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RINGS IN POST ALGEBRAS

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Abstract. Serfati [7] defined a ring structure on every Post algebra. In this paper we determine all the rings that are defined over a Post algebra and share the properties of the Serfati ring. In the case $r = 3$ one of them is equivalent to the Post algebra. This is a term equivalence and it extends the equivalence between a Boolean algebra and the Boolean ring associated with it.

1. Introduction

It is well known that a Boolean algebra $(B, \vee, \cdot, ', 0, 1)$ can be made into a Boolean ring (i.e., commutative, idempotent and of characteristic 2) $(B, +, \cdot, 0, 1)$ where $x + y = xy' \vee x'y$. Conversely, every Boolean ring becomes a Boolean algebra by defining $x \vee y = x + y + xy$ and $x' = x + 1$. Moreover, the above constructions establish a bijection (and together with the identity transformations on morphisms, they determine an isomorphism between the category of Boolean algebras and the category of Boolean rings).

On the other hand, the category of Post algebras is close enough to the category of Boolean algebras; see e.g. [**1**], [**6**]. It is therefore natural to ask whether the above equivalence between Boolean algebras and Boolean rings can be extended to the Post framework. The only result in this direction known so far was obtained by Serfati [**7**], who proved that on every Post algebra one can define a ring in terms of the Post-algebra operations. In this paper we determine all the rings defined over an arbitrary Post algebra and sharing the properties of the Serfati ring. In the case of a Post algebra of order 3 there are 6 such rings and one of them is equivalent to the Post algebra.

The exact formulation of the desired equivalence and of our results needs the following well-known universal algebraic definition. The *term functions* of an algebra are the projection functions, the basic operations of the algebra and all the functions obtained from them by composition (in other words, the *clone* generated by the basic operations). For instance, the term functions of a ring are polynomials of that ring, but the converse does not hold. In [**5**], [**6**] we prefer the denominations *simple Boolean functions* (*simple Post functions*) for the term functions of a Boolean algebra (Post algebra). Thus, the ring operations of a Boolean algebra

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are simple Boolean functions and the operations of the ring defined by Serfati are simple Post functions.

The paper is structured as follows. After the next section recalling all necessary prerequisites, in Section 3 we prove that each ring on the chain of constants of a Post algebra P can be uniquely extended to a ring defined on P by simple Post functions and conversely, every such ring can be defined in this way. In Section 4 we confine to Post algebras of order 3. We prove that there are 6 rings on the chain of constants $\{0, e, 1\}$, all of them isomorphic to the field $\mathbb{Z}_3$. By applying the results of Section 3, it follows that there are 6 rings defined on the whole algebra P by simple Post functions (including, of course, the ring found by Serfati). One of these rings is equivalent to the Post algebra in the same way as a Boolean ring is equivalent to the Boolean algebra having the same support: *it is a commutative ring defined by simple Post functions, having the same* 0 *and* 1 *as the Post algebra and the basic operations of the Post algebra are term functions of the ring.*

2. Prerequisites on Post algebras

Let $(P, \vee, \cdot, 0, 1)$ be a bounded distributive lattice; the meet $\cdot$ is also denoted simply by concatenation. An element $x \in P$ is said to be *complemented* provided there exists $x' \in P$ such that $x \vee x' = 1$ and $xx' = 0$; the element x' is called the *complement* of x and is uniquely determined by x. The set $B(P)$ of all complemented elements is a Boolean algebra and a sublattice of P.

The lattice theoretic definition of Post algebras, given below, is due to Epstein [**2**].

Let r be an integer, $r \geq 2$. Set

$$\langle r \rangle = \{0, 1, \dots, r-1\}\,. \tag{1}$$

A *Post algebra of order* r is an algebra $(P, \vee, \cdot, {}^0, {}^1, \dots, {}^{r-1}, e_0 = 0, e_1, \dots, e_{r-2}, e_{r-1} = 1)$ of type $(2, 2, (1)_{i \in \langle r \rangle}, (0)_{i \in \langle r \rangle})$ such that $(P, \vee, \cdot, 0, 1)$ is a bounded distributive lattice,

$$e_0 = 0 < e_1 < \dots < e_{r-2} < e_{r-1} = 1 \tag{2}$$

and every element $x \in P$ can be uniquely represented in the form

$$x = \bigvee_{i \in \langle r \rangle} e_i x^i \quad \text{with} \quad \bigvee_{i \in \langle r \rangle} x^i = 1 \text{ and } x^i x^j = 0 \quad (\forall\, i, j \in \langle r \rangle, i \neq j). \tag{3}$$

The Post algebras of order 2 coincide with Boolean algebras. In fact, a Boolean algebra $(B, \vee, \cdot, ', 0, 1)$ can be identified with the Post algebra $(B, \vee, \cdot, {}^0, {}^1, 0, 1)$ where $x^0 = x'$ and $x^1 = x$, because the identity function missing in the former algebra is in fact the projection function of one variable, so that the two algebras have the same clone of term functions.

In the sequel we consider $r \geq 3$.

The set

$$E = \{e_0 = 0, e_1, \dots, e_{r-2}, e_{r-1} = 1\} \tag{4}$$

is called the *chain of constants* of the Post algebra P and is a subalgebra of P. The elements $x^0, x^1, \ldots, x^{r-1}$ are called the *disjunctive components* of x.

It follows from (3) that all $x^i \in B(P)$, with $(x^i)' = \bigvee_{j\in\langle r\rangle, j\neq i} x^j$.

It is also easy to see that

$$x \in B(P) \Longleftrightarrow x^0 = x',\ x^{r-1} = x,\ x^j = 0 \quad (j = 1, \ldots, r-2) \tag{5}$$

and that $(e_i)^j$ is the Kronecker δ:

$$(e_i)^j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{6}$$

The *Post functions* are the elements of the clone generated by the $2r+2$ basic operations and all the constant functions. They are the specialization of the universal algebraic concept of algebraic function (in Grätzer's terminology; also called polynomial functions by other authors). As mentioned in the introduction, the *simple Post functions* are the elements of the clone generated by the basic operations, and they are the specialization of the universal algebraic concept of term function (or Grätzer polynomial)[1].

The alternative notation $x^i = x^{e_i}$ $(i \in \langle r\rangle)$ is very useful in the study of Post functions. The main result is that Post functions are characterized by the expansion

$$f(x_1, \ldots, x_n) = \bigvee_{a_1,\ldots,a_n\in E} f(a_1, \ldots, a_n) \cdot x_1^{a_1} \cdot \cdots \cdot x_n^{a_n}, \tag{7}$$

while the simple Post functions are precisely those Post functions for which

$$f(a_1, \ldots, a_n) \in E\ (\forall\, a_1, \ldots, a_n \in E). \tag{8}$$

It follows from this representation that if f and g are Post functions, then

$$f(X) = g(X)\,(\forall\, X \in P^n) \Longleftrightarrow f(A) = g(A)\,(\forall\, A \in E^n). \tag{9}$$

We call this result the *Verification Theorem*, like in the case of Boolean algebras. It also follows from (8) that the expression (7) of a simple Post function makes sense in any Post algebra. Therefore two simple Post functions coincide if and only if they coincide on the Post algebra E. This is a "global" version of the Verification Theorem, again like in the case of Boolean algebras.

The expressions of the form (7) obey the following identities:

$$\Big(\bigvee_{A\in E^n} c_A X^A\Big) \circ \Big(\bigvee_{A\in E^n} d_A X^A\Big) = \bigvee_{A\in E^n} (c_A \circ d_A) X^A \ (\circ = \vee, \cdot), \tag{10$'$}$$

$$\Big(\bigvee_{A\in E^n} c_A X^A\Big)^i = \bigvee_{A\in E^n} (c_A)^i X^A \ (\forall i \in \langle r\rangle), \tag{10$''$}$$

where $c_A, d_A \in P$ and we have set $X = (x_1, \ldots, x_n)$, $A = (a_1, \ldots, a_n)$, $X^A = x_1^{a_1} \cdot \cdots \cdot x_n^{a_n}$.

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[1] So both polynomial function-term function and algebraic function-polynomial function are conventional universal algebraic terminologies, but we prefer the mixed terminology algebraic function-term function.

For every Post function $f : P^n \longrightarrow P$, the equation

$$f(x_1, \dots, x_n) = 0 \tag{11}$$

has solutions in P^n if and only if

$$\prod_{a_1,\dots,a_n \in E} f(a_1, \dots, a_n) = 0, \tag{12}$$

where $\prod$ denotes iterated meet.

The above results and many others can be found in [**6**, Chapter 5] and [**2**, Theorem 13]; the notation and the basic results are due to Serfati [**7**]. See also [**5**] for the case of Boolean algebras.

3. Extending rings in Post algebras

Let P be a Post algebra and E its chain of constants. Our starting point is the remark that the $r!$ bijections between E and $\langle r\rangle$ (cf. (4) and (1)) yield $r!$ rings defined on E and isomorphic to $\mathbb{Z}_r$. Therefore the rings described in the theorem below do exist.

Theorem 1. *Each ring (commutative ring, unitary ring) defined on the chain of constants E can be uniquely extended to a ring (commutative ring, unitary ring) defined on the Post algebra P by simple Post functions; cf. formulae* (13) *and* (14) *below. Conversely, each ring (commutative ring, unitary ring) defined on P by simple Post functions and having the zero (and one) in E is obtained in this way.*

Comment. The ring found by Serfati is indeed commutative, with zero and one in E, and defined by simple Post functions.

Proof. Let $(E, \oplus, \odot)$ be a ring. In view of the representation (7), (8), the functions $\oplus$ and $\odot$ can be uniquely extended to simple Post functions on P, which, by an abuse of notation, we will denote by the same symbols $\oplus$ and $\odot$, respectively:

$$x \oplus y = \bigvee_{i,j\in\langle r\rangle} (e_i \oplus e_j)x^i y^j = \bigvee_{h\in\langle r\rangle} e_h \bigvee_{e_i\oplus e_j=e_h} x^i y^j, \tag{13}$$

$$x \odot y = \bigvee_{i,j\in\langle r\rangle} (e_i \odot e_j)x^i y^j = \bigvee_{h\in\langle r\rangle} e_h \bigvee_{e_i\odot e_j=e_h} x^i y^j. \tag{14}$$

Each ring axiom different from the existence of $-x$ is satisfied by the operations (13), (14) in view of the Verification Theorem (9).

To prove the existence of $-x$, take an element $x \in P$ and look for an element $y \in P$ such that $x \oplus y = e_\alpha$, where e_α is the common zero of the operation $\oplus$ on E and the extended operation $\oplus$ on P. Since $0 \in E$, it follows that the latter equation implies $x \oplus y \oplus 0 = e_\alpha \oplus 0 = 0$. But the existence of e_β such that $0 \oplus e_\beta = e_\alpha$ implies that if $x \oplus z = 0$ then $x \oplus z \oplus e_\beta = e_\alpha$. Therefore it suffices to prove that for each $x \in P$, the equation $x \oplus y = 0$ has a solution $y \in P$.

The latter equation can be written

$$\bigvee_j (\bigvee_i (e_i \oplus e_j) x^i) y^j = 0 \ ,$$

which is of the form (11). Its consistency condition (12) becomes

$$\prod_j \bigvee_i (e_i \oplus e_j) x^i = 0$$

and this should be true for every $x \in P$. In view of the Verification Theorem (9), the latter condition is equivalent to

$$\prod_j \bigvee_i (e_i \oplus e_j)(e_h)^i = 0 \ (\forall\, e_h \in E) \ ,$$

or equivalently, using (6),

$$\prod_j (e_h \oplus e_j) = 0 \ (\forall\, e_h \in E) \ ,$$

which is true because for each $e_h \in E$ there is an element $e_j \in E$ such that $e_h \oplus e_j = 0$.

Conversely, if the operations $\oplus$ and $\odot$ of the ring (commutative ring, unitary ring) P are simple Post functions, they satisfy (13) and (14) and their restrictions to E exist by (8). In view of the Verification Theorem, if the zero of P is in E then these restrictions satisfy each ring axiom different from the existence of $-x$. Besides, for each $e \in E$, the elements $e \oplus e_0, e \oplus e_1, \ldots, e \oplus e_{r-1}$ are r distinct elements of E, therefore one of them is zero.

Clearly if the ring P has unit element in E, then the ring E has the same unit. □

Corollary 1. *There is an algorithm which constructs all the rings (commutative rings, unitary rings) defined on a Post algebra by simple Post functions.*

Proof. Construct algorithmically all the ring structures on the finite set E and apply Theorem 1. □

Corollary 2. *Every subalgebra of a Post algebra is also a subring of each of the rings constructed in Theorem 1. In particular so is E.*

Proof. If S is a subalgebra of P, then $E \subseteq S$ and S is a Post algebra itself (see e.g. [**1**, Remark 4.1.14]), therefore the conclusion folows by formulae (13) and (14). □

Corollary 3. *If the ring* $(E, \oplus, \odot)$ *is isomorphic to* $\mathbb{Z}_r$*, then the ring constructed in Theorem 1 satisfies* $x \oplus x \oplus \ldots \oplus x = 0$ *(r terms) and* $x \odot x \odot \ldots \odot x = x$ *(r factors).*

Proof. By the Verification Theorem (9). □

The Boolean pattern suggests that we should look for rings having the same zero and one as the Post algebra.

Corollary 4. *Let $\pi : \langle r\rangle \longrightarrow E$ be a bijection such that $\pi(0) = 0$ and $\pi(1) = e_{r-1}$. Let $(E, \oplus, \odot)$ be the ring such that $\pi : \mathbb{Z}_r \longrightarrow (E, \oplus, \odot)$ is an isomorphism. Then e_0 and e_{r-1} are the zero and the unit, respectively, of $(E, \oplus, \odot)$ and of the ring $(P, \oplus, \odot)$ associated with it in Theorem 1*

Proof. The first statement is obvious. The identity $e_h \oplus e_0 = e_h$ $(\forall\, e_h \in E)$ implies the identity $x \oplus e_0 = x$ $(\forall\, x \in P)$ by the Verification Theorem (9), and similarly we get $x \odot e_{r-1} = x$ $(\forall\, x \in P)$. □

Serfati [**7**], using another approach, found the ring

$$x \oplus y = \bigvee_{h\in\langle r\rangle} e_h \bigvee_{i+j\equiv h} x^i y^j \ , \tag{15}$$

$$x \odot y = \bigvee_{h\in\langle r\rangle} e_h \bigvee_{ij\equiv h} x^i y^j \ , \tag{16}$$

where $\equiv$ is the congruence mod(r). Note that this is the ring constructed in Theorem 1, when the starting ring is isomorphic to $\mathbb{Z}_r$ via the mapping $e_i \mapsto i$. It follows by Corollary 3 that the ring (15), (16) is of characteristic r and if r is prime then it is also r-potent. Serfati noted these properties, as well as Corollary 2 for his ring. He also noted that for $r = 3$ the restriction of the operation (15) to the Boolean algebra $B(P)$ is not the symmetric difference $x + y = xy' \vee x'y$. Let us prove the following general result for arbitrary r.

Proposition 1. *The restriction of the ring sum* (13) *to the Boolean algebra* $B(P)$ *is the symmetric difference if and only if*

$$e_{r-1} \oplus e_{r-1} = e_0 \ . \tag{17}$$

Proof. It follows from (5) that the only non-null terms of the restriction under investigation are x^0y^0, $x^{r-1}y^{r-1}$, x^0y^{r-1} and $x^{r-1}y^0$, that is, $x'y'$, xy, $x'y$ and xy'. On the other hand, put

$$X_h = \{x^i y^j \mid e_i \oplus e_j = e_h\} \quad (\forall\, h \in \langle r\rangle)$$

and note that these sets of functions are pairwise disjoint.

If the expansion of $x \oplus y$ over $B(P)$ is the symmetric difference, then it does not contain the term $x^{r-1}y^{r-1} = xy$, hence $x^{r-1}y^{r-1} \notin X_k$ $(k = 1, \ldots, r-1)$. Since $\bigcup_{h\in\langle r\rangle} X_h$ consists of all the terms $x^i y^j$, it follows that $x^{r-1}y^{r-1} \in X_0$, which is equivalent to (17).

Conversely, suppose (17) holds true. Then $x^{r-1}y^{r-1} \in X_0$ and since $x^0y^0 \in X_0$ while $x^0y^{r-1}, x^{r-1}y^0 \in X_{r-1}$, it follows that the expansion of $x \oplus y$ over $B(P)$ is

$$x \oplus y = e_{r-1}(x^0y^{r-1} \vee x^{r-1}y^0) = x'y \vee xy'.$$

□

Now we see that Serfati's remark is valid for arbitrary $r \geq 3$:

Corollary 5. *The restriction of the Serfati ring (see* (15) *and* (16)*) to* $B(P)$ *does not reduce to the symmetric difference.*

Proof. Notice that $x^{r-1}y^{r-1} \notin X_0$ because $(r-1)+(r-1) \equiv r-2 \not\equiv 0$. $\square$

So far Theorem 1 and Corollary 4 represent a step forward towards the desideratum of constructing a theory of rings in Post algebras, totally analogous to the theory of Boolean rings. The results of the next section partially fulfil this ambitious plan.

4. The case $r = 3$

Theorem 1 characterizes all the rings (commutative rings, unitary rings) defined on a Post algebra by simple Post functions, while Corollary 1 provides an algorithm which lists all of them. In this section we explicitly determine all these rings in the case $r = 3$.

The chain of constants $E = \{e_0 = 0, e_1, e_2 = 1\}$ will alternatively be written $E = \{0, e, 1\}$ or $E = \{e_\alpha, e_\beta, e_\omega\}$ (where (α, β, ω) is a permutation of $(0, 1, 2)$), as may be convenient.

Propositions 2 and 3 below go back to Moisil [**3**], except that uniqueness is taken for granted.

Proposition 2. *The Abelian groups defined on E are of the form $(E = \{e_\alpha, e_\beta, e_\omega\}, \oplus, e_\alpha)$, where $\oplus$ is defined by the table*

$\oplus$	e_α	e_β	e_ω
e_α	e_α	e_β	e_ω
e_β	e_β	e_ω	e_α
e_ω	e_ω	e_α	e_β

and they are isomorphic to $(\mathbb{Z}_3, +, 0)$.

Proof. The mapping $e_\alpha \mapsto 0,\ e_\beta \mapsto 2,\ e_\omega \mapsto 1$ establishes the isomorphism with $\mathbb{Z}_3$; the routine proof is left to the reader. It remains to prove that any Abelian group on E is of this form.

Let e_α denote the zero of an Abelian group on E. Then

$$e_\alpha \oplus e_\alpha = e_\alpha,\ e_\alpha \oplus e_\beta = e_\beta \oplus e_\alpha = e_\beta,\ e_\alpha \oplus e_\omega = e_\omega \oplus e_\alpha = e_\omega\ .$$

Since $e_\beta \oplus e_\omega = e_\beta \Longrightarrow e_\omega = e_\alpha$ and $e_\beta \oplus e_\omega = e_\omega \Longrightarrow e_\beta = e_\alpha$, it follows that

$$e_\beta \oplus e_\omega = e_\omega \oplus e_\beta = e_\alpha.$$

Since the rows and the columns of the Cayley table of the group are permutations of $(e_\alpha, e_\beta, e_\omega)$, it follows from $e_\beta \oplus e_\alpha = e_\beta$ and $e_\beta \oplus e_\omega = e_\alpha$ that $e_\beta \oplus e_\beta = e_\omega$, while $e_\omega \oplus e_\alpha = e_\omega$ and $e_\omega \oplus e_\beta = e_\alpha$ imply $e_\omega \oplus e_\omega = e_\beta$. $\square$

Proposition 3. *The unitary rings $(E = \{e_\alpha, e_\beta, e_\omega\}, \oplus, \odot, e_\alpha, e_\omega)$ defined on E are of the following form: the operation $\oplus$ is given in Proposition 2, while $\odot$ is defined by the table*

$\odot$	e_α	e_β	e_ω
e_α	e_α	e_α	e_α
e_β	e_α	e_ω	e_β
e_ω	e_α	e_β	e_ω

These rings are fields isomorphic to $\mathbb{Z}_3$.

Proof. The mapping $e_\alpha \mapsto 0$, $e_\beta \mapsto 2$, $e_\omega \mapsto 1$ establishes the asserted isomorphism; the routine proof is again left to the reader. It remains to prove that any unitary ring on E is of this form.

Note that $x \odot e_\omega = e_\omega \odot x = x$ and $x \odot e_\alpha = e_\alpha \odot x = e_\alpha$ for all $x \in E$, while the operation $\oplus$ is given in Proposition 2 (in which the interchange of β and ω is immaterial).

Furthermore, $e_\beta \odot (e_\beta \oplus e_\omega) = e_\beta \odot e_\alpha = e_\alpha$ and since

$$e_\beta \odot e_\beta = e_\alpha \Longrightarrow e_\beta \odot e_\beta \oplus e_\beta \odot e_\omega = e_\alpha \oplus e_\beta = e_\beta \neq e_\alpha \ ,$$

$$e_\beta \odot e_\beta = e_\beta \Longrightarrow e_\beta \odot e_\beta \oplus e_\beta \odot e_\omega = e_\beta \oplus e_\beta = e_\omega \neq e_\alpha \ ,$$

it follows that $e_\beta \odot e_\beta = e_\omega$. □

Theorem 2. *The commutative unitary rings defined on a Post algebra P of order 3 by simple Post functions are of the form $(P, \oplus, \odot, e_\alpha, e_\omega)$, where*

$$\begin{aligned} x \oplus y = {} & e_\alpha(x^\alpha y^\alpha \vee x^\beta y^\omega \vee x^\omega y^\beta \vee e_\beta(x^\omega y^\omega \vee x^\alpha y^\beta \vee x^\beta y^\alpha) \\ & \vee e_\omega(x^\beta y^\beta \vee x^\alpha y^\omega \vee x^\omega y^\alpha) \ , \end{aligned} \tag{18}$$

$$\begin{aligned} x \odot y = {} & e_\alpha(x^\alpha y^\alpha \vee x^\alpha y^\beta \vee x^\beta y^\alpha \vee x^\alpha y^\omega \vee x^\omega y^\alpha) \\ & \vee e_\beta(x^\beta y^\omega \vee x^\omega y^\beta) \vee e_\omega(x^\beta y^\beta \vee x^\omega y^\omega) \ . \end{aligned} \tag{19}$$

Comment. The ring explicitly given by Serfati [**7**] in the case $r = 3$ is obtained for $(\alpha, \beta, \omega) := (0, 2, 1)$, hence $(e_\alpha, e_\beta, e_\omega) = (0, 1, e)$:

$$x \oplus_S y = x^1 y^1 \vee x^0 y^2 \vee x^2 y^0 \vee e(x^2 y^2 \vee x^0 y^1 \vee x^1 y^0) \ ,$$

$$x \odot_S y = x^2 y^1 \vee x^1 y^2 \vee e(x^2 y^2 \vee x^1 y^1) \ .$$

Proof. It follows by Theorem 1 that the sought rings are of the form (13), (14), where the operations $\oplus$ and $\odot$ on E are given in Proposition 3. This yields formulae (18) and (19). □

Corollary 6. *Each of the rings described in Proposition 3 and Theorem 2 is of characteristic 3 and 3-potent.*

Proof. By Proposition 3 and Corollary 3, since the rings in Theorem 2 are obtained by the construction in Theorem 1. □

As explained in the previous section, we wish that the zero and the one of the rings coincide with those of the Post algebra.

Theorem 3. *The unique ring of the form $(P, \oplus, \odot, 0, 1)$ defined on P by simple Post functions is given by*

$$x \oplus y = e(x^2 y^2 \vee x^0 y^1 \vee x^1 y^0) \vee x^1 y^1 \vee x^0 y^2 \vee x^2 y^0 \ , \tag{20}$$

$$x \odot y = e(x^1 y^2 \vee x^2 y^1) \vee x^1 y^1 \vee x^2 y^2 \ , \tag{21}$$

and its restriction on E is the field $\mathbb{Z}_3$ where $2 = e$.

Proof. According to Theorem 1, the zero and one of P coincide with those of E. So we apply Theorem 2 with $e_\alpha := 0 = e_0$ and $e_\omega := 1 = e_2$, hence

$\oplus$	0	e	1
0	0	e	1
e	e	1	0
1	1	0	e

$\odot$	0	e	1
0	0	0	0
e	0	1	e
1	0	e	1

$e_\beta = e_1 = e$. We thus obtain the tables above, which define $\mathbb{Z}_3$, while formulae (18) and (19) reduce to (20) and (21). $\square$

Remark. Moisil [**3**] determined a ring structure on every centred 3-valued Lukasiewicz-Moisil algebra. The centred Lukasiewicz-Moisil algebras coincide with Post algebras; cf. [**1**, Corollary 4.1.9]. The ring in Theorem 3 coincides with the ring found by Moisil, after the translation of the latter into the Post algebra language.

Corollary 7. *The ring in Theorem 3 is unitary, commutative, of characteristic 3 and 3-potent.*

Proof. By Corollary 6. $\square$

Corollary 8. *The restriction of the ring in Theorem 3 to $B(P)$ is not the symmetric difference.*

Proof. By Proposition 1, since $1 \oplus 1 = e$. $\square$

Theorem 4. *The ring in Theorem 3 satisfies the identities*

$$x \vee y = (e \odot x \odot x \odot y \odot y) \oplus (x \odot x \odot y) \oplus (x \odot y \odot y) \oplus (x \odot y) \oplus x \oplus y\ , \tag{22}$$

$$xy = (x \odot x \odot y \odot y) \oplus (e \odot x \odot x \odot y) \oplus (e \odot x \odot y \odot y) \oplus (e \odot x \odot y)\ , \tag{23}$$

$$x^0 = (e \odot x \odot x) \oplus 1\ , \tag{24}$$

$$x^1 = (e \odot x \odot x) \oplus x\ , \tag{25}$$

$$x^2 = (e \odot x \odot x) \oplus (e \odot x)\ , \tag{26}$$

and it is equivalent to the Post algebra, as defined in the Introduction.

Proof. In view of the Verification Theorem (9), it suffices to prove (22)–(26) for the Post algebra E, in which $x \vee y = \max(x, y)$ and $xy = \min(x, y)$. We begin with (22) and (23).

If $x = 0$ or $y = 0$ this is readily checked.

If $x = y$ the right-hand sides of (22) and (23) are

$$(e \odot x \odot x \odot x \odot x) \oplus x \oplus x \oplus (x \odot x) \oplus x \oplus x = ((e \oplus 1) \odot x \odot x) \oplus x = x,$$

$$(x \odot x) \oplus (e \odot x) \oplus (e \odot x) \oplus (e \odot x \odot x) = ((1 \oplus e) \odot x \odot x) \oplus ((e \oplus e) \odot x) = x.$$

If $x = 1$, $y = e$ the right-hand sides of (22) and (23) are

$$(e \odot 1 \odot 1) \oplus (1 \odot e) \oplus (1 \odot 1) \oplus (1 \odot e) \oplus 1 \oplus e = e \oplus e \oplus 1 \oplus e \oplus 1 \oplus e = e \oplus e = 1,$$

$$(1 \odot 1) \oplus (e \odot 1 \odot e) \oplus (e \odot 1 \odot 1) \oplus (e \odot 1 \odot e) = 1 \oplus 1 \oplus e \oplus 1 = e,$$

and similarly for $x = e$, $y = 1$.

Formulae (24)–(26) are readily checked for $x = 0, e, 1$,

Since $e = 1 \oplus 1$, the polynomials (22)–(26) and the constant polynomials 0, e, 1 are term functions of the ring $(P, \oplus, \odot, 0, 1)$. Hence formulae (20)–(26) establish the desired equivalence. □

Remark. Moisil [**3**] proved that formulae (22), (23) with $e = 2$ make $\mathbb{Z}_3$ into a 3-valued centred Lukasiewicz(-Moisil) algebra and sketched the proof of the same result for any 3-ring, i.e., any ring satisfying $x \odot x \odot x = x$ and $x \oplus x = 0$.

5. Conclusions

In a subsequent paper we will tackle the functorial aspect of this equivalence.

The cases $r \geq 4$ remain open.

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