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ON SOME NEW INEQUALITIES OF HADAMARD TYPE INVOLVING h-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some inequalities of Hadamard type for $h-$convex functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

In [**8**], Fejér gave a generalization of the inequality (1.1) as follows.

If $f : [a, b] \to \mathbb{R}$ is a convex function and $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\mathrm{d}x. \tag{1.2}$$

For some results which generalize, improve and extend the inequalities (1.1) and (1.2), we refer the reader to the recent papers (see [**6**], [**7**], [**12**], [**15**]).

Definition 1 ([**9**])**.** We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is nonnegative and for all $x, y \in I$ and $\alpha \in (0, 1)$, we have

$$f(\alpha x + (1-\alpha)y) \leq \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}.$$

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The class $Q(I)$ was firstly described in [**9**] by Godunova and Levin. Some further properties of it are given in [**6**], [**13**] and [**14**]. Among the others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 2 ([**2**])**.** Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex (in the second sense) or f belongs to the class K_s^2, if

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\alpha \in [0,1]$.

In 1978, Breckner introduced s-convex functions as a generalization of convex functions [**2**]. Also, in the paper Breckner proved the important fact that the set-valued map is an s-convex only if the associated support function is s-convex function [**3**]. A number of properties and connections with s-convexity in the first sense is discussed in paper [**11**]. Of course, s-convexity means just convexity when $s = 1$. In [**2**] and [**4**], Berstein-Doetsch type results were proved on rationally s-convex functions, moreover, for the s-Hölder property of s-convex functions.

Definition 3 ([**6**])**.** We say that $f : I \to \mathbb{R}$ is a P-function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0,1]$, we have

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y).$$

Definition 4 ([**16**])**.** Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function. We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is h-convex function, or f belongs to the class $SX(h,I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in (0,1)$, we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y). \tag{1.3}$$

If inequality (1.3) is reversed, then f is said to be h-concave, i.e. $f \in SV(h,I)$.

Obviously, if $h(\alpha) = \alpha$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h,I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h,I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0,1)$, then $SX(h,I) \supseteq K_s^2$.

Proposition 1 ([**16**])**.** *Let f and g be similarly ordered functions on I, i.e.*

$$\left(f\left(x\right) - f\left(y\right)\right)\left(g\left(x\right) - g\left(y\right)\right) \geq 0$$

for all $x, y \in I$. If $f \in SX\left(h_1, I\right)$, $g \in SX\left(h_2, I\right)$ and $h(\alpha) + h(1-\alpha) \leq c$ for all $\alpha \in (0,1)$, where $h\left(t\right) = \max\left\{h_1\left(t\right), h_2\left(t\right)\right\}$ and c is a fixed positive number, then the product fg belongs to $SX\left(ch, I\right)$.

For recent results for h-convex functions, we refer the reader to the recent papers (see [**1**], [**5**], [**10**], [**15**]).

In [**7**], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1 ([**7**]). *Suppose that* $f : [0, \infty) \to [0, \infty)$ *is an* s*-convex function in the second sense, where* $s \in (0, 1)$*, and let* $a, b \in [0, \infty)$*,* $a < b$*. If* $f \in L^1([a, b])$*, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4).

In [**6**], Dragomir et. al. proved two inequalities of Hadamard type for classes of Godunova-Levin functions and P-functions.

Theorem 2 ([**6**]). *Let* $f \in Q(I)$*,* $a, b \in I$ *with* $a < b$ *and* $f \in L_1([a, b])$*. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{4}{b-a} \int_a^b f(x)\mathrm{d}x. \tag{1.5}$$

Theorem 3 ([**6**]). *Let* $f \in P(I)$*,* $a, b \in I$ *with* $a < b$ *and* $f \in L_1([a, b])$*. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{2}{b-a} \int_a^b f(x)\mathrm{d}x \le 2\left[f(a) + f(b)\right]. \tag{1.6}$$

In [**15**], Sarikaya et. al. established a new Hadamard-type inequality for h-convex functions.

Theorem 4 ([**15**]). *Let* $f \in SX(h, I)$*,* $a, b \in I$ *with* $a < b$ *and* $f \in L_1([a, b])$*. Then*

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x \le [f(a) + f(b)] \int_0^1 h(\alpha)d\alpha. \tag{1.7}$$

The main purpose of this paper is to establish new inequalities like those given the in above theorems, but now for the class of h-convex functions.

2. Main Results

In the sequel of the paper, I and J are intervals on $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real nonnegative functions defined on J and I, respectively. Throughout this paper, we suppose that $h(\frac{1}{2}) \neq 0$.

Lemma 1. *Let* $f \in SX(h, I)$*. Then for any* x *in* $[a, b]$*,*

$$f(a + b - x) \le (h(\alpha) + h(1 - \alpha))\left[f(a) + f(b)\right] - f(x), \quad \alpha \in [0, 1]. \tag{2.1}$$

If f *is an* h*-concave function, then also the reversed inequality holds.*

Proof. Any x in $[a,b]$ can be represented as $\alpha a+(1-\alpha)b,\ 0\le\alpha\le 1$. Thus, we obtain

$$\begin{aligned}
f(a+b-x) &= f(a+b-\alpha a-(1-\alpha)b) = f((1-\alpha)a+\alpha b)\\
&\le h(1-\alpha)f(a)+h(\alpha)f(b)\\
&= (h(\alpha)+h(1-\alpha))\,[f(a)+f(b)]-[h(\alpha)f(a)+h(1-\alpha)f(b)]\\
&\le (h(\alpha)+h(1-\alpha))\,[f(a)+f(b)]-f(\alpha a+(1-\alpha)b)\\
&= (h(\alpha)+h(1-\alpha))\,[f(a)+f(b)]-f(x).
\end{aligned}$$

□

Theorem 5. *Let $f\in SX(h,I)$, $a,b\in I$ with $a<b$, $f\in L_1([a,b])$ and $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric about $(a+b)/2$. Then*

$$\int_a^b f(x)g(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}\int_a^b\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x)\mathrm{d}x. \tag{2.2}$$

Proof. Since $f\in SX(h,I)$ and g is nonnegative, integrable and symmetric about $(a+b)/2$, we find that

$$\begin{aligned}
\int_a^b f(x)g(x)\mathrm{d}x &= \frac{1}{2}\left[\int_a^b f(x)g(x)\mathrm{d}x+\int_a^b f(a+b-x)g(a+b-x)\mathrm{d}x\right]\\
&= \frac{1}{2}\left[\int_a^b (f(x)+f(a+b-x))\,g(x)\mathrm{d}x\right]\\
&= \frac{1}{2}\int_a^b\left[f\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)+f\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right)\right]g(x)\mathrm{d}x\\
&\le \frac{1}{2}\int_a^b\left\{h\left(\frac{b-x}{b-a}\right)f(a)+h\left(\frac{x-a}{b-a}\right)f(b)\right.\\
&\quad \left.+\,h\left(\frac{x-a}{b-a}\right)f(a)+h\left(\frac{b-x}{b-a}\right)f(b)\right\}g(x)\mathrm{d}x\\
&= \frac{f(a)+f(b)}{2}\int_a^b\left(h\left(\frac{b-x}{b-a}\right)+h\left(\frac{x-a}{b-a}\right)\right)g(x)\mathrm{d}x.
\end{aligned}$$

The proof is complete. □

Remark 1. In Theorem 5, if we choose $h(\alpha)=\alpha$ and $g(x)=1$, then (2.2) reduces the second inequality in (1.1), and if we take $h(\alpha)=\alpha$, then (2.2) reduces the second inequality in (1.2).

Theorem 6. *Let* $f \in SX(h, I)$, $a, b \in I$ *with* $a < b$, $f \in L_1([a, b])$ *and* $g : [a, b] \to \mathbb{R}$ *is nonnegative, integrable and symmetric about* $(a + b)/2$*. Then*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(x)\mathrm{d}x \le \int_a^b f(x)g(x)\mathrm{d}x \\ &\le \frac{f(a)+f(b)}{2}\left(h(\alpha)+h(1-\alpha)\right)\int_a^b g(x)\mathrm{d}x. \end{aligned} \tag{2.3}$$

Proof. Since $f \in SX(h, I)$ and $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $(a + b)/2$, we have

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(x)\mathrm{d}x &= \frac{1}{2h(\frac{1}{2})} \int_a^b f\left(\frac{a+b}{2}\right) g(x)\mathrm{d}x \\ &= \frac{1}{2h(\frac{1}{2})} \int_a^b f\left(\frac{a+b-x+x}{2}\right) g(x)\mathrm{d}x \\ &\le \frac{1}{2h(\frac{1}{2})} \int_a^b h(\frac{1}{2})\left(f(a+b-x)+f(x)\right)g(x)\mathrm{d}x \\ &= \frac{1}{2}\int_a^b f(a+b-x)g(a+b-x)\mathrm{d}x + \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x \\ &= \int_a^b f(x)g(x)\mathrm{d}x. \end{aligned}$$

This proves the first inequality in (2.3). On the other hand, from Lemma 1, we have

$$\begin{aligned} \int_a^b f(x)g(x)\mathrm{d}x &= \frac{1}{2}\int_a^b f(a+b-x)g(a+b-x)\mathrm{d}x + \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x \\ &= \frac{1}{2}\int_a^b f(a+b-x)g(x)\mathrm{d}x + \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x \\ &\le \frac{1}{2}\int_a^b \left[(h(\alpha)+h(1-\alpha))\left[f(a)+f(b)\right]-f(x)\right]g(x)\mathrm{d}x + \frac{1}{2}\int_a^b f(x)g(x)\mathrm{d}x \\ &= \frac{f(a)+f(b)}{2}\left(h(\alpha)+h(1-\alpha)\right)\int_a^b g(x)\mathrm{d}x. \end{aligned}$$

$\square$

Remark 2. In Theorem 6, if we take $h(\alpha) = \alpha$, then the inequality (2.3) reduces inequality to (1.2).

Remark 3. In Theorem 6, if we take $g(x) = 1$, then the inequality (2.3) reduces to the following inequality

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\left(h(\alpha)+h(1-\alpha)\right).$$

Integrating both sides of the above inequality over $[0,1]$ with α, we have the inequality (1.7).

Remark 4. In Theorem 6, if we take $h(\alpha) = \alpha^s,\ s \in (0,1)$ and $g(x) = 1$, then the inequality (2.3) reduces to the following inequality

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\left(\alpha^s+(1-\alpha)^s\right).$$

Integrating both sides of the above inequality over $[0,1]$ with α, we have the inequality (1.4).

Theorem 7. *Let $fg \in SX(\mathrm{ch}, I)$, $a, b \in I$ with $a < b$ and $fg \in L_1([a,b])$. Then*

$$\begin{aligned}\frac{1}{2\,\mathrm{ch}(\frac{1}{2})}(fg)\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b (fg)(x)\mathrm{d}x \\ &\leq c[(fg)(a)+(fg)(b)]\int_0^1 h(\alpha)d\alpha,\end{aligned} \tag{2.4}$$

where c is fixed positive number.

Proof. Since $fg \in SX(\mathrm{ch}, I)$, $\alpha \in (0,1)$, then

$$(fg)\left(\alpha x+(1-\alpha)y\right) \leq \mathrm{ch}\left(\alpha\right)(fg)\left(x\right)+\mathrm{ch}\left(1-\alpha\right)(fg)\left(y\right). \tag{2.5}$$

For $x = ta+(1-t)b$, $y = (1-t)a+tb$ and $\alpha = \frac{1}{2}$ we obtain

$$(fg)(\frac{a+b}{2}) \leq \mathrm{ch}\left(\frac{1}{2}\right)(fg)\left(ta+(1-t)b\right)+\mathrm{ch}\left(\frac{1}{2}\right)(fg)\left((1-t)a+tb\right).$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$(fg)\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\,\mathrm{ch}\left(\frac{1}{2}\right)\int_a^b (fg)(x)\mathrm{d}x,$$

which completes the proof of the first inequality in (2.4).

The proof of the second inequality follows by using (2.5) with $x = a$ and $y = b$ and integrating with respect to α over $[0, 1]$. That is,

$$\frac{1}{b-a}\int_a^b (fg)(x)\mathrm{d}x \le c[(fg)(a) + (fg)(b)]\int_0^1 h(\alpha)d\alpha. \tag{2.6}$$

We obtain inequalities (2.4) from (2.5) and (2.6).The proof is complete. □

Remark 5. In Theorem 7, if we choose $c = 1$ and $g(x) = 1$, then inequalities of (2.4) reduce to inequalities (1.7).

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