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NON-DEGENERATE SURFACES OF REVOLUTION IN MINKOWSKI SPACE THAT SATISFY THE RELATION $aH + bK = c$

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Abstract. In this work we study spacelike and timelike surfaces of revolution in Minkowski space $\mathbf{E}_1^3$ that satisfy the linear Weingarten relation $aH+bK=c$, where H and K denote the mean curvature and the Gauss curvature of the surface and a, b and c are constants. The classification depends on the causal character of the axis of revolution. We will give a first integral of the equation of the generating curve of the surface and obtain explicit solutions for some particular choices of the constants a, b and c. Also, we completely solve the equation when the axis of revolution of the surface is lightlike.

1. Introduction

Consider the three-dimensional Minkowski space $\mathbf{E}_1^3$, that is, the real vector space $\mathbb{R}^3$ endowed with the Lorentzian metric $\langle,\rangle = (dx)^2 + (dy)^2 - (dz)^2$, where (x, y, z) stand for the usual coordinates of $\mathbb{R}^3$. A vector $v \in \mathbf{E}_1^3$ is said spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$. A submanifold $S \subset \mathbf{E}_1^3$ is said spacelike, timelike or lightlike if the induced metric on S is a Riemannian metric (positive definite), a Lorentzian metric (a metric of index 1) or a degenerate metric, respectively.

An immersion $x : M \to \mathbf{E}_1^3$ of a surface M is said non-degenerate if the induced metric $x^*(\langle,\rangle)$ on M is non-degenerate. There are only two possibilities on non-degenerate surfaces. If the metric $x^*(\langle,\rangle)$ is positive definite, that is, it is a Riemannian metric, the immersion is called *spacelike*. If $x^*(\langle,\rangle)$ is a Lorentzian metric, that is, a metric of index 1, the immersion is called *timelike*. For spacelike surfaces, the tangent planes are spacelike everywhere, and for timelike surfaces, the tangent planes are timelike.

We consider non-degenerate surfaces in $\mathbf{E}_1^3$ that satisfy the relation

$$aH + bK = c, \tag{1}$$

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where H and K are the mean curvature and the Gauss curvature of the surface, respectively, and a, b and c are constants. We then say that the surface is a *linear Weingarten surface* of $\mathbf{E}_1^3$. In general, a Weingarten surface is a surface that satisfies a certain smooth relation $W = W(H, K) = 0$ between H and K. The surfaces that satisfy (1) are the simplest case of W, that is, W is a linear function in its variables. The family of linear Weingarten surfaces include the surfaces with constant mean curvature ($b = 0$) and the surfaces with constant Gauss curvature ($a = 0$). Weingarten surfaces were the interest for geometers in the fifties, such as Chern, Hartman, Hopf and Winter. More recently in Euclidean space, linear Weingarten surfaces were studied for example in [**2, 6, 10, 13, 15**]. When the ambient space is Lorenzt-Minkowski space, we refer, among others, [**1, 3, 4, 5, 8**] with a focus on ruled and helicoidal surfaces.

We study linear Weingarten surfaces that are rotational, that is, invariant by a group of motions of $\mathbf{E}_1^3$ that pointwised fixed a straight-line. In such case, Equation (1) is a second ordinary differential equation that describes the shape of the generating curve of the surface. One can not expect to integrate this equation because even in the trivial cases that $a = 0$ or $b = 0$, this integration is not possible. We will discard the cases that H is constant constant of K which were studied, for example, in [**7, 11, 12**]. The simplest examples of linear Weingarten rotational surfaces are pseudohyperbolic surfaces and pseudospheres which have both H and K constant. Thus these surfaces satisfy (1) for many values a, b and c (see Proposition 2.1).

In this work for rotational surfaces, we will obtain a first integration of the linear Weingarten relation changing (1) into an ordinary differential equation of order 1. The integration will be given in next sections according to the causal character of the axis of revolution: see Theorems 3.1, 4.1, 4.2 and 5.1 below. A first result appears when the axis of revolution is lightlike. In such cases the differential equation that describes the generating curve can be integrated by simple methods.

Theorem 1.1. *Non-degenerate surfaces of revolution in $\mathbf{E}_1^3$ whose axis is lightlike satisfying the linear Weingarten relation* (1) *can be explicitly described by parametrizations.*

This theorem will be proved in Section 5 although the expressions of the parametrizations are cumbersome and it does not deserve to explicit them.

Although we can not completely solve (1) for rotational surfaces in all its generality, we will discuss some particular cases. First, we recover pseudohyperbolic surfaces and pseudospheres for particular choices of the constants of integration. Second, we consider the case that the values of a, b and c in (1) satisfy the relation $a^2 - 4bc\varepsilon = 0$, where throughout this paper $\varepsilon = 1$ indicates that M is spacelike and $\varepsilon = -1$ that M is timelike. In Euclidean ambient space, the sign, of $\Delta := a^2 + 4bc$ makes differences in the study of linear Weingarten surfaces and according to this sign the surfaces are classified into elliptic ($\Delta > 0$), hyperbolic ($\Delta < 0$) and parabolic ($\Delta = 0$). Returning to the Minkowski setting and for parabolic surfaces, we prove the following theorem.

Theorem 1.2. *Let M be a non-degenerate rotational surface in $\mathbf{E}_1^3$ that satisfies* (1). *Assume that $a^2 - 4bc\varepsilon = 0$. After a rigid motion of the ambient space, a parametrization $X(u,v)$ of M is given by:*

1. *If the axis is timelike, $X(u,v) = (u\cos(v), u\sin(v), z(u))$, where*

$$z(u) = \pm\sqrt{\frac{4\varepsilon b^2}{a^2} + \Big(\frac{C}{a} \pm u\Big)^2} + \mu, \quad C = 2\sqrt{b\varepsilon(-b+\lambda)}, \quad \mu, \lambda \in \mathbb{R}. \tag{2}$$

2. *If the axis is spacelike, we have two possibilities:*
   (a) *The parametrization is $X(u,v) = (u, z(u)\sinh(v), z(u)\cosh(v))$, where*

$$z(u) = \pm\frac{C}{a} \pm \sqrt{\frac{4\varepsilon b^2}{a^2} \pm (u\pm\mu)^2}, \quad C = 2\sqrt{b\varepsilon\lambda}, \quad \mu, \lambda \in \mathbb{R}. \tag{3}$$

   (b) *The parametrization is $X(u,v) = (u, z(u)\cosh(v), z(u)\sinh(v))$, where*

$$z(u) = \frac{-C}{a} \pm \sqrt{\frac{4b^2}{a^2} \pm (u\pm\mu)^2}, \quad C = 2\sqrt{b\lambda}, \quad \mu, \lambda \in \mathbb{R}. \tag{4}$$

3. *If the axis is lightlike, $X(u,v) = (-2uv, z(u) + u - uv^2, z(u) - u - uv^2)$, where*

$$z(u) = \frac{1}{48}\Big(\frac{-4ac\lambda + (cC^2 - 2a^2\lambda)u}{\varepsilon c\lambda(2\lambda + cu^2)} + \varepsilon\frac{cC^2 + 2a^2\lambda}{\sqrt{-2c\lambda}}\operatorname{arc\,tanh}\big(\sqrt{-\frac{c}{2\lambda}}\ u\big)\Big) + \mu, \quad \mu, \lambda \in \mathbb{R}. \tag{5}$$

Finally, we study the case that H and K are proportional, that is, $c = 0$ in the relation (1). Even in this case, the integration of (1) is not possible when the axis of revolution is timelike or spacelike. However in this situation, the laborious expressions announced in Theorem 1.1 now are easier.

Theorem 1.3. *The non-degenerate surfaces in $\mathbf{E}_1^3$ with lightlike axis satisfying the relation $H = bK$, $b \neq 0$ that pseudohyperbolic surfaces, pseudospheres are surfaces are locally parametrize as $X(u,v) = (-2uv, z(u)+u-uv^2, z(u)-u-uv^2)$, $u \neq 0$ and $z(u)$ is given by*

$$z(u) = \frac{1}{96\lambda^2}\Big(\varepsilon u^3 + 12b\lambda u \pm (u^2 + 8\varepsilon b\lambda)^{3/2}\Big) + \mu, \tag{6}$$

where $\lambda, \mu \in \mathbb{R}$, $\lambda \neq 0$.

2. Preliminaries

In this section we describe the surfaces of revolution of $\mathbf{E}_1^3$ and we recall the notions of the mean curvature and the Gauss curvature for a non-degenerate surface (for more details see [**9, 14, 16**]). We consider the rigid motions of the ambient space that leave a straight-line pointwised fixed, called the axis of revolution of the surface. Let L be the axis of the surface. Depending on L, there are three types of rotational motions. After an isometry of $\mathbf{E}_1^3$, the expressions of rotational motions with respect to the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbf{E}_1^3$ are as follows:

$$R_v : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cos v & \sin v & 0 \\ -\sin v & \cos v & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$R_v : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh v & \sinh v \\ 0 & \sinh v & \cosh v \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$$R_v : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & -v & v \\ v & 1-\frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Definition 2.1. A surface M in $\mathbf{E}_1^3$ is a surface of revolution (or rotational surface) if M is invariant by some of the above three groups of rigid motions.

In what follows we discard spacelike and timelike planes which are trivially surfaces of revolution and satisfy (1). Our study on the linear Weingarten relation (1) is local. Thus we will work with local parametrizations of rotational surfaces satisfying (1). Given a surface of revolution, there exists a planar curve $\alpha = \alpha(u)$ that generates the surface, that is, M is the set of points given by $\{R_v(\alpha(u)); u \in I, v \in \mathbb{R}\}$. After a rigid motion of $\mathbf{E}_1^3$, we describe the parametrizations of a rotational surface.

1. *The axis L is timelike.* The generating curve is $\alpha(u) = (u, 0, z(u))$ and the parametrization of the surface is

$$X(u,v) = (u\cos(v), u\sin(v), z(u)),\ u \neq 0. \tag{7}$$

2. *The axis L is spacelike.* We have two types of surfaces:
   (a) (Type I) The generating curve is $\alpha(u) = (u, 0, z(u))$ and the surface is

$$X(u,v) = (u, z(u)\sinh(v), z(u)\cosh(v)),\ u \neq 0. \tag{8}$$

   (b) (Type II) The generating curve is $\alpha(u) = (u, z(u), 0)$ and the surface is

$$X(u,v) = (u, z(u)\cosh(v), z(u)\sinh(v)),\ u \neq 0. \tag{9}$$

3. *The axis L is lightlike.* The generating curve is $\alpha(u) = (0, u + z(u), -u + z(u))$ and the parametrization of the surface is

$$X(u,v) = (-2uv, z(u) + u - uv^2, z(u) - u - uv^2),\ u \neq 0. \tag{10}$$

We now define the mean curvature H and the Gauss curvature K of a surface in $\mathbf{E}_1^3$. Let M be an orientable surface and let $x : M \to \mathbf{E}_1^3$ be a spacelike or timelike immersion. Denote by N the Gauss map of M.

Let U, V be vector fields to M and we denote by ∇^0 and ∇ the Levi-Civitta connections of $\mathbf{E}_1^3$ and M respectively. The Gauss formula says

$$\nabla^0_U V = \nabla_U V + \mathrm{II}(U,V),$$

where II is the second fundamental form of the immersion. The Weingarten endomorphism is $A_p : T_pM \to T_pM$ defined as $A_p(U) = -(\nabla^0_U N)^\top_p = (-dN)_p(U)$. Then we have

$$\mathrm{II}(U, V) = -\varepsilon\langle \mathrm{II}(U, V), N\rangle N = -\varepsilon\langle AU, V\rangle N.$$

The mean curvature vector $\vec{H}$ is defined as $\vec{H} = (1/2)\mathrm{trace}(\mathrm{II})$ and the Gauss curvature K as the determinant of II computed in both cases with respect to an orthonormal basis. The mean curvature H is the function given by $\vec{H} = HN$, that is, $H = -\varepsilon\langle \vec{H}, N\rangle$. If $\{e_1, e_2\}$ is an orthonormal basis at each tangent plane, with $\langle e_1, e_1\rangle = 1$, $\langle e_2, e_2\rangle = \varepsilon$, then

$$\begin{aligned}\vec{H} &= \frac{1}{2}(\mathrm{II}(e_1, e_1) + \mathrm{II}(e_2, e_2)) = -\varepsilon\frac{1}{2}(\langle Ae_1, e_1\rangle + \varepsilon\langle Ae_2, e_2\rangle)N\\ &= -\varepsilon\left(\frac{1}{2}\mathrm{trace}(A)\right)N\\ K &= -\varepsilon\det(A).\end{aligned}$$

The expressions of H and K using a parametrization X of the surface are:

$$H = -\varepsilon\frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2}, \qquad K = -\varepsilon\frac{eg - f^2}{EG - F^2}, \tag{11}$$

where $\{E, F, G\}$ and $\{e, f, g\}$ are the coefficients of I and II, respectively. Here the Gauss map N is

$$N = \frac{X_u \times X_v}{\sqrt{\varepsilon(EG - F^2)}}.$$

We recall that

$$W := EG - F^2 = \varepsilon|X_u \times X_v|^2 \begin{cases} \text{is positive if } M \text{ is spacelike,}\\ \text{is negative if } M \text{ is timelike.}\end{cases}$$

In Minkowski ambient space, the role of spheres is played by pseudohyperbolic surfaces and pseudospheres. If $p_0 \in \mathbf{E}^3_1$ and $r > 0$, the pseudohyperbolic surface centered at p_0 with radius $r > 0$ is $\mathbf{H}^{2,1}(r; p_0) = \{p \in \mathbf{E}^3_1; \langle p - p_0, p - p_0\rangle = -r^2\}$ and the pseudosphere centered at p_0 and radius $r > 0$ is $\mathbf{S}^{2,1}(r; p_0) = \{p \in \mathbf{E}^3_1; \langle p - p_0, p - p_0\rangle = r^2\}$. If M is spacelike (resp. timelike) then the Gauss map N is timelike (resp. spacelike), and $N : M \to \mathbf{H}^{2,1}(1; O)$ (resp. $N: M \to \mathbf{S}^{2,1}(1; O)$), where O is the origin of $\mathbf{E}^3_1$. Taking the orientation $N(p) = (p - p_0)/r$, we obtain for these surfaces that $H = \varepsilon/r$ and $K = -\varepsilon/r^2$. Because both H and K are constant, we obtain the following proposition.

Proposition 2.1. *Pseudohyperbolic surfaces and pseudospheres are linear Weingarten surfaces. Exactly, for any $a, b \in \mathbb{R}$ (resp. $b, c \in \mathbb{R}$ or $a, c \in \mathbb{R}$), there exists $c \in \mathbb{R}$ (resp. $a \in \mathbb{R}$ or $b \in \mathbb{R}$) such that the surfaces satisfy the relation $aH + bK = c$.*

3. Rotational surfaces with timelike axis

If the axis is timelike, we know that a parametrization of a rotational surface is given by (7) and the generating curve is $\alpha(u) = (u, 0, z(u))$, $u > 0$. Here the function $W = u^2(1 - z'^2)$. Thus $z'^2 < 1$ if the surface is spacelike, and $z'^2 > 1$ if M is timelike. Using the expressions of H and K in (11), we have

$$H = -\frac{1}{2}\left(\frac{\varepsilon z'}{u\sqrt{\varepsilon(1-z'^2)}} + \frac{z''}{(\varepsilon(1-z'^2))^{3/2}}\right), \qquad K = -\frac{z'z''}{u(1-z'^2)^2}.$$

Then the relation (1) writes as

$$\frac{a}{2}\left(\frac{\varepsilon z'}{u\sqrt{\varepsilon(1-z'^2)}} + \frac{z''}{(\varepsilon(1-z'^2))^{3/2}}\right) + b\frac{z'z''}{u(1-z'^2)^2} = -c.$$

Multiplying both sides by u, we obtain

$$a\left(u\frac{\varepsilon z'}{\sqrt{\varepsilon(1-z'^2)}}\right)' + b\left(\frac{1}{1-z'^2}\right)' = -2cu.$$

A first integral of this equation is

$$\varepsilon\frac{auz'}{\sqrt{\varepsilon(1-z'^2)}} + \frac{b}{1-z'^2} = -cu^2 + \lambda, \tag{12}$$

where λ is a constant of integration. Let

$$\phi = \frac{z'}{\sqrt{\varepsilon(1-z'^2)}}.$$

Then $1+\varepsilon\phi^2 = 1/(1-z'^2)$ and Equation (12) writes as $b\phi^2+au\phi+\varepsilon(b+cu^2-\lambda) = 0$. Hence we can obtain the value of ϕ and thus we get following theorem.

Theorem 3.1. *The linear Weingarten rotational surfaces in* $\mathbf{E}_1^3$ *whose axis is timelike are locally parametrized as* $X(u,v) = (u\cos(v), u\sin(v), z(u))$, *where* $z = z(u)$ *satisfies*

$$\frac{z'}{\sqrt{\varepsilon(1-z'^2)}} = \frac{-au \pm \sqrt{(a^2-4bc\varepsilon)u^2 + 4b\varepsilon(-b+\lambda)}}{2b}, \quad \lambda \in \mathbb{R}. \tag{13}$$

We study this differential equation in three particular cases:

1. Consider $\lambda = b$. From (13) we have

$$\frac{z'}{\sqrt{\varepsilon(1-z'^2)}} = \frac{-a \pm \sqrt{a^2-4bc\varepsilon}}{2b}u = Cu, \qquad C = \frac{-a \pm \sqrt{a^2-4bc\varepsilon}}{2b}.$$

Then a solution of this equation is

$$z(u) = \pm\frac{\sqrt{\varepsilon + C^2u^2}}{C} + \mu, \quad \mu \in \mathbb{R}.$$

From the parametrization (7) of the surface, one concludes that M satisfies the equation $x^2 + y^2 - (z-\mu)^2 = -\varepsilon/C^2$. Letting $p_0 = (0, 0, \mu)$, if $\varepsilon = 1$, the surface M is the pseudohyperbolic surface $\mathbf{H}^{2,1}(1/|C|;p_0)$ and when $\varepsilon=-1$, M is the pseudosphere $\mathbf{S}^{2,1}(1/|C|;p_0)$.

2. Assume $a^2-4bc\varepsilon=0$. Then Equation (13) writes as

$$\frac{z'}{\sqrt{\varepsilon(1-z'^2)}}=\frac{-au\pm C}{2b},\qquad C=2\sqrt{b\varepsilon(-b+\lambda)}.$$

The solution of this equation is given by (2) in Theorem 1.2. See Figure 1.

3. If H and K are proportional, then $c=0$ in (1). Rewriting as $H=bK$, then Equation (13) writes as

$$\frac{z'}{\sqrt{\varepsilon(1-z'^2)}}=\frac{u\pm\sqrt{u^2-4b\varepsilon(b+\lambda)}}{2b},\quad \lambda\in\mathbb{R}.$$

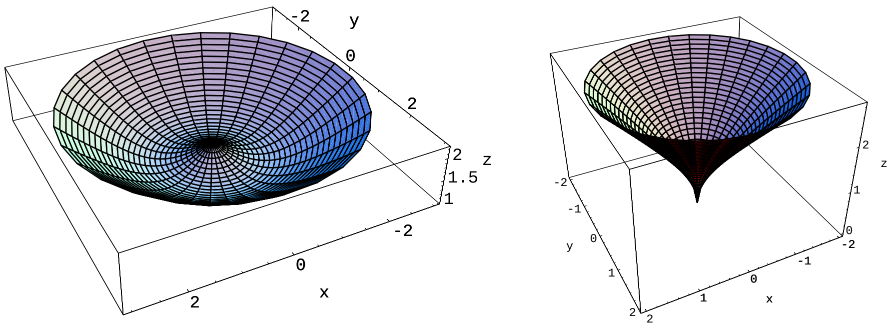

Figure 1. Rotational surfaces with timelike axis where $a^2-4bc\varepsilon=0$. Here $a=2$, $b=\varepsilon$ and $\mu=0$. On the left, the surface is spacelike with $\lambda=2$ and on the right, the surface is timelike with $\lambda=0$.

4. Rotational surfaces with spacelike axis

We distinguish two cases according to the two possible parametrizations.

Type I. Assume that the parametrization is given by (8). The relation (1) writes as

$$\frac{a}{2}\left(\frac{\varepsilon}{z\sqrt{\varepsilon(1-z'^2)}}+\frac{z''}{(\varepsilon(1-z'^2))^{3/2}}\right)+b\frac{z''}{z(1-z'^2)^2}=-c.$$

Multiplying by zz', we obtain a first integral. So there exists a constant of integration $\lambda\in\mathbb{R}$ such that

$$\varepsilon\frac{az}{\sqrt{\varepsilon(1-z'^2)}}+\frac{b}{1-z'^2}=-cz^2+\lambda.$$

Theorem 4.1. *The linear Weingarten rotational surfaces in $\mathbf{E}_1^3$ whose axis is spacelike and of Type I are locally parametrized as*

$$X(u,v)=(u,z(u)\sinh(v),z(u)\cosh(v)),$$

where $z = z(u)$ satisfies

$$\frac{1}{\sqrt{\varepsilon(1-z'^2)}} = \frac{-az \pm \sqrt{(a^2-4bc\varepsilon)z^2+4b\varepsilon\lambda}}{2b}, \quad \lambda \in \mathbb{R}. \tag{14}$$

We solve this differential equation in three particular cases:

1. Consider $\lambda = 0$. From (14) we have
$$\frac{1}{\sqrt{\varepsilon(1-z'^2)}} = \frac{-a \pm \sqrt{a^2-4bc\varepsilon}}{2b} z = Cz, \qquad C = \frac{-a \pm \sqrt{a^2-4bc\varepsilon}}{2b}.$$
The solution of this differential equation is
$$z(u) = \pm\sqrt{\frac{\varepsilon}{C^2} \pm (u \pm C\mu)^2}, \quad \mu \in \mathbb{R}\}.$$
From the parametrization (8) of the surface, one concludes that M satisfies the equation $(x - C\mu)^2 + y^2 - z^2 = -\frac{\varepsilon}{C^2}$. Thus, if we set $p_0 = (\pm C\mu, 0, 0)$, for $\varepsilon = 1$, we obtain that M is the pseudohyperbolic surface $\mathbf{H}^{2,1}(1/|C|; p_0)$ and for $\varepsilon = -1$, M is the pseudosphere $\mathbf{S}^{2,1}(1/|C|; p_0)$.
2. Assume $a^2 - 4bc\varepsilon = 0$. Then
$$\frac{1}{\sqrt{\varepsilon(1-z'^2)}} = \frac{-az \pm C}{2b}, \qquad C = 2\sqrt{b\varepsilon\lambda}.$$
The integration of this equation is the function $z = z(u)$ defined in (3) of Theorem 1.2. See Figure 2.
3. If the Weingarten relation is $H = bK$, then Equation (14) is
$$\frac{1}{\sqrt{\varepsilon(1-z'^2)}} = \frac{z \pm \sqrt{z^2-4b\varepsilon\lambda}}{2b}, \quad \lambda \in \mathbb{R}.$$

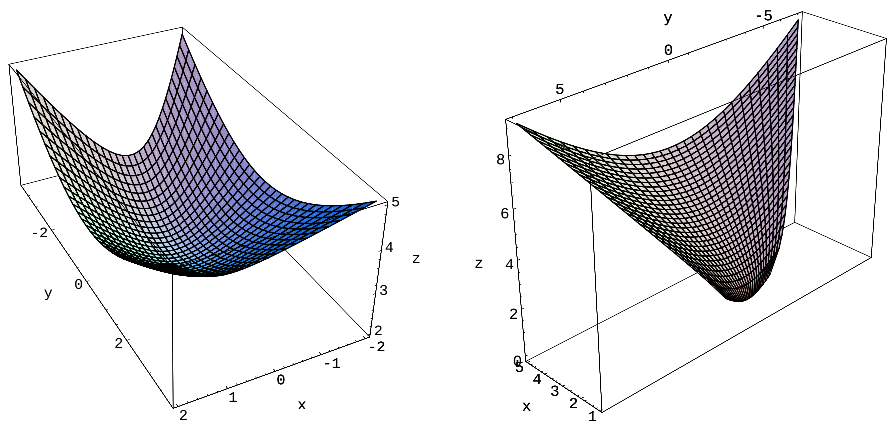

Figure 2. Rotational surfaces with spacelike axis of type I where $a^2 - 4bc\varepsilon = 0$. Here $a = 2$, $b = \varepsilon$, $\lambda = 1$ and $\mu = 0$. On the left, the surface is spacelike and on the right, it is timelike.

Type II. The expression of the parametrization is (9). In this case, the surface is timelike, since $EG-F^2=-z^2(1+z'^2)$. The Weingarten relation (1) is

$$\frac{a}{2}\left(\frac{-1}{z\sqrt{1+z'^2}}+\frac{z''}{(1+z'^2)^{3/2}}\right)-b\frac{z''}{z(1+z'^2)^2}=c$$

and doing a similar reasoning as in the previous case, it follows the existence of a constant of integration $\lambda\in\mathbb{R}$ such that

$$-\frac{az}{\sqrt{1+z'^2}}+\frac{b}{1+z'^2}=cz^2+\lambda.$$

Hence one obtains the following theorem.

Theorem 4.2. *The linear Weingarten rotational surfaces in $\mathbf{E}_1^3$ whose axis is spacelike and of type II are locally parametrized as*

$$X(u,v)=(u,z(u)\cosh(v),z(u)\sinh(v)),$$

where $z=z(u)$ satisfies

$$\frac{1}{\sqrt{1+z'^2}}=\frac{az\pm\sqrt{(a^2+4bc)z^2+4b\lambda}}{2b},\quad \lambda\in\mathbb{R}. \tag{15}$$

As in the previous case, we solve this equation in the next three cases:

1. If $\lambda=0$, then (15) simplifies into
$$\frac{1}{\sqrt{1+z'^2}}=\frac{-a\pm\sqrt{a^2+4bc}}{2b}z=Cz,\qquad C=\frac{a\pm\sqrt{a^2+4bc}}{2b}.$$
The solution of this equation is
$$z(u)=\pm\sqrt{\frac{1}{C^2}-(u\pm C\mu)^2},\quad \mu\in\mathbb{R}\}.$$
This surface is the pseudosphere $\mathbf{S}^{2,1}(1/|C|;p_0)$, with $p_0=(\pm C\mu,0,0)$.
2. If $a^2+4bc=0$, then Equation (15) leads to
$$\frac{1}{\sqrt{1+z'^2}}=\frac{az\pm C}{2b},\quad C=2\sqrt{b\lambda}.$$
The solution of this equation is given by (4) in Theorem 1.2.
3. If H and K are proportional, with $H=bK$, then the function z satisfies
$$\frac{1}{1+z'^2}=\frac{-z\pm\sqrt{z^2-4b\lambda}}{2b},\ \lambda\in\mathbb{R}.$$

5. Rotational surfaces with lightlike axis

Consider the parametrization given in (10). Then $EG-F^2=16u^2z'$ and the relation (1) writes as

$$\frac{a}{2}\left(\frac{1}{2u\sqrt{\varepsilon z'}}-\frac{\varepsilon z''}{4(\varepsilon z')^{3/2}}\right)+b\frac{z''}{8uz'^2}=c.$$

Multiplying by u, we obtain the first integral. There exists a constant of integration $\lambda \in \mathbb{R}$ such that

$$\frac{a}{4}\frac{u}{\sqrt{\varepsilon z'}} - \frac{b}{8z'} = \frac{c}{2}u^2 + \lambda.$$

Theorem 5.1. *The linear Weingarten rotational surfaces in* $\mathbf{E}_1^3$ *whose axis is lightlike are locally parametrized as* $X(u,v) = (-2uv, z(u)+u-uv^2, z(u)-u-uv^2)$, *where* $z = z(u)$ *satisfies*

$$\sqrt{\varepsilon z'} = \frac{a\varepsilon u \pm \sqrt{(a^2-4bc\varepsilon)u^2 - 8b\varepsilon\lambda}}{4\varepsilon(cu^2+2\lambda)}, \quad \lambda \in \mathbb{R}. \tag{16}$$

Differential equation (16) can be solved by direct integrations. However, the variety of cases that appear make that it is not possible to give a general formula for solutions due to the resulting expressions are cumbersome. We give some details proving Theorem 1.1. First, we discard the cases that $\lambda = 0$, $a^2 - 4bc\varepsilon = 0$ and $c = 0$ which will be considered below. Then (16) writes as

$$z' = \varepsilon\left(\frac{au \pm \sqrt{(a^2-4bc)u^2 - 8b\lambda}}{4(cu^2+2\lambda)}\right)^2, \tag{17}$$

where $\lambda c(a^2 - 4bc) \neq 0$. We develop the right hand side in (17) obtaining the next types of integrals (in what follows, by (ct) we indicate a constant):

1. Integrals of type $\displaystyle\int \frac{u^2}{(cu^2+2\lambda)^2}$ and $\displaystyle\int \frac{1}{(cu^2+2\lambda)^2}\,\mathrm{d}u$. Depending on the signs of the constants c and λ, the solutions are functions of type,

$$(ct)\frac{u}{cu^2+2\lambda} + (ct)\arctan((ct)u) \quad \text{or} \quad (ct)\frac{u}{cu^2+2\lambda} + (ct)\,\mathrm{arctanh}((ct)u).$$

2. Integrals of type $\displaystyle\int \frac{u\sqrt{(a^2-4bc)u^2-8b\lambda}}{(cu^2+2\lambda)^2}\,\mathrm{d}u$. According to the signs of the constants again, we have that the solutions are

$$(ct)\frac{u\sqrt{(a^2-4bc)u^2-8b\lambda}}{cu^2+2\lambda} + (ct)\arctan\left(\frac{au}{\sqrt{(a^2-4bc)u^2-8b\lambda}}\right)$$

or

$$(ct)\frac{u\sqrt{(a^2-4bc)u^2-8b\lambda}}{cu^2+2\lambda} + (ct)\,\mathrm{arctanh}\left(\frac{au}{\sqrt{(a^2-4bc)u^2-8b\lambda}}\right).$$

In the following, we show two examples of choices of constants in Equation (17). We consider spacelike surfaces ($\varepsilon = 1$) and take the sign '+' in '$\pm$' that appears before the square root $\sqrt{(a^2-4bc\varepsilon)u^2-8b\lambda}$. For the computations, we have used *Mathematica* to obtain the explicit integrals of (17).

Example 5.1. Let $a = b = c = -\lambda = 1$. Then the solution of (17) is

$$z(u) = \frac{1}{64}\left(\frac{4u}{2-u^2} + \frac{4\sqrt{8-3u^2}}{2-u^2} + 3\sqrt{2}\left(2\,\mathrm{arctanh}\left(\sqrt{4-\frac{3u^2}{2}}\right) - \log\left(\frac{\sqrt{2}+u}{\sqrt{2}-u}\right)\right)\right).$$

Example 5.2. Let $a = -b = c = 1$ and $\lambda = 1/2$. Then the solution of (17) is

$$z(u) = \frac{5(1+u^2)(\arctan(u) + \arctan(\sqrt{4+5u^2}) - u - \sqrt{4+5u^2}}{16(1+u^2)}.$$

As in the previous sections, we distinguish three particular cases. In all of them, the previous integrals are easily solved.

1. If $\lambda = 0$, then $\sqrt{\varepsilon z'} = \dfrac{a \pm \varepsilon\sqrt{a^2-4bc\varepsilon}}{4c}\dfrac{1}{u} := \dfrac{C}{u}$ and $C = \dfrac{a \pm \varepsilon\sqrt{a^2-4bc\varepsilon}}{4c}$. We solve this equation obtaining
$$z(u) = -\frac{\varepsilon C^2}{u} + \mu, \quad \mu \in \mathbb{R}.$$
From the parametrization (10), we see that M satisfies the equation $x^2+y^2-(z-\mu)^2 = -4\varepsilon C^2$. Thus, if $p_0 = (0,0,\mu)$, we have that $M = \mathbf{H}^{2,1}(2|C|;p_0)$ if $\varepsilon = 1$, and $M = \mathbf{S}^{2,1}(2|C|;p_0)$ if $\varepsilon = -1$.
2. Assume $a^2 - 4bc\varepsilon = 0$. Then $\sqrt{\varepsilon z'} = \dfrac{a\varepsilon u \pm C}{4\varepsilon(cu^2+2\lambda)}$ and $C = \sqrt{-8b\varepsilon\lambda}$. We point out that $-8b\varepsilon\lambda > 0$ and that combining with $a^2 - 4bc\varepsilon = 0$, we have $c\lambda \leq 0$. The solution of this equation is (5). See Figure 3.
3. Suppose that H and K are proportional with $H = bK$, $b \in \mathbb{R}$. Then Equation (16) simplifies into
$$\sqrt{\varepsilon z'} = \frac{\varepsilon u \pm \sqrt{u^2+8b\varepsilon\lambda}}{8\varepsilon\lambda}.$$

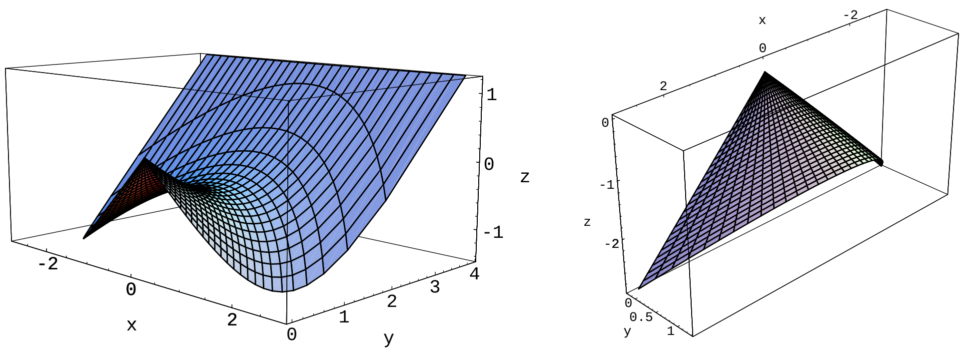

Figure 3. Rotational surfaces with lightlike axis where $a^2 - 4bc\varepsilon = 0$. Here $a = 2$, $b = -\varepsilon$, $\lambda = 1$ and $\mu = 0$. On the left, the surface is spacelike and on the right, the surface is timelike.

The solutions of this differential equation are given in (6) proving Theorem 1.3.

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