Acta Math. Univ. Comenianae
Vol. LXXVII, 2(2008), pp. 305–312

ON THE HILBERT INEQUALITY

ZHOU YU AND GAO MINGZHE

ABSTRACT. In this paper it is shown that the Hilbert inequality for double series can be improved by introducing a weight function of the form $\frac{\sqrt{n}}{n+1}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1}-\frac{\ln n}{\pi}\right)$, where $n \in N$. A similar result for the Hilbert integral inequality is also given. As applications, some sharp results of Hardy-Littlewood's theorem and Widder's theorem are obtained.

1. INTRODUCTION

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. It is all-round known that the inequality

$$\left|\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m\bar{b}_n}{m+n}\right|^2 \le \pi^2\sum_{n=1}^{\infty}|a_n|^2\sum_{n=1}^{\infty}|b_n|^2 \tag{1.1}$$

is called the Hilbert inequality for double series, where $\sum\limits_{n=1}^{\infty}|a_n|^2 < +\infty$ and $\sum\limits_{n=1}^{\infty}|b_n|^2 < +\infty$, and that the constant factor π^2 in (1.1) is the best possible. The equality in (1.1) holds if and only if $\{a_n\}$, or $\{b_n\}$ is a zero-sequence (see [**?**]). The corresponding integral form of (1.1) is that

$$\left|\int_0^{\infty}\int_0^{\infty}\frac{f(x)\overline{g}(y)}{x+y}\mathrm{d}x\mathrm{d}y\right|^2 \le \pi^2\left(\int_0^{\infty}|f(x)|^2\,\mathrm{d}x\right)\left(\int_0^{\infty}|g\,(x)|^2\,\mathrm{d}x\right) \tag{1.2}$$

where $\int\limits_0^{\infty}|f\,(x)|^2\,\mathrm{d}x < +\infty$ and $\int\limits_0^{\infty}|g\,(x)|^2\,\mathrm{d}x < +\infty$, and that the constant factor π^2 in (1.2) is also the best possible. The equality in (1.2) holds if and only if $f\,(x) = 0$, or $g\,(x) = 0$. Recently, various improvements and extensions of (1.1) and (1.2) appeared in a great deal of papers (see [**?**]). The purpose of the present paper is to build the Hilbert inequality with the weights by means of a monotonic function

Received August 3, 2007; revised February 21, 2008.
2000 *Mathematics Subject Classification.* Primary 26D15.
Key words and phrases. Hilbert's inequality; weight function; double series; monotonic function; Hardy-Littlewood's theorem; Widder's theorem.
A Project Supported by Scientific Research Fund of Hunan Provincial Education Department (06C657).

of the form $\frac{\sqrt{x}}{1+\sqrt{x}}$, thereby new refinements of (1.1) and (1.2) are established, and then to give some of their important applications.

For convenience, we need the following lemmas.

Lemma 1.1. *Let $n \in \mathbb{N}$. Then*

$$\int_0^\infty \frac{\mathrm{d}x}{(n+x^2)(1+x)} = \frac{1}{n+1}\left(\frac{\pi}{2\sqrt{n}} + \frac{1}{2}\ln n\right) \tag{1.3}$$

Proof. Let a, e and f be real numbers. Then

$$\begin{aligned}&\int \frac{\mathrm{d}x}{(a^2+x^2)(e+fx)}\\ &= \frac{1}{e^2+a^2f^2}\left\{f\ln|e+fx| - \frac{1}{2}\ln(a^2+x^2) + \frac{e}{a}\arctan\frac{x}{a}\right\} + C\end{aligned}$$

where C is an arbitrary constant. This result has been given in the papers (see [**3**]–[**4**]). Based on this indefinite integral it is easy to deduce that the equality (1.3) is true. □

Lemma 1.2. *Let $n \in \mathbb{N}$, $x \in (0,+\infty)$. Define two functions by*

$$f(x) = \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1 - \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)$$

$$g(x) = \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1 + \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right),$$

then $f(x)$ and $g(x)$ are monotonously decreasing in $(0, +\infty)$, and

$$\int_0^\infty f(x)\,\mathrm{d}x = \pi - \pi\omega(n) \tag{1.4}$$

$$\int_0^\infty g(x)\,\mathrm{d}x = \pi + \pi\omega(n) \tag{1.5}$$

where the weight function ω is defined by

$$\omega(n) = \frac{\sqrt{n}}{n+1}\left(\frac{\sqrt{n}-1}{\sqrt{n}+1} - \frac{\ln n}{\pi}\right) \tag{1.6}$$

Proof. At first, notice that $1 - \frac{\sqrt{x}}{1+\sqrt{x}} = \frac{1}{1+\sqrt{x}}$, hence we can write $f(x)$ in form $f(x) = f_1(x) + f_2(x)$, where

$$f_1(x) = \left(\frac{1}{(x+n)\sqrt{x}}\right)\left(\frac{n}{1+\sqrt{n}}\right), \qquad f_2(x) = \frac{\sqrt{n}}{(x+n)(1+\sqrt{x})\sqrt{x}}.$$

It is obvious that $f_1(x)$ and $f_2(x)$ are monotonously decreasing in $(0, +\infty)$. Hence $f(x)$ is monotonously decreasing in $(0, +\infty)$. Next, notice that $1 - \frac{\sqrt{n}}{1+\sqrt{n}} = \frac{1}{1+\sqrt{n}}$, we can write $g(x)$ in form $g(x) = g_1(x) + g_2(x)$, where

$$g_1(x) = \frac{\sqrt{n}}{(1+\sqrt{n})(x+n)\sqrt{x}}, \qquad g_2(x) = \frac{\sqrt{n}}{(x+n)(1+\sqrt{x})}.$$

It is obvious that $g_1(x)$ and $g_2(x)$ are monotonously decreasing in $(0, +\infty)$. Hence $g(x)$ is also monotonously decreasing in $(0, +\infty)$. Further we need only to compute two integrals.

$$\begin{aligned}
\int_0^\infty f(x)\,\mathrm{d}x &= \int_0^\infty \left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1+\frac{\sqrt{n}}{1+\sqrt{n}}-\frac{\sqrt{x}}{1+\sqrt{x}}\right)\mathrm{d}x \\
&= \left(1+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\int_0^\infty\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\mathrm{d}x-\int_0^\infty\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)\mathrm{d}x \\
&= \left(1+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\pi-\int_0^\infty\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)\mathrm{d}x \\
&= \pi-\left\{2\sqrt{n}\left(\int_0^\infty\frac{1}{(n+t^2)}\mathrm{d}t-\int_0^\infty\frac{1}{(n+t^2)(1+t)}\mathrm{d}t\right)-\frac{\sqrt{n}\,\pi}{1+\sqrt{n}}\right\} \\
&= \pi-\left\{\pi-2\sqrt{n}\int_0^\infty\frac{1}{(n+t^2)(1+t)}\mathrm{d}t-\frac{\sqrt{n}\,\pi}{1+\sqrt{n}}\right\}
\end{aligned}$$

By Lemma 1.1, we obtain

$$\int_0^\infty f(x)\,\mathrm{d}x = \pi-\left\{\pi-\left(\frac{\pi}{n+1}+\frac{\sqrt{n}\ln n}{n+1}\right)-\frac{\sqrt{n}\,\pi}{1+\sqrt{n}}\right\} \tag{1.7}$$

The equality (1.4) follows from (1.7) at once after some simple computations and simplifications.

Similarly, the equality (1.5) can be obtained.

$\square$

2. Main Results

First, we establish a new refinement of (1.1).

Theorem 2.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. If $\sum\limits_{n=1}^{\infty}|a_n|^2<+\infty$ and $\sum\limits_{n=1}^{\infty}|b_n|^2<+\infty$, then*

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m\bar{b}_n}{m+n}\right|^4\le\pi^4\left\{\left(\sum_{n=1}^{\infty}|a_n|^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)|a_n|^2\right)^2\right\}$$
$$\times\left\{\left(\sum_{n=1}^{\infty}|b_n|^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)|b_n|^2\right)^2\right\} \tag{2.1}$$

where the weight function $\omega(n)$ is defined by (1.6).

Proof. Let $c(x)$ be a real function and satisfy the condition $1-c(n)+c(m)\ge 0$, $(n,m\in N)$. Firstly we suppose that $b_n=a_n$. Applying Cauchy's inequality we have

$$\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m\bar{a}_n}{m+n}\right|^2=\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m\bar{a}_n}{m+n}(1-c(n)+c(m))\right|^2$$
$$=\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{a_m(1-c(n)+c(m))^{1/2}}{(m+n)^{1/2}}\left(\frac{m}{n}\right)^{1/4}\right)\right.$$
$$\left.\times\left(\frac{\overline{a}_n(1-c(n)+c(m))^{1/2}}{(m+n)^{1/2}}\left(\frac{n}{m}\right)^{1/4}\right)\right|^2$$
$$\le J_1J_2 \tag{2.2}$$

where
$$J_1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|^2}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{2}}(1-c(n)+c(m))$$
$$J_2=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|\bar{a}_n|^2}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{2}}(1-c(n)+c(m))$$

We can write the double series J_1 in the following form:

$$J_1=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{2}}(1-c(m)+c(n))\right)|a_n|^2.$$

Let $c(x)=\frac{\sqrt{x}}{1+\sqrt{x}}$. It is obvious that $1-\frac{\sqrt{x}}{1+\sqrt{x}}+\frac{\sqrt{n}}{1+\sqrt{n}}\ge 0$. It is known from Lemma 1.2 that the function $f(x)$ is monotonously decreasing. Hence we have

$$J_1=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{2}}\left(1-\frac{\sqrt{m}}{1+\sqrt{m}}+\frac{\sqrt{n}}{1+\sqrt{n}}\right)\right)|a_n|^2$$
$$\le\sum_{n=1}^{\infty}\left\{\int_0^{\infty}\left(\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{2}}\right)\left(1-\left(\frac{\sqrt{x}}{1\,+\,\sqrt{x}}-\frac{\sqrt{n}}{1\,+\,\sqrt{n}}\right)\right)\mathrm{d}x\right\}|a_n|^2$$
$$=\pi\sum_{n=1}^{\infty}|a_n|^2-\pi\sum_{n=1}^{\infty}\omega(n)|a_n|^2$$

where the weight function $\omega(n)$ is defined by (1.6).

Similarly,

$$J_2 \le \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} \frac{1}{x+n} \left(\frac{n}{x}\right)^{\frac{1}{2}} \left(1 + \left(\frac{\sqrt{x}}{1+\sqrt{x}} - \frac{\sqrt{n}}{1+\sqrt{n}}\right)\right) \mathrm{d}x \right\} |\bar{a}_n|^2$$

$$= \pi \sum_{n=1}^{\infty} |a_n|^2 + \pi \sum_{n=1}^{\infty} \omega(n) |a_n|^2.$$

Whence $J_1 J_2 \le \pi^2 \left\{ \left(\sum\limits_{n=1}^{\infty} |a_n|^2 \right)^2 - \left(\sum\limits_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\}$.

Consequently, we have

$$\left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 \le \pi^2 \left\{ \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^2 - \left(\sum_{n=1}^{\infty} \omega(n) |a_n|^2 \right)^2 \right\} \tag{2.3}$$

where the weight function $\omega(n)$ is defined by (1.6).

If $b_n \ne a_n$, then we can apply Schwarz's inequality to estimate the right-hand side of (2.1) as follows:

$$\begin{aligned} \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{b}_n}{m+n} \right|^4 &= \left\{ \left| \int_0^1 \left(\sum_{m=1}^{\infty} a_m t^{m-\frac{1}{2}} \right) \left(\sum_{n=1}^{\infty} \bar{b}_n t^{n-\frac{1}{2}} \right) \mathrm{d}t \right|^2 \right\}^2 \\ &\le \left| \int_0^1 \left(\sum_{m=1}^{\infty} |a_m| \, t^{m-\frac{1}{2}} \right)^2 \mathrm{d}t \int_0^1 \left(\sum_{n=1}^{\infty} |b_n| \, t^{n-\frac{1}{2}} \right)^2 \mathrm{d}t \right|^2 \\ &= \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m \bar{a}_n}{m+n} \right|^2 \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{b_m \bar{b}_n}{m+n} \right|^2 \end{aligned} \tag{2.4}$$

And then by using the relation (2.3), from (2.4) and the inequality (2.1), we obtain at once. □

Similarly, we can establish a new refinement of (1.2).

Theorem 2.2. *Let $f(x)$ and $g(x)$ be two functions in complex number field. If $\int\limits_0^{\infty} |f(x)|^2 \, \mathrm{d}x < +\infty$, $\int\limits_0^{\infty} |g(x)|^2 \, \mathrm{d}x < +\infty$, then*

$$\begin{aligned} \left| \int_0^{\infty} \int_0^{\infty} \frac{f(x)\overline{g}(x)}{x+y} \mathrm{d}x\mathrm{d}y \right|^4 &\le \pi^4 \left\{ \left(\int_0^{\infty} |f(x)|^2 \, \mathrm{d}x \right)^2 - \left(\int_0^{\infty} \omega(x) \, |f(x)|^2 \, \mathrm{d}x \right)^2 \right\} \\ &\times \left\{ \left(\int_0^{\infty} |g(x)|^2 \, \mathrm{d}x \right)^2 - \left(\int_0^{\infty} \omega(x) \, |g(x)|^2 \, \mathrm{d}x \right)^2 \right\} \end{aligned} \tag{2.5}$$

where the weight function ω *is defined by*

$$\omega(x)=\begin{cases}0 & x=0\\ \dfrac{\sqrt{x}}{x+1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}-\dfrac{\ln x}{\pi}\right) & x>0\end{cases}\tag{2.6}$$

Its proof is similar to that of Theorem 2.1, it is omitted here.
For the convenience of the applications, we list the following result.

Corollary 2.3. *Let* $f(x)$ *be a function in complex number field. If* $\int_0^\infty |f(x)|^2\,\mathrm{d}x<+\infty$, *then*

$$\begin{aligned}&\left|\int_0^\infty\int_0^\infty \frac{f(x)\overline{f}(y)}{x+y}\mathrm{d}x\mathrm{d}y\right|^2\\ &\le \pi^2\left\{\left(\int_0^\infty |f(x)|^2\mathrm{d}x\right)^2-\left(\int_0^\infty \omega(x)\,|f(x)|^2\,\mathrm{d}x\right)^2\right\}\end{aligned}\tag{2.7}$$

where the weight function ω *is defined by* (2.6).

3. Applications

As applications, we shall give some new refinements of Hardy-Littlewood's theorem and Widder's theorem.

Let $f(x)\in L^2(0,1)$ and $f(x)\neq 0$ for all x. Define a sequence $\{a_n\}$ by $a_n=\int_0^1 x^n f(x)\mathrm{d}x$, $n=0,\ 1,\ 2,\ \ldots$. Hardy-Littlewood ([**1**]) proved that

$$\sum_{n=0}^\infty a_n^2<\pi\int_0^1 f^2(x)\mathrm{d}x,\tag{3.1}$$

where π is the best constant that the inequality (3.1) keeps valid.

Theorem 3.1. *Let* $f(x)\in L^2(0,1)$ *and* $f(x)\neq 0$ *for all* x. *Define a sequence* $\{a_n\}$ *by* $a_n=\int_0^1 x^{n-1/2}f(x)\mathrm{d}x\quad n=\ 1,2,\ \ldots$. *Then*

$$\left(\sum_{n=1}^\infty a_n^2\right)^2\le\pi\left\{\left(\sum_{n=1}^\infty a_n^2\right)^2-\left(\sum_{n=1}^\infty \omega(n)\,a_n^2\right)^2\right\}^{\frac{1}{2}}\int_0^1 f^2(x)\,\mathrm{d}x\tag{3.2}$$

where $\omega(n)$ *is defined by* (1.6).

Proof. By our assumptions, we may write a_n^2 in the form

$$a_n^2=\int_0^1 a_n x^{n-1/2}f(x)\mathrm{d}x.$$

Applying Cauchy-Schwarz's inequality we estimate the right hand side of (3.2) as follows

$$\begin{aligned}\left(\sum_{n=1}^{\infty} a_n^2\right)^2 &= \left(\sum_{n=1}^{\infty}\int_0^1 a_n x^{n-1/2} f(x)\mathrm{d}x\right)^2 = \left\{\int_0^1 \left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right) f(x)\mathrm{d}x\right\}^2 \\ &\le \int_0^1 \left(\sum_{n=1}^{\infty} a_n x^{n-1/2}\right)^2 \mathrm{d}x \int_0^1 f^2(x)\mathrm{d}x \\ &= \int_0^1 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m a_n x^{m+n-1}\mathrm{d}x \int_0^1 f^2(x)\mathrm{d}x \\ &= \left(\sum_{m=}^{\infty}\sum_{n=1}^{\infty} \frac{a_m a_n}{m+n}\right)\int_0^1 f^2(x)\mathrm{d}x \qquad (3.3)\end{aligned}$$

It is known from (2.3) and (3.3) that the inequality (3.2) is valid. Therefore the theorem is proved. □

Let $a_n \ge 0$ $(n = 0, 1, 2, \dots.)$, $A(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, $A^*(x) = \sum\limits_{n=0}^{\infty} \frac{a_n x^n}{n!}$. Then

$$\int_0^1 A^2(x)\mathrm{d}x \le \pi \int_0^{\infty} \left(\mathrm{e}^{-x} A^*(x)\right)^2 \mathrm{d}x \qquad (3.4)$$

This is Widder's theorem (see [**1**]).

Theorem 3.2. *With the assumptions as the above-mentioned, it yields*

$$\begin{aligned}&\left(\int_0^1 A^2(x)\,\mathrm{d}x\right)^2 \\ &\le \pi^2\left\{\left(\int_0^{\infty}\left(\mathrm{e}^{-x} A^*(x)\right)^2 \mathrm{d}x\right)^2 - \left(\int_0^{\infty}\omega(x)\left(\mathrm{e}^{-x} A^*(x)\right)^2 \mathrm{d}x\right)^2\right\}\end{aligned} \qquad (3.5)$$

where $\omega(x)$ is defined by (2.6).

Proof. At first we have the following relation:

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{-t} A^*(tx)\mathrm{d}t &= \int_0^{\infty} \mathrm{e}^{-t}\sum_{n=0}^{\infty}\frac{a_n (xt)^n}{n!}\mathrm{d}t \\ &= \sum_{n=0}^{\infty}\frac{a_n x^n}{n!}\int_0^{\infty} t^n\,\mathrm{e}^{-t}\,\mathrm{d}t = \sum_{n=0}^{\infty} a_n x^n = A(x)\end{aligned}$$

Let $tx = s$. Then we have

$$\int_0^1 A^2(x)\mathrm{d}x = \int_0^1 \left\{ \int_0^\infty \mathrm{e}^{-t} A^*(tx)\,\mathrm{d}t \right\}^2 \mathrm{d}x = \int_0^i \left(\int_0^\infty \mathrm{e}^{-\frac{s}{x}} A^*(s)\,\mathrm{d}s \right)^2 \frac{1}{x^2}\mathrm{d}x$$

$$= \int_1^\infty \left(\int_0^\infty \mathrm{e}^{-sy} A^*(s)\mathrm{d}s \right)^2 \mathrm{d}y = \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-s(u+1)} A^*(s)\,\mathrm{d}s \right)^2 \mathrm{d}u$$

$$= \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-su} f(s)\,\mathrm{d}s \right)^2 \mathrm{d}u = \int_0^\infty \int_0^\infty \frac{f(s) f(t)}{s+t} \mathrm{d}s\mathrm{d}t \tag{3.6}$$

where $f(x) = \mathrm{e}^{-x} A^*(x)$. By Corollary 2.3, the inequality (3.5) follows from (3.6) at once. □

References

1. Hardy G. H., Littlewood J. E. and Polya G., *Inequalities*, Cambridge Univ. Press, Cambridge, U.K., 1952.

2. Gao Mingzhe and Hsu Lizhi, *A survey of various refinements and generalizations of Hilbert's inequalities*, Journal of Mathematical Research and Exposition **25(2)** (2005), 227–243.

3. Zwillinger D. et al., *CRC Standard Mathematical Tables and Formulae*, CRC Press, 1988.

4. Gradshteyn I. S. and Ryzhik I. M., *Table of Integrals*, Series, and Products, Academic Press, 2000.

Zhou Yu, Department of Mathematics and Computer Science Normal College, Jishou University Jishou Hunan 416000, P. R. China, *e-mail*: `hong2990@163.com`

Gao Mingzhe, Department of Mathematics and Computer Science Normal College, Jishou University Jishou Hunan 416000, P. R. China, *e-mail*: `mingzhegao@163.com`