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POLYA CONDITIONS FOR MULTIVARIATE BIRKHOFF INTERPOLATION: FROM GENERAL TO RECTANGULAR SETS OF NODES

M. CRAINIC AND N. CRAINIC

Abstract. Polya conditions are certain algebraic inequalities that regular Birkhoff interpolation schemes must satisfy, and they are useful in deciding if a given scheme is regular or not. Here we review the classical Polya condition and then we show how it can be strengthened in the case of rectangular nodes.

1. Introduction

The Birkhoff interpolation problem is one of the most general problems in multivariate polynomial interpolation. For clarity of the exposition, we will restrict here to the bivariate case.

1.1. Uniform Birkhoff interpolations

A Birkhoff interpolation scheme depends on

- A finite set $Z \subset \mathbb{R}^2$ (of "nodes").
- For each $z \in Z$, a set $A(z) \subset \mathbb{N}^2$ (of "derivatives at the node z").
- A lower set $S \subset \mathbb{N}^2$, defining the interpolation space

$$\mathcal{P}_S = \left\{ P \in \mathbb{R}[x, y] : P = \sum_{(i,j)\in S} a_{i,j} x^i y^j \right\}.$$

The fact that L is lower means that if $(i, j) \in S$, then S contains all the pairs of positive integers (i', j') with $i' \le i$, $j' \le j$. It is convenient to denote the set of all such pairs (i', j') by $R(i, j)$. Hence the condition is such that $R(i, j) \subset S$ for all $(i, j) \in S$.

We will make the further simplification that the problem is uniform, i.e. $A(z) = A$ does not depend on z, and we refer to (Z, A, S) as a *uniform Birkhoff (interpolation) scheme*, or simply scheme. When Z is understood from the context or is not fixed, one also talks about the pair (A, S) as a uniform Birkhoff scheme.

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Given a scheme (Z, A, S), the interpolation problem consists of finding polynomials $P \in \mathcal{P}_S$ satisfying the equations

$$\frac{\partial^{\alpha+\beta} P}{\partial x^{\alpha} \partial y^{\beta}}(z) = c_{\alpha,\beta}(z), \tag{1.1}$$

for all $z \in Z$, $(\alpha, \beta) \in A$, where $c_{\alpha,\beta}(z)$ are given arbitrary constants.

1.2. Regularity

One says that (Z, A, S) is *regular* if it has a unique solution $P \in \mathcal{P}_S$ for any choice of the constants $c_{\alpha,\beta}(z)$ in (1.1). If Z is not fixed, we say that a scheme (A, S) is *almost regular* with respect to sets of n nodes if there exists a set Z of n nodes such that (Z, A, S) is regular. It then follows that (Z, A, S) is regular for almost all choices of Z.

Since the interpolation problem is just a (very complicated) system of linear equations with un-known the coefficients of P, its regularity is controlled by the corresponding matrix which we denote by $M(Z, A, S)$ that has $|S|$ columns and $|A||Z|$ rows. Note that the matrix $M(Z, A, S)$ is usually very large and difficult to work with (even notationally). To describe its rows, we introduce the generic row $r(x, y)$, depending on the variables x and y, which has as entries the monomials

$$x^u y^v \qquad \text{with } (u, v) \in S,$$

ordered lexicographically. For $(\alpha, \beta) \in A$, we take the (α, β)-derivatives of these monomials:

$$\frac{u!}{(u-\alpha)!} \frac{v}{(v-\beta)!} x^{u-\alpha} y^{v-\beta} \qquad \text{with } (u, v) \in S.$$

They form a new row, denoted $\partial_x^{\alpha} \partial_y^{\beta} r(x, y)$. Varying (α, β) in A and (x, y) in Z, we obtain in total $|Z||A|$ rows of length $|S|$. Together, they form the matrix $M(Z, A, S)$.

The regularity of (Z, A, S) clearly forces the equation $|S| = |A||Z|$, i.e. in the terminology of [**9**], the scheme must be normal. In this case, the regularity is controlled by the determinant of $M(Z, A, S)$ which we denote by $D(Z, A, S)$. Viewing the points in Z as variables, $D(Z, A, S)$ is a polynomial function on the coordinates of these points and the almost regularity of (A, S) is equivalent to the non-vanishing of this function.

1.3. Polya conditions

We have already mentioned that the immediate consequence of the regularity of (Z, S, A) is the normality of the scheme. Polya type conditions [**9**] are further algebraic conditions that are forced by regularity. As we shall explain below, they arise by looking at the determinant of the problem and realizing that if $D(Z, A, S)$ is non-zero, then the matrix $M(Z, A, S)$ cannot have "too many" vanishing entries (Lemma 2.1 below). The resulting Polya conditions are very useful in detecting regular schemes; see [**9**] and also our Example 2.1 .

1.4. Rectangular sets of nodes

Although interesting results are available in the multivariate case (see e.g. [**8, 9**] and the references therein) in comparison with the univariate case however, much still has to be understood. For instance, it appears that the shape of Z strongly influences the regularity of the scheme, and even less is known about schemes where Z has a special shape (in contrast, for generic Z's, very useful criteria can be found in [**9**]). The simplest particular shape is the rectangular one. We say that Z is (p, q)-rectangular (or just rectangular when we do not want to emphasize the integers p and q) if it can be represented as

$$Z = \{(x_i, y_j) : 0 \leq i \leq p, 0 \leq j \leq q\},$$

where the x_i's and the y_j's are real number with $x_a \neq x_b$ and $y_a \neq y_b$ for $a \neq b$. Similar to the discussion above, one says that (A, S) is *almost regular with respect to (p, q)-rectangular sets of nodes* if there exists a set Z such that (Z, A, S) is regular.

1.5. This paper

The study of uniform Birkhoff schemes with rectangular sets of nodes has been initiated in [**2**]. The present work belongs to this program. Here we study Polya-type conditions, proving the Polya inequalities which were already announced in **loc. cit.** (Theorem 3.1 below). We emphasize that, in contrast to the regularity criteria found in [**4, 5, 6**] (which can be used to prove regularity), the role of the Polya conditions is different: they can be used to rule out non-regular schemes. I.e., in practice, for a given scheme, these are the first conditions one has to check; if they are satisfied, then one can move on and apply the other regularity criteria (see Example 3.2 and 3.3).

2. General sets of nodes

In this section we recall and we re-interpret the standard Polya conditions [**9**]; we show that they arise because of a very simple reason: a non-zero determinant cannot have "too many zeros". More precisely, one has the following simple observation.

Lemma 2.1. *Assume that $M \in M_n(\mathbb{R})$ has a rows and b columns with the property that ab elements situated at the intersection of these rows and columns are all zero. If $\det(M) \neq 0$, then $a + b \leq n$.*

Remark 2.1. By removing the intersection elements (ab zeros from the statement) from a rows, one obtains a matrix with a rows and $n - b$ columns, denoted M_1. Similarly, doing the same along the columns, one gets a matrix with $n - a$ rows and b columns, denoted M_2. In the limit case of the lemma (i.e. when $a + b = n$), then both M_1 and M_2 are square matrices, and a simple form of the Laplace formula tells us that $\det(M) = \det(M_1)\det(M_2)$ (up to a sign).

We apply this lemma to the matrix $M(Z, A, S)$ associated with an uniform Birkhoff interpolation scheme. The extreme (and obvious) cases of this lemma show that if (A, S) is almost regular, then A must be contained in S and must also contain the origin. Staying in the context of generic Z's, one immediately obtains the known Polya condition [**9**] which appears as the most general necessary condition for the almost regularity of pairs (A, S) that one can obtain "by counting zeros"

Corollary 2.1 ([**9**])**.** *If the pair (A, S) is almost regular with respect to sets of n nodes, then for any lower set $L \subset S$, $n|L \cap A| \geq |L|$.*

Proof. Indeed, the monomials in $M(Z, A, S)$ which sit in the columns corresponding to L become zeros when taking derivatives coming from $A \setminus L$. These derivatives define $n|A \setminus L|$ rows, hence the previous lemma implies that $|L| + n|A \setminus L| \leq |S|$. Since $|S| = n|A|$, and $|A \setminus L| = |A| - |A \cap L|$, the result follows. $\square$

Also, the limit case described by Remark 2.1 immediately implies

Corollary 2.2 ([**9**])**.** *If (Z, A, S) is a regular scheme and $L \subset S$ is a lower set satisfying $|L| = n|A \cap L|$ (where $n = |Z|$), then $(Z, A \cap L, L)$ must be regular, too.*

Remark 2.2. This corollary applies to the univariate case as well. Writing $A = \{a_0, a_1, \ldots, a_s\}$ with $a_0 < a_1 < \ldots < a_s$, the Polya conditions become:

$$a_i \leq n \cdot i, \quad \forall\ 0 \leq i \leq s.$$

Moreover, this condition actually insures regularity for almost all sets of nodes Z. More precisely, given (A, S) with $|S| = n|A|$, (A, S) is almost regular if and only if it satisfies the Polya conditions. Moreover, if $n = 2$, then the Polya conditions are sufficient also for regularity. For details, see [**7**].

Example 2.1. Given $A = \{(0, 0), (1, 0)\}$ and a lower set S, then the regularity of (A, S) implies that $|S| = 2n$ and that S contains at most n elements on the OY axis. This follows from the Polya condition applied to $L \cap OY$. Conversely, using the regularity criteria based on shifts of [**9**], one can show that these two conditions do imply almost regularity. To see explicit examples, choose $n = 3$. Then we could take S as shown in Figure 1 (in total, there are seven possibilities).

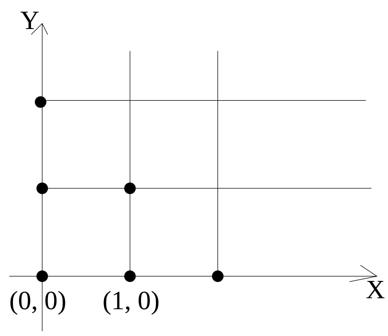

Figure 1.

Denoting by (x_i, y_i) the coordinates of the points of Z, $i \in \{1,2,3\}$, $M(Z, A, S)$ is the six by six matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2, & y_1 & x_1y_1 & y_1^2 \\ 1 & x_2 & x_2^2, & y_3 & x_2y_3 & y_2^2 \\ 1 & x_3 & x_3^2, & y_3 & x_3y_3 & y_3^2 \\ 0 & 1 & 2x_1 & 0 & y_1 & 0 \\ 0 & 1 & 2x_2 & 0 & y_2 & 0 \\ 0 & 1 & 2x_3 & 0 & y_3 & 0 \end{pmatrix}$$

(the first three rows contain monomials supported by S, i.e of type $(1, x, x^2, y, xy, y^2)$; the last three rows contain the derivatives of these monomials with respect to x, i.e. the $(1,0)$-derivative where we used $(1,0) \in A$). One can also compute the resulting determinant explicitly and obtain, up to a sign,

$$2(y_1 - y_2)(y_1 - y_3)(y_2 - y_3)(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3).$$

Example 2.2. Let us look at schemes with

$$A = \{(0,0), (1,1)\}, \qquad |Z| = 6.$$

Then, there exist only two schemes (A, S) which are almost regular with respect to sets of six nodes, namely the ones with $S = R(2,3)$ or $S = R(3,2)$.

Proof. Assume first that (A, S) is almost regular with respect to sets of six nodes. Let a be the maximal integer with the property that $(a, 1) \in S$, let b be the maximal integer with the property that $(1, b) \in S$ and let L be the set of the elements of S on the coordinate axes. Since S is lower and $(1,1)$ must be in S, it follows that $a, b \geq 1$ and $|L| \geq a + b + 1$. But Corollary 2.1 forces $|L| \leq 6$, hence $a + b \leq 5$. On the other hand, $S \setminus L$ is contained on the rectangle with vertexes $(1,1)$, $(a,1)$, $(1,b)$ and (a,b), hence $12 - |L| = |S \setminus L| \leq ab$. Since $|L| \leq 6$, we must have $ab \geq 6$. But this together with $a + b \leq 5$ can only hold when (a,b) is either $(2,3)$ or $(3,2)$. Moreover, in both cases equality holds, hence all the inclusions used on deriving those inequalities must become equalities. In particular, L must contain $a+1$ elements on OX, $b+1$ elements on OY and $S \setminus L$ must coincide with the rectangle mentioned above. This forces $S = R(a,b)$ in each of the cases. To prove that $S = R(a,b)$ for $\{a,b\} = \{2,3\}$ do induce almost regular schemes, one can either proceed directly or use the regularity criteria based on shifts of [**9**]. □

3. Rectangular sets of nodes

In this section we look at Polya conditions on schemes with rectangular sets of nodes.

First, we have to discuss the boundary points of a lower set L. Given L, a point $(u,v) \in L$ is called a *boundary point* if $(u+1, v+1) \notin L$. We denote by ∂L the set of such points. We consider the following two possibilities:

(i) $(u, v+1) \in L$;
(ii) $(u+1, v) \in L$.

We denote by (see Figure 2):

- $\partial_e L$ the set of boundary points (u, v) for which any two conditions above are not satisfied ("exterior boundary points"),
- $\partial_i L$ the set of those which satisfy both conditions ("interior boundary points"),
- $\partial_x L$ the set of those for which only (ii) holds true ("x-direction boundary points"),
- $\partial_y L$ the set of those for which only (i) holds true ("y-direction boundary points").

These four sets form a partition of the boundary $\partial(L)$ of L.

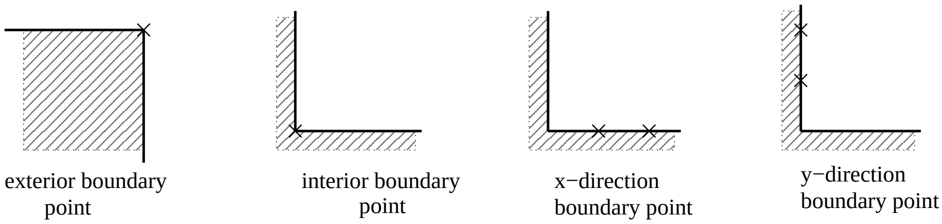

Figure 2. Boundary points

Example 3.1. If L is as shown in Figure 3, it has three exterior boundary points, two interior ones, three which are x-direction and two which are y-direction-labelled in the picture by the letters e, i, x and y, respectively.

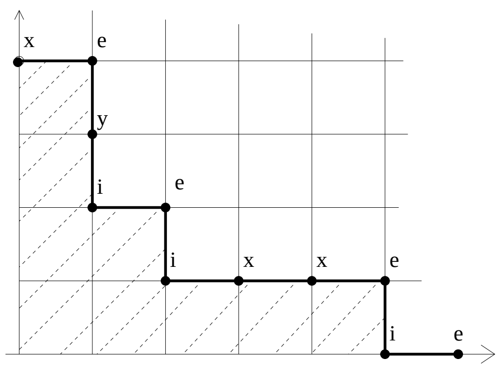

Figure 3.

Note that, in general, the number of exterior boundary points equals the number of interior boundary points plus one. Also, denoting

$$L_x = L \cap OX, \ L_y = L \cap OY,$$

one has $|\partial_x L| = |L_x| - |\partial_e L|$ and $|\partial_y L| = |L_y| - |\partial_e L|$. In particular, for the total number of boundary points,

$$|\partial L| = |L_x| + |L_y| - 1.$$

Finally, the set $\partial_e L$ of exterior boundary points determines L uniquely since

$$L = \bigcup_{(u,v)\in\partial_e L} R(u,v).$$

This should be clear from Figure 4 where

$$\partial_e L = \{(a_1, b_k), (a_2, b_{k-1}), \ldots, (a_k, b_1)\}.$$

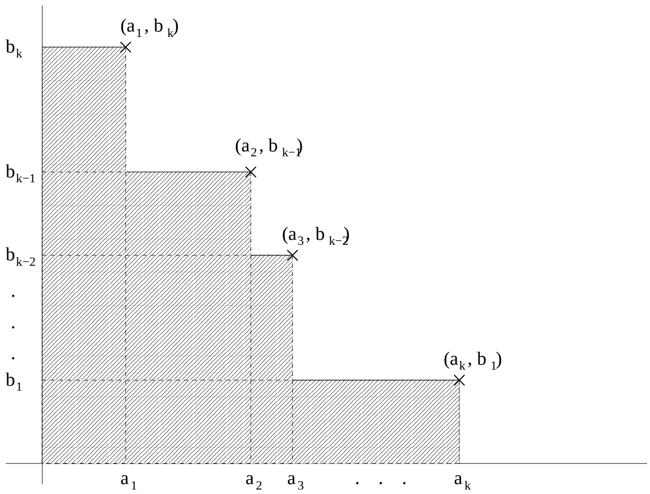

Figure 4. Exterior boundary points

With these we have:

Theorem 3.1. *If (A, S) is almost regular with respect to (p, q)-rectangular sets of nodes, $n = (p + 1)(q + 1)$, then, for any lower subset $L \subset S$,*

$$n|A \cap L| \geq |L| + pq|A \cap \partial L| + (p + q)|A \cap \partial_e L| + p|A \cap \partial_y L| + q|A \cap \partial_x L|.$$

The idea of the proof is to start with the matrix $M(Z, A, S)$ and, depending on the lower set L, perform certain elementary transformations along the rows or columns of the matrix, so that a large number of its entries vanish and then apply Lemma 2.1. But before we give the proof, we illustrate how the Theorem can be used.

Example 3.2. We emphasize that these inequalities form a collection of conditions on the scheme (A, S), one condition for each lower set L inside S. It is not always clear what the best choice of L is. For an explicit example, consider $p = 2$,

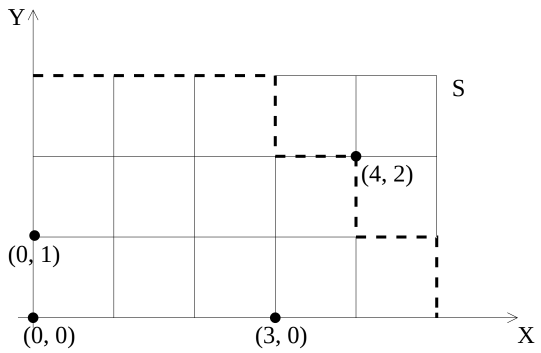

Figure 5. Example 3.2

$q = 1$ (so that the total number of nodes is $n = 6$), the lower set $S = R(5, 3)$ and the set of orders of derivatives

$$A = \{(0, 0), (0, 1), (3, 0), (4, 2)\},$$

see Figure 5.

The Polya inequalities become

$$6|A \cap L| \geq |L| + 2|A \cap \partial L| + 3|A \cap \partial_e L| + 2|A \cap \partial_y L| + |A \cap \partial_x L|.$$

Choose first $L = S_x$. Then

$$A \cap L = A \cap \partial L = A \cap \partial_x L = \{(0, 0), (0, 3)\}, A \cap \partial_e L = A \cap \partial_y L = \emptyset$$

and the inequality becomes $6 \cdot 2 \geq 6 + 2 \cdot 2 + 3 \cdot 0 + 2 \cdot 0 + 1 \cdot 1$, i.e. $12 \geq 11$, which is true, hence no conclusion can be drawn. Let us now choose L consisting of the only first four points on the Ox axis. Then the inequality becomes $6 \cdot 2 \geq 4 + 2 \cdot 2 + 3 \cdot 1 + 2 \cdot 0 + 1 \cdot 1$, i.e. $12 \geq 12$. Hence, again, no conclusion can be drawn. Finally, we choose L to be set drawn in Figure 5 by dotted lines. In this case the inequality becomes

$$6 \cdot 4 \geq 20 + 2 \cdot 1 + 3 \cdot 1 + 2 \cdot 0 + 1 \cdot 0,$$

i.e. $24 \geq 25$, which is false. In conclusion, (A, S) is not almost regular with respect to $(2, 1)$-rectangular sets of nodes.

Roughly speaking, the reason for this scheme not being regular comes from the fact that $(4, 2) \in A$ is "too large". To avoid the previous type of argument, one may replace $(4, 2)$ by one of its smaller neighbors, i.e. by $(3, 2)$ or $(4, 1)$. Then one cannot find any L for which the Polya condition is false. Actually, as an immediate application of the criteria in [**4**], one obtains that the scheme is indeed almost regular.

Proof of the Theorem 3.1. From the general description of the matrix $M(Z,A,S)$ (see the introduction) we see that its rows are indexed by the pairs $(i, j) \in R(p, q)$

(which give the nodes (x_i, y_j)) and elements $(\alpha, \beta) \in A$, and consist of the derivatives of order (α, β) of the monomials in $\mathcal{P}_S$, evaluated at (x_i, y_j):

$$\partial_x^\alpha \partial_y^\beta r(x_i, y_j) : \frac{u!}{(u-\alpha)!} \frac{v}{(v-\beta)!} x^{u-\alpha} y^{v-\beta} \quad \text{with } (u, v) \in S.$$

Next, we consider the columns corresponding to L and look for those rows which intersected with these columns produce zeros (possibly after some elementary operations). We distinguish four types of derivatives depending on the position of (α, β) relative to A.

(i) $(\alpha, \beta) \in A \setminus L$. Clearly, each of the rows $\partial_x^\alpha \partial_y^\beta r(x_i, y_j)$ is of the type we are looking for, for each $(x_i, y_j) \in Z$. This produces $n|A \setminus L|$ rows of type we are looking for.

(ii) $(\alpha, \beta) \in A \cap \partial_e L$. If we subtract one of these rows (say the one corresponding to (x_0, y_0)) from all others, we obtain $n - 1$ new rows that intersected with the columns corresponding to L give zeros. In total, $(n-1)|A \cap \partial_e L|$ new rows.

(iii) $(\alpha, \beta) \in A \cap \partial_x L$. Looking at the corresponding intersections of a row defined by such a derivative (and by a pair $(i, j) \in R(p, q)$) with the columns defined by L, the only possible non-zero elements are powers of x.

Then, for each x_i, we subtract the row corresponding to (x_i, y_0) from the rows corresponding to (x_i, y_j), $j \geq 1$ to get rid of the non-zero elements containing x. This produces q new rows which do have zero at the intersection with the L-columns. We do this for each $0 \leq i \leq p$ and for each derivative $(\alpha, \beta) \in A \cap \partial_x L$, hence we end up with $(p+1)q|A \cap \partial_x L|$ new rows of the type we are looking for.

(iv) $(\alpha, \beta) \in A \cap \partial_y L$ is similar to (iii) and produces $p(q+1)|A \cap \partial_y L|$ rows.

(v) $(\alpha, \beta) \in A \cap \partial_i L$. We basically apply twice the subtraction that we did in the previous two cases. Looking at the corresponding intersections of a row defined by such a derivative (and by a pair $(i, j) \in R(p, q)$) with the columns defined by L, the only possible non-zero elements are powers of x or powers of y (evaluated at (x_i, y_j)). Then, for each x_i, we subtract the row corresponding to (x_i, y_0) from the rows corresponding to (x_i, y_j), $j \geq 1$ to get rid of the non-zero elements containing x, and then we do the same with to get rid of y's. The outcome consists of $pq|A \cap \partial_i L|$ new rows of the type we are looking for.

Adding up and using the Lemma 2.1, we get

$$\begin{aligned} |L| + n|A \setminus L| + (n-1)|A \cap \partial_e L| + (p+1)q|A \cap \partial_x L| \\ + p(q+1)|A \cap \partial_y L| + pq|A \cap \partial_i L| \leq n|A|, \end{aligned}$$

and since $\partial L = \partial_e L \cup \partial_x L \cup \partial_y L \cup \partial_i L$, this can easily be transformed into the inequality in the statement. □

Example 3.3. The example below explains [**2**, Example 2.7]. To compare with the generic case, let us take A as in Example 2.1 above and use the (stronger) Polya condition applied to the same $L = S_x$. Then we obtain $|S_x| \leq 2(p+1)$. Similarly, for $L = S_y$, we obtain $|S_y| \leq (q+1)$. On the other hand, since S is lower, $|S| \leq |S_x||S_y|$. Combining these, and the fact that $|S| = 2n$, we deduce that the regularity of (A, S) with respect to (p, q)-rectangular sets of nodes can only happen when $S = R(2p+1, q)$ (and one can prove that, indeed, $(A, R(2p+1, q))$ is almost regular).

On the other hand, taking A as in Example 2.2 and $p = 2$, $q = 1$ (so that the total number of nodes is indeed six), the same argument as above shows that there is no S for which (A, S) is almost regular with respect to $(2, 1)$-rectangular sets of nodes.

Other applications are presented in [**2**].

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M. Crainic, Department of Mathematics, Utrecht University, 3508 TA Utrecht, The Netherlands, *e-mail*: `m.crainic@uu.nl`

N. Crainic, 1 Decembrie 1918 University, Alba Iulia, Romania,
e-mail: `crainic@math.uu.nl`