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EXISTENCE AND POSNER'S THEOREM FOR α-DERIVATIONS IN PRIME NEAR-RINGS

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Abstract. In this paper we define α-derivation for near-rings and extend some results for derivations of prime rings or near-rings to a more general case for α-derivations of prime near-rings. To initiate the study of the theory, the existence of such derivation is shown by an example. It is shown that if d is an α-derivation of a prime near-ring N such that d commutes with α, then $d^2 = 0$ implies $d = 0$. Also a Posner-type result for the composition of α-derivations is obtained.

1. Introduction

A left (right) near-ring is a set N with two operations $+$ and $\cdot$ such that $(N, +)$ is a group and $(N, \cdot)$ is a semigroup satisfying the left distributive law: $x(y + z) = xy + xz$ (right distributive law: $(x + y)z = xz + yz$) for all $x, y, z \in N$. Zero symmetric left (right) near-rings satisfy $x \cdot 0 = 0 \cdot x = 0$ for all $x \in N$. Throughout this note, unless otherwise specified, N will stand for a zero symmetric left near-ring. Let α be an automorphism of N. An additive endomorphism $d : N \to N$ is said to be an α-derivation if $d(xy) = \alpha(x)d(y) + d(x)y$ for all $x, y \in N$. According to [**1**], a near-ring N is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$. As there were only a few papers on derivations of near-rings and none (to the knowledge of the author) on α-derivations of near-rings, it seems that the present paper would initiate and develop the study of the subject in this direction. On the way to this aim, we construct an example of this type of derivation that would make sense of the theory we are dealing with. Furthermore, an analogous version of a well-known result of Posner for the composition of derivations of rings is obtained for the case of near-rings. Also, some properties for α-derivations of near-rings are given.

In the next section we show that such derivations on near-rings do exist.

2. Examples

The following consideration provides a class of near-rings on which we can define an α-derivation. Let M be a near-ring which is not a ring such that $(M, +)$ is abelian. Let R be a commutative ring. Take N to be the direct sum of M and

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R. So, we have the near-ring $N = M \bigoplus R$. Observe that N is not a ring, R is an ideal of N and its elements commute with all elements of N.
Let α be a non-trivial automorphism of N and take $a \in R$. Define $d_a^\alpha : N \longrightarrow N$ by $d_a^\alpha(x) = \alpha(x)a - xa$. Then d_a^α is an α-derivation on N. This can be verified as follows. Let $x, y \in N$. Then

$$\begin{aligned} d_a^\alpha(xy) &= \alpha(xy)a - xya \\ &= \alpha(xy)a - \alpha(x)ya + \alpha(x)ya - xya \\ &= \alpha(xy)a - \alpha(x)ya + ya\alpha(x) - yax \\ &= \alpha(xy)a - \alpha(x)ya + y\alpha(x)a - yxa \\ &= \alpha(x)\alpha(y)a - \alpha(x)ya + y[\alpha(x)a - xa] \\ &= \alpha(x)[\alpha(y)a - ya] + [\alpha(x)a - xa]y \\ &= \alpha(x)d_a^\alpha(y) + d_a^\alpha(x)y. \end{aligned}$$

This shows that

$$d_a^\alpha(xy) = \alpha(x)d_a^\alpha(y) + d_a^\alpha(x)y, \qquad \text{for all } x, y \in N.$$

Hence d_a^α is an α-derivation on N.

3. Results

Our main goal in this section is to prove the following result which deals with composition of α-derivations on prime near-rings. In fact, it is an analog of a well-known theorem of Posner [**5**] for the case of α-derivations on prime near-rings.

Theorem. *Let d_1 be an α-derivation and d_2 be a β-derivation on a 2-torsion-free prime near-ring N such that α, β commute with d_1 and with d_2. Then $d_1 d_2$ is an $\alpha\beta$-derivation if and only if $d_1 = 0$ or $d_2 = 0$.*

Before we proceed to prove the theorem, we derive some properties for α-derivations on prime near-rings.

Although the underlying group of the near-ring N is not necessarily commutative, the first result gives an equivalent definition of α-derivation on N which involves a sort of commutativity on N.

Proposition 1. *Let d be an additive endomorphism of a near-ring N. Then d is an α-derivation if and only if $d(xy) = d(x)y + \alpha(x)d(y)$ for all $x, y \in N$.*

Proof. By definition, if d is an α-derivation then for all $x, y \in N$, $d(xy) = \alpha(x)d(y) + d(x)y$. Then

$$\begin{aligned} d(x(y+y)) &= \alpha(x)d(y+y) + d(x)(y+y) \\ &= 2\alpha(x)d(y) + 2d(x)y, \end{aligned}$$

and

$$d(xy + xy) = 2d(xy) = 2(\alpha(x)d(y) + d(x)y),$$

so that $\alpha(x)d(y) + d(x)y = d(x)y + \alpha(x)d(y)$. The converse is easy. $\square$

Now, we show that the near-ring N satisfies some partial distributive laws which will be used in the sequel.

Proposition 2. *Let d be an α-derivation on a near-ring N. Then for all $x, y, z \in N$,*

(i) $(\alpha(x)d(y) + d(x)y)z = \alpha(x)d(y)z + d(x)yz;$

(ii) $(d(x)y + \alpha(x)d(y))z = d(x)yz + \alpha(x)d(y)z.$

Proof. (i) Let $x, y, z \in N$. Then

$$\begin{aligned} d(x(yz)) &= \alpha(x)d(yz) + d(x)(yz) \\ &= \alpha(x)(\alpha(y)d(z) + d(y)z) + d(x)(yz) \\ &= (\alpha(x)\alpha(y))d(z) + \alpha(x)d(y)z + d(x)(yz) \\ &= \alpha(xy)d(z) + \alpha(x)d(y)z + (d(x)y)z. \end{aligned} \tag{1}$$

Also,

$$\begin{aligned} d((xy)z) &= \alpha(xy)d(z) + d(xy)z \\ &= \alpha(xy)d(z) + (\alpha(x)d(y) + d(x)y)z. \end{aligned} \tag{2}$$

From (1) and (2), we get

$$(\alpha(x)d(y) + d(x)y)z = \alpha(x)d(y)z + d(x)yz.$$

(ii) It follows similarly by Proposition 1. □

Remark 3. A similar distributivity result can be obtained for the case of right near-rings.

Proposition 4. *Let d be an α-derivation of a prime near-ring N and $a \in N$ such that $ad(x) = 0$ (or $d(x)a = 0$) for all $x \in N$. Then $a = 0$ or $d = 0$.*

Proof. For all $x, y \in N$,

$$\begin{aligned} 0 = ad(xy) &= a(\alpha(x)d(y) + d(x)y) \\ &= a\alpha(x)d(y) + ad(x)y \\ &= a\alpha(x)d(y) + 0 \\ &= a\alpha(x)d(y). \end{aligned}$$

Thus $aNd(y) = 0$. Since N is prime, we get $a = 0$ or $d = 0$. To prove the case when $d(x)a = 0$, we need Proposition 2. So if $d(x)a = 0$ for all $x \in N$, then for all $x, y \in N$, we have

$$\begin{aligned} 0 = d(yx)a &= (\alpha(y)d(x) + d(y)x)a \\ &= \alpha(y)d(x)a + d(y)xa, \qquad \text{by Proposition 2,} \\ &= 0 + d(y)xa. \end{aligned}$$

Thus $d(y)Na = 0$. Now the primeness of N implies that $d = 0$ or $a = 0$. □

Proposition 5. *Let N be a 2-torsion-free prime near-ring. Let d be an α-derivation on N such that $d\alpha = \alpha d$. Then $d^2 = 0$ implies $d = 0$.*

Proof. Suppose that $d^2 = 0$. Let $x, y \in N$. Then

$$\begin{aligned} d^2(xy) = 0 &= d(d(xy)) \\ &= d(\alpha(x)d(y) + d(x)y) \\ &= d(\alpha(x)d(y)) + d(d(x)y) \\ &= \alpha^2(x)d^2(y) + d(\alpha(x))d(y) + \alpha(d(x))d(y) + d^2(x)y \\ &= d(\alpha(x))d(y) + \alpha(d(x))d(y) \\ &= 2d(\alpha(x))d(y). \end{aligned}$$

Hence, $2d(\alpha(x))d(y) = 0$. Since N is 2-torsion-free, we have

$$d(\alpha(x))d(y) = 0.$$

Since α is onto, we get $d(x)d(y) = 0$ and hence by Proposition 4, $d = 0$. □

The following proposition displays, in some way, a sort of commutativity of automorphisms of the near-ring N and the derivation we are considering on N.

Proposition 6. *Let d be an α-derivation on a near-ring N. Let β be an automorphism of N which commutes with d. Then*

$$\alpha\beta(x)d\beta(y) = \beta\alpha(x)\beta d(y) \qquad \textit{for all} \;\; x, y \in N.$$

Proof. Let $x, y \in N$. Then

$$\beta d(xy) = \beta(\alpha(x)d(y) + d(x)y) = \beta\alpha(x)\beta d(y) + \beta d(x)\beta(y). \tag{3}$$

And,

$$d\beta(xy) = d(\beta(x)\beta(y)) = \alpha\beta(x)d(\beta(y)) + d(\beta(x))\beta(y). \tag{4}$$

Since β commutes with d, equations (3) and (4) imply that

$$\alpha\beta(x)d\beta(y) = \beta\alpha(x)\beta d(y), \qquad \text{as required.}$$

□

Now we are ready to prove the theorem.

Proof of the Theorem. Let $d_1 d_2$ be an $\alpha\beta$-derivation. For $x, y \in N$, we have

$$(d_1 d_2)(xy) = (\alpha\beta)(x)d_1 d_2(y) + (d_1 d_2)(x)y. \tag{5}$$

Also,

$$\begin{aligned} (d_1 d_2)(xy) &= d_1(d_2(xy)) \\ &= d_1[\beta(x)d_2(y) + d_2(x)y] \\ &= d_1[\beta(x)d_2(y)] + d_1[d_2(x)y] \\ &= (\alpha\beta)(x)d_1 d_2(y) + (d_1\beta)(x)d_2(y) + (\alpha d_2)(x)d_1(y) + d_1 d_2(x)y. \end{aligned} \tag{6}$$

From (5) and (6), we get

$$(d_1\beta)(x)d_2(y) + (\alpha d_2)(x)d_1(y) = 0. \tag{7}$$

Replacing x by $xd_2(z)$ in (7), we get

$$(d_1\beta)(xd_2(z))d_2(y) + (\alpha d_2)(xd_2(z))d_1(y) = 0,$$

and so,

$$(\beta d_1)(xd_2(z))d_2(y) + (\alpha d_2)(xd_2(z))d_1(y) = 0. \tag{8}$$

Using Proposition 1, equation (8) becomes

$$\beta[d_1(x)d_2(z) + \alpha(x)d_1d_2(z)]d_2(y) + \alpha[\beta(x)d_2^2(z) + d_2(x)d_2(z)]d_1(y) = 0,$$

$$\begin{aligned}[\beta d_1(x)\beta d_2(z) + \beta\alpha(x)\beta d_1d_2(z)]d_2(y)\\ + [\alpha\beta(x)\alpha d_2^2(z) + \alpha d_2(x)\alpha d_2(z)]d_1(y) = 0\end{aligned} \tag{9}$$

Using Proposition 6 and the hypothesis, equation (9) becomes

$$\begin{aligned}[d_1\beta(x)d_2\beta(z) + \alpha\beta(x)d_1(\beta d_2(z))]d_2(y)\\ + [\alpha\beta(x)d_2^2\alpha(z) + d_2\alpha(x)d_2(\alpha(z))]d_1(y) = 0.\end{aligned} \tag{10}$$

Using Proposition 2, equation (10) becomes

$$\begin{aligned}d_1\beta(x)d_2\beta(z)d_2(y) + \alpha\beta(x)d_1(\beta d_2(z))d_2(y) + \alpha\beta(x)d_2^2(\alpha(z))d_1(y)\\ + d_2(\alpha(x))d_2(\alpha(z))d_1(y) = 0,\end{aligned}$$

$$\begin{aligned}d_1\beta(x)d_2\beta(z)d_2(y) + (\alpha\beta)(x)[d_1(\beta d_2(z))d_2(y) + d_2^2(\alpha(z))d_1(y)]\\ + d_2(\alpha(x))d_2(\alpha(z))d_1(y) = 0.\end{aligned} \tag{11}$$

Replacing x by $d_2(z)$ in (7), we get

$$(d_1\beta)(d_2(z))d_2(y) + (\alpha d_2)(d_2(z))d_1(y) = 0,$$

or

$$d_1(\beta d_2(z))d_2(y) + d_2^2(\alpha(z))d_1(y) = 0. \tag{12}$$

Since N is zero symmetric, equations (11) and (12) imply that

$$d_1\beta(x)d_2\beta(z)d_2(y) + d_2(\alpha(x))d_2(\alpha(z))d_1(y) = 0. \tag{13}$$

Replacing now x by z in (7), we get

$$(d_1\beta)(z)d_2(y) + (\alpha d_2)(z)d_1(y) = 0,$$

or

$$\alpha d_2(z)d_1(y) = -d_1(\beta(z))d_2(y). \tag{14}$$

Replacing y by $\beta(z)$ in (7), we get

$$(d_1\beta)(x)d_2(\beta(z)) + (\alpha d_2)(x)d_1(\beta(z)) = 0.$$

So,

$$d_1(\beta(x))d_2(\beta(z)) = -d_2(\alpha(x))d_1(\beta(z)). \tag{15}$$

Combining (13), (14) and (15) we get

$$\{-[d_2(\alpha(x))d_1(\beta(z))]\}d_2(y) + d_2(\alpha(x))\{-[d_1(\beta(z))d_2(y)]\} = 0. \tag{16}$$

To simplify notations, we put $u = d_2(\alpha(x)), v = d_1(\beta(z))$, and $w = d_2(y)$. Then

$$\begin{aligned}\{-[uv]\}w + u\{-[vw]\} &= 0,\\ u[-v]w + u\{-[vw]\} &= 0,\\ u[-v]w - u[vw] &= 0,\\ -uvw - uvw &= 0,\\ uvw + uvw &= 0,\\ u(2vw) &= 0.\end{aligned}$$

If $u \neq 0$ (i.e. $d_2 \neq 0$), then by Proposition 4, $2vw = 0$, that is, $v[2w] = 0$. Again if $w \neq 0$ (i.e. $d_2 \neq 0$), then by hypothesis $2w \neq 0$, and then by Proposition 4 we have $v = 0$; that is $d_1 = 0$. This shows that if $d_2 \neq 0$ then $d_1 = 0$ which completes the proof. $\square$

Remark 7. In the above result, the hypothesis that N is 2-torsion-free may be weakened by assuming instead the existence of an element y in the near-ring N such that $2d_2(y) \neq 0$. Then the same proof will lead to the conclusion that $d_1 = 0$.

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