(x,y) := {} & f_1(x+y)+f_2(x-y)-m^2 f_3\left(\frac{1}{m}x+y\right)-m^2 f_4\left(\frac{1}{m}x-y\right) \\ & -2m^2(m^2-1)f_5\left(\frac{1}{m}x\right)+2(m^2-1)f_6(y), \end{aligned}$$

$$\begin{aligned}Q_2^P(f_1,\cdots,f_6)(x,y) := {}& f_1\left(\frac{1}{b}x+\frac{1}{a}y\right)+f_2\left(\frac{1}{b}x-\frac{1}{a}y\right)\\&-(ab)^2\left[f_3\left(\frac{1}{ab}x+\frac{1}{ab}y\right)+f_4\left(\frac{1}{ab}x-\frac{1}{ab}y\right)\right]\\&-2a^2(a^2-b^2)f_5\left(\frac{1}{ab}x\right)+2b^2(a^2-b^2)f_6\left(\frac{1}{ab}y\right),\end{aligned}$$

where a, b and m are fixed integers with $a,b\neq 0, a\pm b\neq 0$ and $m\neq 0,\pm 1$.

Definition 1.2. The operators $Q_1, Q_2 : X_\lambda \to X_\lambda^2$ are defined by

$$\begin{aligned}Q_1(f)(x,y) := {}& f(x+y)+f(x-y)-m^2f(\frac{1}{m}x+y)-m^2f\left(\frac{1}{m}x-y\right)\\&-2m^2(m^2-1)f\left(\frac{1}{m}x\right)+2(m^2-1)f(y),\end{aligned}$$

$$\begin{aligned}Q_2(f)(x,y) := {}& f\left(\frac{1}{b}x+\frac{1}{a}y\right)+f\left(\frac{1}{b}x-\frac{1}{a}y\right)\\&-(ab)^2\left[f\left(\frac{1}{ab}x+\frac{1}{ab}y\right)+f\left(\frac{1}{ab}x-\frac{1}{ab}y\right)\right]\\&-2a^2(a^2-b^2)f\left(\frac{1}{ab}x\right)+2b^2(a^2-b^2)f\left(\frac{1}{ab}y\right)\end{aligned}$$

where a,b and m are fixed integers with a, $b\neq 0, a\pm b\neq 0$ and $m\neq 0,\pm 1$.

In this paper, we will give the exact norms of the operators Q_1^P, Q_2^P on the function space Z_λ^6 and norms of the operators Q_1, Q_2 on the function space X_λ.

2. Main results

Throughout this section, a,b and m are fixed integers with $a,b\neq 0, a\pm b\neq 0$, and $m\neq 0,\pm 1$. In the following theorems give us the exact norms of operators Q_1^P, Q_2^P, Q_1 and Q_2.

Theorem 2.1. *The operator $Q_1^P : Z_\lambda^6 \to X_\lambda^2$ is a bounded linear operator with*

$$\|Q_1^P\| = 2m^2(m^2+1). \tag{2.1}$$

Proof. First, we show that $\|Q_1^P\| \le 2m^2(m^2+1)$. Since it holds that

$$\max\left\{\|x+y\|, \|x-y\|, \left\|\frac{1}{m}x+y\right\|, \left\|\frac{1}{m}x-y\right\|, \left\|\frac{1}{m}x\right\|, \|y\|\right\} \le \|x\|+\|y\|$$

for all $x, y \in X$, we obtain

$$\begin{aligned}
&\|Q_1^P(f_1,\cdots,f_6)\| \\
&= \sup_{x,y\in X} \mathrm{e}^{-\lambda(\|x\|+\|y\|)} \left\| f_1(x+y)+f_2(x-y)-m^2 f_3\left(\frac{1}{m}x+y\right)\right. \\
&\quad \left. -m^2 f_4\left(\frac{1}{m}x-y\right) - 2m^2(m^2-1)f_5\left(\frac{1}{m}x\right) + 2(m^2-1)f_6(y)\right\| \\
&\le \sup_{x,y\in X} \mathrm{e}^{-\lambda\|x+y\|}\|f_1(x+y)\| + \sup_{x,y\in X} \mathrm{e}^{-\lambda\|x-y\|}\|f_2(x-y)\| \\
&\quad + m^2 \sup_{x,y\in X} \mathrm{e}^{-\lambda\left\|\frac{1}{m}x+y\right\|}\left\|f_3\left(\frac{1}{m}x+y\right)\right\| \\
&\quad + m^2 \sup_{x,y\in X} \mathrm{e}^{-\lambda\left\|\frac{1}{m}x-y\right\|}\left\|f_4\left(\frac{1}{m}x-y\right)\right\| \\
&\quad + 2m^2(m^2-1) \sup_{x\in X} \mathrm{e}^{-\lambda\left\|\frac{1}{m}x\right\|}\left\|f_5\left(\frac{1}{m}x\right)\right\| + 2(m^2-1)\sup_{y\in X}\mathrm{e}^{-\lambda\|y\|}\|f_6(y)\| \\
&= \|f_1\|+\|f_2\|+m^2\|f_3\|+m^2\|f_4\|+2m^2(m^2-1)\|f_5\|+2(m^2-1)\|f_6\| \\
&\le 2m^2(m^2+1)\max\{\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|,\|f_5\|,\|f_6\|\} \\
&= 2m^2(m^2+1)\|(f_1,\cdots,f_6)\|
\end{aligned}$$

for each $(f_1,\cdots,f_6) \in Z_\lambda^6$. This implies that

$$\|Q_1^P\| \le 2m^2(m^2+1). \tag{2.2}$$

For a fixed $\nu \in Y$ with $\|\nu\| = 1$ and a sequence $\{\xi_n\}_n$ of positive real numbers decreasing to 0, we define

$$f_n(x) = \begin{cases} \mathrm{e}^{2\lambda\xi_n}\,\nu, & \text{if } \|x\| = 2\xi_n, \|x\| = 0 \quad \text{or } \|x\| = \xi_n, \\ -\mathrm{e}^{2\lambda\xi_n}\,\nu, & \text{if } \|x\| = \left|1 \pm \frac{1}{m}\right|\xi_n \quad \text{or } \|x\| = \left|\frac{1}{m}\right|\xi_n, \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

for all $x \in X$. Then we have

$$\mathrm{e}^{-\lambda\|x\|}\,\|f_n(x)\| = \begin{cases} \mathrm{e}^{2\lambda\xi_n}, & \text{if } \|x\| = 0, \\ \mathrm{e}^{\lambda\xi_n}, & \text{if } \|x\| = \xi_n, \\ 1, & \text{if } \|x\| = 2\xi_n, \\ \mathrm{e}^{(2-\left|1+\frac{1}{m}\right|)\lambda\xi_n}, & \text{if } \|x\| = \left|1+\dfrac{1}{m}\right|\xi_n, \\ \mathrm{e}^{(2-\left|1-\frac{1}{m}\right|)\lambda\xi_n}, & \text{if } \|x\| = \left|1-\dfrac{1}{m}\right|\xi_n, \\ \mathrm{e}^{(2-|\frac{1}{m}|)\lambda\xi_n}, & \text{if } \|x\| = \left|\dfrac{1}{m}\right|\xi_n, \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

for all $x \in X$, so that $f_n \in X_\lambda$ for all positive integers n with

$$\|f_n\| = e^{2\lambda\xi_n}. \tag{2.5}$$

Let $u \in X$ be such that $\|u\| = 1$ and take $x, y \in X$ as $x = y = \xi_n u$. Then it follows from (2.3) that

$$\begin{aligned} \|Q_1^P(f_n, \cdots, f_n)\| = {} & \sup_{x,y\in X} \mathrm{e}^{-\lambda(\|x\|+\|y\|)}\,\|f_n(x+y) + f_n(x-y) \\ & - m^2 f_n\left(\frac{1}{m}x + y\right) - m^2 f_n\left(\frac{1}{m}x - y\right) \\ & - 2m^2(m^2-1) f_n\left(\frac{1}{m}x\right) + 2(m^2-1) f_n(y)\| \\ \geq {} & \mathrm{e}^{-2\lambda\xi_n}\,\|2\,\mathrm{e}^{2\lambda\xi_n}\,\nu + 2m^2\,\mathrm{e}^{2\lambda\xi_n}\,\nu + 2(m^4-1)\,\mathrm{e}^{2\lambda\xi_n}\,\nu\| \\ = {} & 2m^2(m^2+1). \end{aligned} \tag{2.6}$$

If we assume that $\|Q_1^P\| < 2m^2(m^2+1)$, then we can choose a positive constant ε with

$$\|Q_1^P(f_n, \cdots, f_n)\| \leq (2m^2(m^2+1) - \varepsilon)\|(f_n, \cdots, f_n)\| \tag{2.7}$$

for all positive integers n. So it follows from (2.5), (2.6) and (2.7) that

$$2m^2(m^2+1) \leq \|Q_1^P(f_n, \cdots, f_n)\| \leq (2m^2(m^2+1) - \varepsilon)\,\mathrm{e}^{2\lambda\xi_n} \tag{2.8}$$

for all positive integers n. Since $\lim_{n\to\infty} \mathrm{e}^{2\lambda\xi_n} = 1$, the right-hand side of (2.8) tends to $2m^2(m^2+1)-\varepsilon$ as $n \to \infty$, whence $2m^2(m^2+1) \leq 2m^2(m^2+1)-\varepsilon$, which leads to a contradiction. Hence we have $\|Q_1^P\| = 2m^2(m^2+1)$. This completes the proof of the theorem. □

Corollary 2.1. *The operator $Q_1\colon X_\lambda \to X_\lambda^2$ is a bounded linear operator with*

$$\|Q_1\| = 2m^2(m^2+1). \tag{2.9}$$

Proof. The result follows from the proof of Theorem 2.1. □

The following corollary is a result of Theorem 2.1 for $m = 2$.

Corollary 2.2. *The Pexiderized quartic operator* $Q_1^P\colon Z_\lambda^6 \to X_\lambda^2$ *given by*

$$\begin{aligned} Q_1^P(f_1, \cdots, f_6)(x,y) := {} & f_1(x+y) + f_2(x-y) - 4f_3\left(\frac{1}{2}x + y\right) - 4f_4\left(\frac{1}{2}x - y\right) \\ & - 24f_5\left(\frac{1}{2}x\right) + 6f_6(y) \end{aligned}$$

is a bounded linear operator with $\|Q_1^P\| = 40$.

The following corollary is a result of Corollary 2.1 for $m = 2$.

Corollary 2.3. *The quartic operator* $Q_1\colon X_\lambda \to X_\lambda^2$ *given by*

$$\begin{aligned} Q_1(f)(x,y) := {} & f(x+y) + f(x-y) - 4f\left(\frac{1}{2}x + y\right) - 4f\left(\frac{1}{2}x - y\right) \\ & - 24f\left(\frac{1}{2}x\right) + 6f(y) \end{aligned}$$

is a bounded linear operator with $\|Q_1\| = 40$.

Theorem 2.2. *The operator* $Q_2^P\colon Z_\lambda^6 \to X_\lambda^2$ *is a bounded linear operator with*

$$\|Q_2^P\| = 2|a^4 - b^4| + 2(ab)^2 + 2. \tag{2.10}$$

Proof. First, we prove that $\|Q_2^P\| \le 2|a^4 - b^4| + 2(ab)^2 + 2$. By the assumption we obtain

$$\max\left\{\left\|\frac{1}{b}x \pm \frac{1}{a}y\right\|, \left\|\frac{1}{ab}x \pm \frac{1}{ab}y\right\|, \left\|\frac{1}{ab}x\right\|, \left\|\frac{1}{ab}y\right\|\right\} \le \|x\| + \|y\|$$

for all $x, y \in X$. Hence we obtain

$$\begin{aligned}
&\|Q_2^P(f_1,\cdots,f_6)\| \\
&= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)} \left\| f_1\left(\frac{1}{b}x+\frac{1}{a}y\right) + f_2\left(\frac{1}{b}x-\frac{1}{a}y\right) \right. \\
&\quad - (ab)^2\left[f_3\left(\frac{1}{ab}x+\frac{1}{ab}y\right) + f_4\left(\frac{1}{ab}x-\frac{1}{ab}y\right)\right] \\
&\quad \left. -2(a^2-b^2)\left[a^2 f_5\left(\frac{1}{ab}x\right) - b^2 f_6\left(\frac{1}{ab}y\right)\right]\right\| \\
&\le \sup_{x,y\in X} e^{-\lambda\|\frac{1}{b}x+\frac{1}{a}y\|}\left\|f_1\left(\frac{1}{b}x+\frac{1}{a}y\right)\right\| \\
&\quad + \sup_{x,y\in X} e^{-\lambda\|\frac{1}{b}x-\frac{1}{a}y\|}\left\|f_2\left(\frac{1}{b}x-\frac{1}{a}y\right)\right\| \\
&\quad + (ab)^2(\sup_{x,y\in X} e^{-\lambda\|\frac{1}{ab}x+\frac{1}{ab}y\|}\left\|f_3\left(\frac{1}{ab}x+\frac{1}{ab}y\right)\right\| \\
&\quad + \sup_{x,y\in X} e^{-\lambda\|\frac{1}{ab}x-\frac{1}{ab}y\|}\left\|f_4\left(\frac{1}{ab}x-\frac{1}{ab}y)\right\|\right) \\
&\quad + 2|a^2-b^2|\left(a^2\sup_{x\in X} e^{-\lambda\|\frac{1}{ab}x\|}\left\|f_5\left(\frac{1}{ab}x\right)\right\| + b^2\sup_{y\in X} e^{-\lambda\|\frac{1}{ab}y\|}\left\|f_6\left(\frac{1}{ab}y\right)\right\|\right) \\
&= \|f_1\|+\|f_2\|+(ab)^2(\|f_3\|+\|f_4\|)+2a^2|a^2-b^2|\|f_5\|+2b^2|a^2-b^2|\|f_6\| \\
&\le (2|a^4-b^4|+2(ab)^2+2)\max\{\|f_1\|,\|f_2\|,\|f_3\|,\|f_4\|,\|f_5\|,\|f_6\|\} \\
&= (2|a^4-b^4|+2(ab)^2+2)\|(f_1,\cdots,f_6)\|
\end{aligned}$$

for each $(f_1,\cdots,f_6) \in Z_\lambda^6$. This implies that

$$\|Q_2^P\| \le 2|a^4-b^4|+2(ab)^2+2. \tag{2.11}$$

Let η be a real number such that

$$\eta \notin \left\{0,\frac{1}{2},1,\pm\frac{1-a}{1-b},\pm\frac{1-a}{1+b},\pm\frac{1-a}{b},\frac{a}{1-b},\frac{a}{1+b}\right\}. \tag{2.12}$$

Let $u \in X$, $\nu \in Y$ be such that $\|u\| = \|\nu\| = 1$ and let $\{\xi_n\}_n$ be a sequence of positive real numbers decreasing to 0. We define

$$f_n(x) = \begin{cases} e^{\lambda(1+|\eta|)\xi_n}\,\nu, & \text{if } x = (\frac{1}{b}\pm\frac{\eta}{a})\xi_n u, \\ -e^{\lambda(1+|\eta|)\xi_n}\,\nu, & \text{if } x = (\frac{1}{ab}\pm\frac{\eta}{ab})\xi_n u, \\ -\frac{|a^2-b^2|}{a^2-b^2} e^{\lambda(1+|\eta|)\xi_n}\,\nu, & \text{if } x = \frac{1}{ab}\xi_n u, \text{ or } x = \frac{\eta}{ab}\xi_n u, \\ 0, & \text{otherwise} \end{cases} \tag{2.13}$$

for all $x \in X$. Then we obtain

$$\text{(2.14)}\quad \mathrm{e}^{-\lambda\|x\|}\,\|f_n(x)\| = \begin{cases} \mathrm{e}^{(1+|\eta|-|\frac{1}{b}+\frac{\eta}{a}|)\lambda\xi_n}, & \text{if } x = \left(\frac{1}{b}+\frac{\eta}{a}\right)\xi_n u, \\ \mathrm{e}^{(1+|\eta|-|\frac{1}{b}-\frac{\eta}{a}|)\lambda\xi_n}, & \text{if } x = \left(\frac{1}{b}-\frac{\eta}{a}\right)\xi_n u, \\ \mathrm{e}^{(1+|\eta|-|\frac{1}{ab}+\frac{\eta}{ab}|)\lambda\xi_n}, & \text{if } x = \left(\frac{1}{ab}+\frac{\eta}{ab}\right)\xi_n u, \\ \mathrm{e}^{(1+|\eta|-|\frac{1}{ab}-\frac{\eta}{ab}|)\lambda\xi_n}, & \text{if } x = \left(\frac{1}{ab}-\frac{\eta}{ab}\right)\xi_n u, \\ \mathrm{e}^{(1+|\eta|-|\frac{1}{ab}|)\lambda\xi_n}, & \text{if } x = \frac{1}{ab}\xi_n u, \\ \mathrm{e}^{(1+|\eta|-|\frac{\eta}{ab}|)\lambda\xi_n}, & \text{if } x = \frac{\eta}{ab}\xi_n u, \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in X$, so that $f_n \in X_\lambda$ for all positive integers n with

$$\text{(2.15)}\quad \begin{aligned} \|f_n\| = \ & \max\{\mathrm{e}^{(1+|\eta|-|\frac{1}{b}+\frac{\eta}{a}|)\lambda\xi_n}, \mathrm{e}^{(1+|\eta|-|\frac{1}{b}-\frac{\eta}{a}|)\lambda\xi_n}, \mathrm{e}^{(1+|\eta|-|\frac{1}{ab}+\frac{\eta}{ab}|)\lambda\xi_n}, \\ & \mathrm{e}^{(1+|\eta|-|\frac{1}{ab}-\frac{\eta}{ab}|)\lambda\xi_n}, \mathrm{e}^{(1+|\eta|-|\frac{1}{ab}|)\lambda\xi_n}, \mathrm{e}^{(1+|\eta|-|\frac{\eta}{ab}|)\lambda\xi_n}\}. \end{aligned}$$

Let $x, y \in X$ be such that $x = \xi_n u$ and $y = \eta\xi_n u$. Then it follows from the definition of f_n that

$$\text{(2.16)}\quad \begin{aligned} \|Q_2^P(f_n, \cdots, f_n)\| &= \sup_{x,y\in X} \mathrm{e}^{-\lambda(\|x\|+\|y\|)} \Big\| f_1\left(\frac{1}{b}x+\frac{1}{a}y\right) + f_2\left(\frac{1}{b}x-\frac{1}{a}y\right) \\ &\quad - (ab)^2\left[f_3\left(\frac{1}{ab}x+\frac{1}{ab}y\right) + f_4\left(\frac{1}{ab}x-\frac{1}{ab}y\right)\right] \\ &\quad -2a^2(a^2-b^2)f_5\left(\frac{1}{ab}x\right) + 2b^2(a^2-b^2)f_6\left(\frac{1}{ab}y\right)\Big\| \\ &\geq \mathrm{e}^{-\lambda(1+|\eta|)\xi_n}\,\|2e^{-\lambda(1+|\eta|)\xi_n} + 2(ab)^2\,\mathrm{e}^{-\lambda(1+|\eta|)\xi_n} \\ &\quad + 2(a^2+b^2)|a^2-b^2|e^{-\lambda(1+|\eta|)\xi_n}\| \\ &= 2|a^4-b^4| + 2(ab)^2 + 2. \end{aligned}$$

If on the contrary $\|Q_2^P\| < 2|a^4-b^4| + 2(ab)^2 + 2$, then there exists $\varepsilon > 0$ such that

$$\text{(2.17)}\quad \|Q_2^P(f_n, \cdots, f_n)\| \leq (2|a^4-b^4| + 2(ab)^2 + 2 - \varepsilon)\|(f_n, \cdots, f_n)\|$$

for all positive integers n. So it follows from (2.16) and (2.17) that

$$\begin{aligned} 2|a^4 - b^4| + 2(ab)^2 + 2 &\le \|Q_2^P(f_n, \cdots, f_n)\| \\ &\le (2|a^4 - b^4| + 2(ab)^2 + 2 - \varepsilon)\|f_n\| \end{aligned} \tag{2.18}$$

for all positive integers n. Since $\lim_{n\to\infty} \xi_n = 0$, it follows from (2.15) that $\lim_{n\to\infty} \|f_n\| = 1$, so the right-hand side of (2.18) tends to $2|a^4 - b^4| + 2(ab)^2 + 2 - \varepsilon$ as $n \to \infty$, whence

$$2|a^4 - b^4| + 2(ab)^2 + 2 \le 2|a^4 - b^4| + 2(ab)^2 + 2 - \varepsilon, \tag{2.19}$$

which is a contradiction. Hence we have $\|Q_2^P\| = 2|a^4 - b^4| + 2(ab)^2 + 2$. This completes the proof of the theorem. □

Corollary 2.4. *The operator $Q_2\colon X_\lambda \to X_\lambda^2$ is a bounded linear operator with*

$$\|Q_2\| = 2|a^4 - b^4| + 2(ab)^2 + 2. \tag{2.20}$$

Proof. The result follows from the proof of Theorem 2.2. □

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