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ANALYSIS TOOLS FOR FINITE VOLUME SCHEMES

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Abstract. This paper is devoted to a review of the analysis tools which have been developed for the the mathematical study of cell centred finite volume schemes in the past years. We first recall the general principle of the method and give some simple examples. We then explain how the analysis is performed for elliptic equations and relate it to the analysis of the continuous problem; the lack of regularity of the approximate solutions is overcome by an estimate on the translates, which allows the use of the Kolmogorov theorem in order to get compactness. The parabolic case is treated with the same technique. Next we introduce a co-located scheme for the incompressible Navier–Stokes equations, which requires the definition of some discrete derivatives. Here again, we explain how the continuous estimates can guide us for the discrete estimates. We then give the basic ideas of the convergence analysis for non linear hyperbolic conservation laws, and conclude with an overview of the recent domains of application.

1. Introduction

Finite volume methods (FVM) are known to be well suited for the discretisation of conservation laws; these conservation laws may yield partial differential equations (PDE's) of different nature (elliptic, parabolic or hyperbolic) and also to coupled systems of equations of different nature. Consequently, the functional spaces in which the solutions of the continuous problems are sought may be quite different: H_0^1, $L^2(0,T,H^1)$, $L^\infty\dots$, so that it might seem rash to think of approximating them all equally with piecewise constant functions, as with the cell centred FVM considered here; indeed, even though it seems natural that the space L^∞ should be approximated by the discrete space consisting of piecewise constant functions on the control volumes, this is no longer the case when the continuous functional space is H_0^1. Surprisingly, the cell centred approximation is quite efficient even in the case of elliptic and parabolic equations, as a number of works have proved in the past fifteen years. Indeed, analysis tools have been developed for all types of equations, most of them adapted from tools used in the study of the respective continuous partial differential equations. The unified theory of these discrete analysis tools, which was initiated in the late 80's, allows to tackle the numerical analysis of the discretisation of more complex systems. The aim of the present paper is to give a

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unified presentation of the cell centred FVM analysis for different types of PDE's, and give a review of the main analysis tools which were developed for different model problems, and relate them to their continuous counterparts.

The first question that is often asked by a layman is: what is the difference between finite volumes and, say, finite elements or finite differences? The answer truly lies in the concepts of the methods, but indeed, in some cases, these methods yield similar schemes (this may be seen on the simple example $-u'' = f$ discretized by the three above mentioned methods with a constant mesh step). The concepts, however, are quite different. Roughly speaking, one could say that the finite element method is based on a weak formulation coupled with a convenient approximation of the functional spaces while the finite difference method relies on an approximation of the original differential operators by Taylor expansions; and the finite volume method is constructed from a balance equation, rather than the PDE itself, with a consistent approximation of the fluxes defined on the boundary of the control volumes.

Confusion between the finite volume method and the finite difference method arises from the fact that the FVM has often been called finite differences methods when the flux on the boundary of the control volumes are approximated by finite differences. This is the case, for instance, in oil reservoir simulations, where rectangular cartesian grids are used, so that the diffusion flux can easily be dicretised by a differential quotient, at least in the isotropic case. Moreover, numerous schemes which have been designed for hyperbolic equations and systems, and cast in the finite difference family, are also of the finite volume type, since they are based on a suitable approximation of the fluxes at the interfaces of the discretisation cells. Links between the FVM and the finite element method (FEM) can also be mentioned. Indeed, for particular problems, the FVM may be written as a FEM with some particular integration rule. Conversely, there are cases where the FEM can be seen as a FVM. For instance, the piecewise linear finite element method for the discretisation of the Laplace operator on a triangular mesh satisfying the weak Delaunay condition yields a matrix which is the same as that of the FVM on the dual Voronoï mesh, see [**38**] for details. The FVM may also be seen as a discontinuous Galerkin method (DGM) of lowest order; although the DGM, derived from the finite element ideas, is also based on a weak formulation, the approximation of the continuous space is no longer conforming, as is also the case in the cell centred FVM. However, the tools used to analyse the DGM of higher order do not seem to apply to the FVM. Let us also mention that other families of FVM's have been developed, such as vertex centered schemes, box or co–volume schemes, finite volume element methods: see [**6, 3, 15, 23, 33, 68, 26, 58, 59**] and references therein. Our interest for cell centred schemes is primarily motivated by the fact that they are probably the most widely used in industrial codes.

The outline of this paper is as follows. In Section 2, we shall give the principle of the cell centred FVM for general conservation laws. Section 3 is devoted to the convergence analysis of the FVM approximations for steady state convection diffusion equations. We show that one of the key ingredients is an estimate on the translates of the approximate solutions, which allows the use of the Kolmogorov theorem. Time dependent convection diffusion problems are then tackled in Section 4, where estimates on the time translates are also developed. Sections 5 and 6 are devoted to more recent works on the incompressible Stokes and Navier-Stokes equations. Discrete derivatives are introduced to handle the gradient and divergence terms. In Section 7, we give the main ideas which lie behind the (difficult) analysis of cell centred FVM's for hyperbolic equations. Finally we conclude in Section 8 by mentioning the different problems which have been studied in the past, along with some of the ongoing works.

2. Principle of the finite volume method

Let Ω be a polygonal open subset of $\mathbb{R}^d$, $T \in \mathbb{R}$, and let us consider a balance law written under the general form:

$$u_t + \operatorname{div}(F(u, \nabla u)) + s(u) = 0 \qquad \text{on } \Omega \times (0, T), \tag{1}$$

where $F \in C^1(\mathbb{R} \times \mathbb{R}^d, \mathbb{R})$ and $s \in C(\mathbb{R}, \mathbb{R})$. Let $\mathcal{T}$ be a finite volume mesh of Ω. For the time being, we shall only assume that $\mathcal{T}$ is a collection of convex polygonal control volumes K, disjoint one to another, and such that: $\bar{\Omega} = \cup_{K \in \mathcal{T}} \bar{K}$. The balance equation is obtained from the above conservation law by integrating it over a control volume K and applying the Stokes formula:

$$\int_K u_t \, \mathrm{d}x + \int_{\partial K} F(u, \nabla u) \cdot \mathbf{n}_K \, \mathrm{d}\gamma(x) + \int_K s(u) \, \mathrm{d}x = 0,$$

where $\mathbf{n}_K$ stands for the unit normal vector to the boundary ∂K outward to K and γ denotes the integration with respect to the $(d-1)$–dimensional Lebesgue measure. Let us denote by $\mathcal{E}$ the set of edges (faces in 3D) of the mesh, and $\mathcal{E}_K$ the set of edges which form the boundary ∂K of the control volume K. With these notations, the above equation reads:

$$\int_K u_t \, \mathrm{d}x + \sum_{\sigma \in \mathcal{E}_K} \int_\sigma F(u, \nabla u) \cdot \mathbf{n}_K \, \mathrm{d}\gamma(x) + \int_K s(u) \, \mathrm{d}x = 0.$$

Let $k = T/M$, where $M \in \mathbb{N}, M \geq 1$, and let us perform an explicit Euler discretization of the above equation (an implicit or semi-implicit discretization could also be performed, and is sometimes preferable, depending on the type of equation). We then get:

$$\int_K \frac{u^{(m+1)} - u^{(m)}}{k} \, \mathrm{d}x \; + \sum_{\sigma \in \mathcal{E}_K} \int_\sigma F(u^{(m)}, \nabla u^{(m)}) \cdot \mathbf{n}_K \, \mathrm{d}\gamma(x) + \int_K s(u^{(m)}) \, \mathrm{d}x = 0,$$

where $u^{(m)}$ denotes an approximation of $u(\cdot, t^{(m)})$, with $t^{(m)} = mk$. Let us then introduce the discrete unknowns (one per control volume and time step) $(u_K^{(m)})_{K \in \mathcal{T},\ m \in \mathbb{N}}$; assuming the existence of such a set of real values, we may define a piecewise constant function by:

$$u_{\mathcal{T}}^{(m)} \in H_{\mathcal{T}}(\Omega) : \; u_{\mathcal{T}}^{(m)} = \sum_{K \in \mathcal{T}} u_K^{(m)} 1_K,$$

where $H_{\mathcal{T}}(\Omega)$ denotes the space of functions from Ω to $\mathbb{R}$ which are constant on each control volume of the mesh $\mathcal{T}$, and 1_K the characteristic function of K, that is $1_K(x) = 1$ if $x \in K$, $1_K(x) = 0$ otherwise. In order to define the scheme, the fluxes $\int_\sigma F(u^{(m)}, \nabla u^{(m)}) \cdot \mathbf{n}_K \, \mathrm{d}\gamma(x)$ need to be approximated as a function of the discrete unknowns. We denote by $F_{K,\sigma}(u_{\mathcal{T}}^{(m)})$ the resulting numerical flux, the expression of which depends on the type of flux to be approximated. Let us now give this expression for various simple examples.

First we consider the case of a linear convection equation, that is equation (1) where the flux $F(u, \nabla u)$ reduces to $F(u, \nabla u) = \mathbf{v}u$, $\mathbf{v} \in \mathbb{R}^d$, and $s(u) = 0$:

$$u_t + \operatorname{div}(\mathbf{v}u) = 0 \qquad \text{on } \Omega. \tag{2}$$

In order to approximate the flux $\mathbf{v}u \cdot \mathbf{n}$ on the edges of the mesh, one needs to approximate the value of u on these edges, as a function of the discrete unknowns u_K associated to each control volume K. This may be done in several ways. A straightforward choice is to approximate the value of u on the edge $\sigma = \sigma_{KL}$ separating the control volumes K and L by the mean value $\frac{1}{2}(u_K + u_L)$. This yields the following numerical flux:

$$F_{K,\sigma}^{(cv,c)}(u_{\mathcal{T}}) = v_{K,\sigma} \frac{u_K + u_L}{2}$$

where $v_{K,\sigma} = \int_\sigma \mathbf{v} \cdot \mathbf{n}_{K,\sigma}$, and $\mathbf{n}_{K,\sigma}$ denotes the unit normal vector to the edge σ outward to K. This centred choice is known to lead to stability problems, and is therefore often replaced by the so–called upstream choice, which is given by:

$$F_{K,\sigma}^{(cv,u)}(u_{\mathcal{T}}) = v_{K,\sigma}^{+} u_K - v_{K,\sigma}^{-} u_L, \tag{3}$$

where $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$.

If we now consider a linear convection diffusion reaction equation, that is equation (1) with $F(u, \nabla u) = -\nabla u + \mathbf{v}u, \mathbf{v} \in \mathbb{R}^d$, and $s(u) = bu$, $b \in \mathbb{R}$:

$$u_t - \Delta u + \operatorname{div}(\mathbf{v}u) + bu = 0 \qquad \text{on } \Omega, \tag{4}$$

the flux through a given edge then reads:

$$\int_\sigma F(u) \cdot \mathbf{n}_{K,\sigma} = \int_\sigma -\nabla u \cdot \mathbf{n}_{K,\sigma} + \mathbf{v} \cdot \mathbf{n}_{K,\sigma} \; u,$$

so that we now need to discretize the additional term $\int_\sigma -\nabla u \cdot \mathbf{n}_{K,\sigma}$; this diffusion flux involves the normal derivative to the boundary, for which a possible discretization is obtained by considering the differential quotient between the value of $u_{\mathcal{T}}$ in K and in the neighbouring control volume, let say L:

$$F_{K,\sigma}^{(d)}(u_{\mathcal{T}}) = -\frac{|\sigma|}{d_{KL}}(u_L - u_K). \tag{5}$$

where $|\sigma|$ stands for the $(d-1)$–dimensional Lebesgue measure of σ and d_{KL} is the distance between some points of K and L, which will be defined further. Using the above upstream scheme (3) for the convective part of the scheme, we then obtain the following numerical flux:

$$F_{K,\sigma}^{(cvd)}(u_{\mathcal{T}}) = -\frac{|\sigma|}{d_{KL}}(u_L - u_K) + v_{K,\sigma}^{+} u_K - v_{K,\sigma}^{-} u_L.$$

However, we are able to prove that this choice for the discretization of the diffusion flux yields accurate results only if the mesh satisfies the so-called orthogonality condition, that is, there exists a family of points $(x_K)_{K\in\mathcal{T}}$, such that for a given edge σ_{KL}, the line segment $x_K x_L$ is orthogonal to this edge (see Figure 1). The length d_{KL} is then defined as the distance between x_K and x_L. This geometrical feature of the mesh will be exploited to prove the consistency of the flux, a notion which is detailed in the next section. Of course, this orthogonality condition is not satisfied for any mesh. Such a family of points exists for instance in the case of triangles, rectangles or Voronoï meshes. We refer to [**38**] for more details.

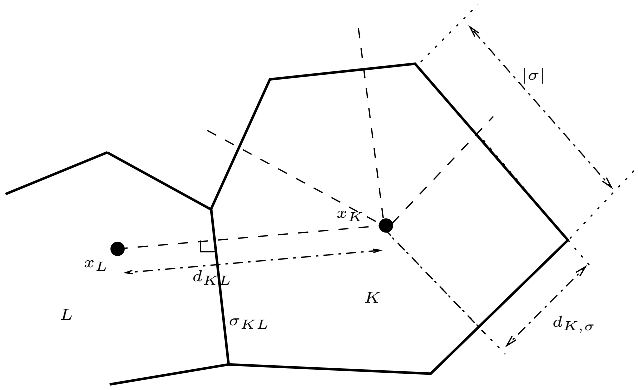

Figure 1. Notations for a control volume.

3. Convergence analysis for the steady state reaction convection diffusion equation

3.1. The continuous and discrete problems

Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, $f \in L^2(\Omega)$, $\mathbf{v} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, and let us consider the following steady–state linear reaction convection diffusion equation:

$$-\Delta u + \operatorname{div}(\mathbf{v}u) + bu = f \qquad \text{on } \Omega, \tag{6}$$

with homogeneous boundary conditions on $\partial\Omega$. A weak formulation of this problem is:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla\phi \,\mathrm{d}x + \int_\Omega \operatorname{div}(\mathbf{v}u)\phi \,\mathrm{d}x + \int_\Omega bu\phi \,\mathrm{d}x = \int_\Omega f\phi \,\mathrm{d}x, \\ \forall \phi \in H_0^1(\Omega). \end{cases} \tag{7}$$

Let $(\mathcal{T},\mathcal{E},\mathcal{P})$ be a discretization of Ω: $\mathcal{T}$ denotes the set of control volumes, $\mathcal{E}$ the set of edges of the mesh, $\mathcal{P}$ the set of points satisfying the above mentioned orthogonality condition. The finite volume scheme may be written under the following weak form:

$$\left\{\begin{array}{l}\text{Find } u_{\mathcal{T}} \in H_{\mathcal{T}}(\Omega) \text{ such that}\\ [u_{\mathcal{T}},\phi]_{\mathcal{T}} + c_{\mathcal{T}}(u_{\mathcal{T}},\phi) + \displaystyle\int_\Omega b u_{\mathcal{T}}\phi \,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x, \quad \forall \phi \in H_{\mathcal{T}}(\Omega).\end{array}\right. \tag{8}$$

where:

1. $H_{\mathcal{T}}(\Omega)$ is the space of piecewise constant functions on the control volumes of $\mathcal{T}$,
2. the inner product $[\cdot,\cdot]_{\mathcal{T}}$ is defined by:
$$[u,v]_{\mathcal{T}} = \sum_{\sigma_{KL}\in\mathcal{E}_{\mathrm{int}}} \frac{|\sigma_{KL}|}{d_{KL}}(u_L-u_K)(v_L-v_K) + \sum_{\sigma\in\mathcal{E}_{\mathrm{ext}}} \frac{|\sigma|}{d_{K,\sigma}}\, u_K\, v_K,$$
where $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$, $\mathcal{E}_K$) denotes the set of edges included in Ω (resp. $\partial\Omega$, ∂K), $|\sigma|$ the $(d-1)$–dimensional Lebesgue measure of σ, d_{KL} the distance between x_K and x_L (see Figure 1) and $d_{K,\sigma}$ the distance between x_K and σ; in the first summation, σ_{KL} denotes the edge separating the control volumes K and L, and in the last summation, the volume K is the unique volume of which σ is an edge.
3. the bilinear convective form is defined by:
$$c_{\mathcal{T}}(u_{\mathcal{T}},\phi) = \sum_{K\in\mathcal{T}} \phi_K \left[\sum_{\sigma_{KL}\in\mathcal{E}_K} (v^+_{K,\sigma_{KL}} u_K - v^-_{K,\sigma_{KL}} u_L) + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} v^+_{K,\sigma} u_K\right].$$

The finite volume scheme may equivalently be written under its more classical flux form:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u_{\mathcal{T}}) + b|K|u_K = |K| f_K, \qquad \forall K \in \mathcal{T}, \tag{9}$$

where $|K|$ denotes the d dimensional Lebesgue measure of K and:

$$F_{K,\sigma}(u_{\mathcal{T}}) = \left\{\begin{array}{ll} -\dfrac{|\sigma|}{d_{KL}}(u_L-u_K) + v^+_{K,\sigma}u_K - v^-_{K,\sigma}u_L, & \text{if } \sigma=\sigma_{KL},\\ -\dfrac{|\sigma|}{d_{KL}}(-u_K) + v^+_{K,\sigma}u_K, & \text{if } \sigma \text{ is an edge of K located on } \partial\Omega.\end{array}\right. \tag{10}$$

Indeed, taking $\phi = 1_K$ in (8), it is easily seen that (8) implies (9). Conversely, let $\phi \in H_{\mathcal{T}}(\Omega)$. Multiplying (9) by ϕ_K, summing the resulting equations for all $K\in\mathcal{T}$ and reordering the summations leads to (8).

One may also define a discrete Laplace operator in $H_{\mathcal{T}}$ in the following way. For $v\in H_{\mathcal{T}}$, let $\Delta_{\mathcal{T}} v \in H_{\mathcal{T}}$ be defined by:

$$(\Delta_{\mathcal{T}} v)_K = -\frac{1}{|K|}\sum_{\sigma\in\mathcal{E}_K} F^{(d)}_{K,\sigma}(v),$$

where:

$$F^{(d)}_{K,\sigma}(v) = \begin{cases} -\dfrac{|\sigma|}{d_{KL}}(v_L - v_K) & \text{if } \sigma = \sigma_{KL}, \\ -\dfrac{|\sigma|}{d_{KL}}(-v_K) & \text{if } \sigma \subset \partial\Omega. \end{cases} \tag{11}$$

Then one may remark that, thanks to the property of conservativity of the flux (that is $F_{K,\sigma} = -F_{L,\sigma}$ if $\sigma = \sigma_{KL}$), one has:

$$[u, v]_{\mathcal{T}} = -\int_\Omega \Delta_{\mathcal{T}} u \; v \,\mathrm{d}x = -\int_\Omega u \; \Delta_{\mathcal{T}} v \,\mathrm{d}x, \qquad \forall\, u, v \in H_{\mathcal{T}}(\Omega). \tag{12}$$

The formulation (8) highlights a property of finite volume schemes for elliptic equations, namely the fact that, as Galerkin methods, they may be derived from a coercive variational formulation. However, because of the non-conforming nature of finite volumes, going further in the analogy with Galerkin methods does not seem to be of practical interest: the coercivity of the formulation is not inherited from the coercivity of the continuous problem but rather stems from the conservativity of the fluxes; even if the convergence of the method is proven by an analogue of the second Strang lemma, classical in the finite element framework, it relies *in fine* on the consistency of the fluxes, at least in the presently available analyses.

Note that, thanks to the following Poincaré inequality which holds for $u \in H_{\mathcal{T}}$ (see e.g. [**38**, Lemma 9.1]):

$$\|u\|_{L^2(\Omega)} \le \mathrm{diam}(\Omega)\; ||u||_{1,\mathcal{T}}, \tag{13}$$

we may define a mesh dependent "discrete H^1_0 norm" using the inner product introduced above:

(14)

$$\|u\|_{1,\mathcal{T}} = ([u,u]_{\mathcal{T}})^{1/2} = \left(\sum_{\sigma_{KL} \in \mathcal{E}_{\text{int}}} \frac{|\sigma|}{d_{KL}} (u_L - u_K)^2 + \sum_{\sigma \in \mathcal{E}_{\text{ext}}} \frac{|\sigma|}{d_{K,\sigma}}\; u_K^2 \right)^{1/2}.$$

3.2. Convergence results

The mathematical analysis of any numerical scheme must address the question of existence of a solution, which is rather easy here since the problem is linear, and the question of convergence (i. e. "does the approximate solution converge to the solution of the continuous problem as the mesh size tends to 0?"). A related question is the obtention of a rate of convergence, through error estimates, usually conditionned to regularity assumptions on the continuous solution. The proof of the convergence of the finite volume scheme for a semi-linear equation generalizing (6) was first proven in [**37**] (see also [**38**]). We shall state the result here for the linear case, and explain the main steps of the proof, since the presented techniques extend to nonlinear problems.

Under the assumptions given at the beginning of this section, it is easily seen that the system (8) (resp. (9)) has a unique solution $u_{\mathcal{T}} \in H_{\mathcal{T}}$ (resp. $(u_K)_{K\in\mathcal{T}}$).

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of finite volume discretizations satisfying the orthogonality condition, and let $h_{\mathcal{T}_n}$ be the size of the mesh $\mathcal{T}_n$, that is the maximum of the diameters of the control volumes of $\mathcal{T}_n$. We suppose that $h_{\mathcal{T}_n} \to 0$ as $n \to +\infty$ and we are going to show that, in this case, the corresponding sequence $(u_{\mathcal{T}_n})_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to the unique solution of (7). The proof of this result may be decomposed into four steps:

1. We first get some *a priori* estimates on the approximate solution in the $H_{\mathcal{T}}$ norm and the L^2 norm which yield existence (and uniqueness) of $u_{\mathcal{T}}$ solution of the scheme. We can then also deduce the weak convergence of $(u_{\mathcal{T}_n})_{n\in\mathbb{N}}$ in $L^2(\Omega)$, up to a subsequence, to some $\bar{u} \in L^2(\Omega)$.
2. Strong convergence and regularity of the limit, that is $\bar{u} \in H^1_0(\Omega)$, are obtained through a kind of discrete Rellich theorem, which we shall describe hereafter.
3. The fact that the limit $\bar{u}$ is a weak solution of the continuous problem is obtained by a passage to the limit in the scheme (as $h_{\mathcal{T}} \to 0$).
4. We then use a classical argument of uniqueness to show that the whole sequence converges.

Note that we do not need to assume the existence of the solution to the continuous problem: we get it as a by-product of the convergence of the scheme. In the present easy linear case, this is quite useless, since existence is well-known. For more complicated nonlinear problems, obtaining the existence of the solution *via* the convergence of the numerical scheme may come in handy (see e.g. [**9**]).

These four steps will be detailed in the following paragraphs for the pure diffusion operator, for the sake of simplicity. We also sketch the proof of order h convergence in L^2 and $H_{\mathcal{T}}$ norms, under regularity conditions on the solution, namely $u \in H^2(\Omega)$. Note that the upstream scheme for the convection flux does not lead to any additional difficulty, see [**37, 53**].

Order 2 convergence in the L^2 norm may be proven for the pure diffusion operator on uniform grids. However, the same result on triangular meshes, which is observed in numerical experiments, remains an open problem; recall that higher convergence rates in weaker norms (including this special case) are known and proven for most Galerkin methods via duality arguments (the so-called Aubin-Nitsche lemma, [**24**]).

3.3. A priori estimate

Definition 1 (Discrete H^{-1} norm)**.** Let ψ be a function of $L^2(\Omega)$, then

$$\|\psi\|_{-1,\mathcal{T}} = \sup_{v\in H_{\mathcal{T}}(\Omega), v\neq 0} \frac{\displaystyle\int_\Omega \psi v \,\mathrm{d}x}{\|v\|_{1,\mathcal{T}}}. \tag{15}$$

Note that, by the discrete Poincaré inequality (13), we have:

$$\|\psi\|_{-1,\mathcal{T}} \leq \mathrm{diam}(\Omega)\ \|\psi\|_{L^2(\Omega)}.$$

Assuming $\mathbf{v} = 0$ and $b = 0$ and using the notation (8), the finite volume scheme reads:

$$[u_{\mathcal{T}}, v]_{\mathcal{T}} = \int_{\Omega} fv \, \mathrm{d}x, \qquad \forall v \in H_{\mathcal{T}}(\Omega).$$

Choosing $v = u_{\mathcal{T}}$, we get by definition (1):

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}} \leq \|f\|_{-1,\mathcal{T}}. \tag{16}$$

Taking $f = 0$, we thus obtain uniqueness (and therefore existence) of the discrete solution. This estimate also yields weak convergence of a subsequence of approximate solutions in $L^2(\Omega)$.

3.4. Convergence theorem

In order to prove strong convergence of the approximate solutions, we need some control on their oscillations. In the finite element framework, the family of approximate solutions is bounded in $H^1(\Omega)$, and one may therefore use the Rellich theorem to obtain compactness in $L^2(\Omega)$. This is not the case here, but we note that the Rellich theorem derives from the Kolmogorov theorem, which gives a necessary and sufficient condition for a bounded family of $L^p(\Omega)$, $p < +\infty$, to be relatively compact. Because of the lack of regularity of our approximate solutions, the Kolmogorov theorem is an adequate tool. In order to use it, we need some estimates on the translates of functions of $H_{\mathcal{T}}(\Omega)$. Indeed, one may show, in a way which is close to that of the continuous case (replacing the derivatives by differences) that for any function $v \in H_{\mathcal{T}}(\Omega)$, one has:

$$\|v(\cdot + \eta) - v\|^2_{L^2(\Omega)} \leq |\eta| \ (|\eta| + 4h_{\mathcal{T}}))\, \|v\|^2_{1,\mathcal{T}}, \qquad \forall \eta \in \mathbb{R}^d.$$

From this estimate, we may deduce the following result.

Theorem 1 (Discrete Rellich theorem)**.** *Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of finite volume discretizations satisfying the orthogonality condition, such that $h_{\mathcal{T}_n} \to 0$, and let $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ such that $u_n \in H_{\mathcal{T}_n}$ and $\|u_n\|_{1,\mathcal{T}_n} \leq C$, where $C \in \mathbb{R}$. Then there exists a subsequence $(u_n)_{n\in\mathbb{N}}$ and $\overline{u} \in H^1_0(\Omega)$ such that $u_n \to \overline{u}$ in $L^2(\Omega)$ as $n \to +\infty$.*

From the discrete H^1 estimate (16), we then deduce from the above theorem the strong convergence of a subsequence of the approximate solutions in $L^2(\Omega)$, to some function $\bar{u} \in H^1_0(\Omega)$.

3.5. Passage to the limit in the scheme

We now need to show that the limit $\bar{u}$ is solution to the continuous variational problem. Let $(\mathcal{T}_n)$ be a sequence of discretizations such that $h_{\mathcal{T}_n} \to 0$. For each mesh $\mathcal{T}_n$, the finite volume scheme reads:

$$[u_{\mathcal{T}_n}, v]_{\mathcal{T}_n} = \int_{\Omega} fv \, \mathrm{d}x, \qquad \forall v \in H_{\mathcal{T}_n}(\Omega). \tag{17}$$

Lemma 1 (Consistency of the discrete Laplace operator)**.** *Let $\mathcal{T}$ be a finite volume mesh satisfying the orthogonality condition. We denote by $P_{\mathcal{T}}$ and $\Pi_{\mathcal{T}}$ the following interpolation operators:*

$$P_{\mathcal{T}} \ : \ C(\Omega) \to H_{\mathcal{T}}(\Omega), \quad P_{\mathcal{T}}\varphi(x) = \varphi(x_K), \qquad \forall x \in K, \quad \forall K \in \mathcal{T}, \tag{18}$$

$$\Pi_{\mathcal{T}} \ : \ L^2(\Omega) \to H_{\mathcal{T}}(\Omega), \quad \Pi_{\mathcal{T}}\varphi(x) = \frac{1}{|K|}\int_K \varphi \ \mathrm{d}x, \quad \forall x \in K, \quad \forall K \in \mathcal{T}. \tag{19}$$

For $\varphi \in C_c^\infty(\Omega)$, let us define the consistency error $R_{\Delta,\mathcal{T}}(\varphi) \in H_{\mathcal{T}}(\Omega)$ on the discrete Laplace operator by:

$$R_{\Delta,\mathcal{T}}(\varphi) = \Delta_{\mathcal{T}} P_{\mathcal{T}}\varphi - \Pi_{\mathcal{T}}(\Delta\varphi).$$

Then there exists C_φ depending only on φ such that:

$$\|R_{\Delta,\mathcal{T}}(\varphi)\|_{-1,\mathcal{T}} \le C_\varphi h_{\mathcal{T}}. \tag{20}$$

Proof. For $\varphi \in C_c^\infty(\Omega)$, one has:

$$\|R_{\Delta,\mathcal{T}}(\varphi)\|_{-1,\mathcal{T}} = \sup_{v \in H_{\mathcal{T}}(\Omega), \|v\|_{1,\mathcal{T}}=1} X(v),$$

with:

$$X(v) = \sum_{K\in\mathcal{T}} |K| \left[(\Delta_{\mathcal{T}} P_{\mathcal{T}}\varphi)_K \, v_K - (\Pi_{\mathcal{T}}(\Delta\varphi))_K \, v_K \right].$$

For $h_{\mathcal{T}}$ small enough, φ vanishes in all the control volumes having an edge on the boundary of the domain so that, by definition of $\Delta_{\mathcal{T}}$, $P_{\mathcal{T}}$ and $\Pi_{\mathcal{T}}$, one has:

$$\begin{aligned} X(v) \ &= \sum_{K\in\mathcal{T}} v_K \left[\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(P_{\mathcal{T}}\varphi) - \int_\sigma \nabla\varphi \cdot \mathbf{n}_{K,\sigma} \ \mathrm{d}\gamma(x) \right] \\ &= \sum_{\sigma_{KL}\in\mathcal{E}_{\mathrm{int}}} |\sigma| \ R_{K,\sigma}(\varphi) \ (v_K - v_L), \end{aligned} \tag{21}$$

where $R_{K,\sigma}(\varphi)$ is the consistency error on the fluxes, defined by:

$$R_{K,\sigma}(\varphi) = \frac{1}{|\sigma|}\big(F_{K,\sigma}(P_{\mathcal{T}}\varphi) - \int_\sigma \nabla\varphi \cdot \mathbf{n}_{K,\sigma} \ \mathrm{d}\gamma(x)\big).$$

Now we use the property of consistency of the fluxes, namely that for a regular function φ, there exists $c_\varphi \in \mathbb{R}$ depending only on φ such that:

$$|R_{K,\sigma}(\varphi)| \le c_\varphi h_{\mathcal{T}}.$$

This result, proven in [**38**], is a central argument of the proof. It relies on the orthogonality condition for the mesh, and is obtained by Taylor's expansions. By the Cauchy–Schwarz inequality, we then obtain from (21) that:

$$X(v) \le C_\varphi h_{\mathcal{T}} \|v\|_{1,\mathcal{T}},$$

which concludes the proof. ☐

An immediate consequence is the following corollary.

Corollary 1. *Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a family of meshes satisfying the orthogonality property and such that $h_{\mathcal{T}_n} \to 0$. Let $(u_{\mathcal{T}_n})_{n\in\mathbb{N}} \subset L^2(\Omega)$ and $\bar{u} \in H^1(\Omega)$ such that $\|u_{\mathcal{T}_n}\|_{1,\mathcal{T}} \leq C$, where $C \in \mathbb{R}_+$, and $u_{\mathcal{T}_n} \to \bar{u}$ in $L^2(\Omega)$ as $n \to +\infty$, then:*

$$\int_\Omega u_{\mathcal{T}_n}\ \Delta_{\mathcal{T}_n}(P_{\mathcal{T}_n}\varphi)\,\mathrm{d}x \to \int_\Omega \bar{u}\ \Delta\varphi\ \mathrm{d}x \text{ as } n \to +\infty, \quad \forall\ \varphi \in C_c^\infty(\Omega).$$

We now sketch the proof of convergence of the scheme. Let us now take $v = P_{\mathcal{T}_n}\varphi$ in (17). Thanks to (12), we have:

$$-\int_\Omega u_{\mathcal{T}_n}\Delta_{\mathcal{T}_n}(P_{\mathcal{T}_n}\varphi)\ \mathrm{d}x = \int_\Omega f P_{\mathcal{T}_n}\varphi\ \mathrm{d}x.$$

Let us then pass to the limit as $n \to +\infty$. From Corollary 1 and the fact that the right hand side converges to $\int_\Omega \varphi\ \mathrm{d}x$, we get that:

$$-\int_\Omega \bar{u}\Delta\varphi\ \mathrm{d}x = \int_\Omega f\varphi\ \mathrm{d}x.$$

Since we know from the previous step that $\bar{u} \in H_0^1(\Omega)$, we obtain that $\bar{u}$ is indeed the solution to (7).

3.6. Error analysis

An error estimate for convection diffusion equations was first obtained in [**60**] in the case of continuous data and triangular meshes. It was extended to L^2 data and general admissible meshes and general boundary conditions in [**53**]. The key argument for the error analysis is the fact that the consistency Lemma (1) still holds, under regularity assumptions for the mesh, for ϕ in $H^2(\Omega)$. Using the variational form of the scheme (17), we have:

$$[u_{\mathcal{T}_n} - P_{\mathcal{T}_n}u, v]_{\mathcal{T}_n} = \int_\Omega fv\ \mathrm{d}x - [P_{\mathcal{T}_n}u, v]_{\mathcal{T}_n}, \qquad \forall v \in H_{\mathcal{T}_n}(\Omega),$$

where u is the solution to the continuous problem. Integrating the continuous equation $-\Delta u = f$ over each control volume to replace the first term of the right hand side of the above relation, we get:

$$[u_{\mathcal{T}_n} - P_{\mathcal{T}_n}u, v]_{\mathcal{T}_n} = \int_\Omega R_{\Delta,\mathcal{T}_n}(u)v\ \mathrm{d}x, \qquad \forall v \in H_{\mathcal{T}_n}(\Omega).$$

A first order convergence result in the $H_{\mathcal{T}}$ norm then follows by the stability estimate (16); first order convergence is also obtained in the L^2 norm, thanks to the discrete Poincaré inequality.

4. The parabolic case

4.1. The continuous problem

We now consider a transient convection diffusion equation. Let $T > 0$, $u_0 \in L^2(\Omega)$ and $\mathbf{v} \in \mathbb{R}^d$ be given; the partial derivative equation at hand reads:

$$\left\{\begin{array}{ll} u : \Omega \times [0,T] \to \mathbb{R}; & \\ u_t + \operatorname{div}(\mathbf{v}u) - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(.,0) = u_0 & \text{in } \Omega. \end{array}\right. \tag{22}$$

A weak formulation of this problem is:

$$\left\{\begin{array}{l} \text{Find } u \in L^2(0,T;\, H_0^1(\Omega)) \text{ such that } u_t \in L^2(0,T;\, H^{-1}(\Omega)) \text{ and} \\ \qquad < u_t, \varphi >_{H^{-1},H_0^1} + \displaystyle\int_\Omega \nabla u(x,\cdot)\ \nabla\varphi(x,\cdot)\,\mathrm{d}x = 0, \quad \forall\ \varphi \in H_0^1(\Omega), \\ \qquad u(\cdot,0) = u_0. \end{array}\right. \tag{23}$$

As in the steady state case, we shall use some estimates on the translates of u in order to get compactness properties, despite the lack of regularity of the approximate finite volume solutions. To get some insight into what kind of estimates we should be aiming at, it is informative to look at the estimates that can be obtained on the continuous solution. First, we see that since $u \in L^2(0,T;\, H_0^1(\Omega))$, we have the following estimate on the translates in space:

$$\|u(\cdot + \eta, \cdot) - u(\cdot,\cdot)\|_{L^2(0,T;\, L^2(\Omega))} \leq C|\eta|, \qquad \forall \eta \in \mathbb{R}^d.$$

Then, since $u \in L^2(0,T;\, H_0^1(\Omega))$ and $u_t \in L^2(0,T;\, H^{-1}(\Omega))$, the following estimate on the time translates holds:

$$\|u(\cdot, \cdot + \tau) - u(\cdot,\cdot)\|_{L^2(0,T;\, L^2(\Omega))} \leq C|\tau|^{\frac{1}{2}}, \qquad \forall \tau \in \mathbb{R}.$$

We shall therefore look for the same kind of estimates in the discrete framework.

4.2. The finite volume scheme

Let $k = 1/M$ be the (uniform) time step. The finite volume scheme, using an implicit Euler discretization in time, reads:

$$\left\{\begin{array}{l} |K|\dfrac{u_K^{n+1} - u_K^n}{k} + \displaystyle\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_{\mathcal{T}}^{n+1}) = 0, \qquad 0 \leq n \leq M-1, \\ u_K^0 = \dfrac{1}{|K|}\displaystyle\int_K u_0(x)\,\mathrm{d}x. \end{array}\right. \tag{24}$$

with $F_{K,\sigma}(u_{\mathcal{T}}^{n+1}) = -\dfrac{|\sigma|}{d_{KL}}(u_L^{n+1} - u_K^{n+1}) + v_{K,\sigma}^+ u_K^{n+1} - v_{K,\sigma}^- u_L^{n+1}$.

The existence and uniqueness of a solution $(u_K^n)_{n\in\mathbb{N}}$ to (24) is easily deduced from the steady state case. Let us denote by $H_{\mathcal{D}}(\Omega \times (0,T))$ the set of functions of $L^2(\Omega \times (0,T))$ which are piecewise constant on the subsets $K \times [t_n, t_{n+1})$. We define the approximate solution $u_{\mathcal{D}} \in H_{\mathcal{D}}(\Omega \times (0,T))$ by $u_{\mathcal{D}}(x,t) = u_K^n$, $\forall x \in K$, $\forall t \in [t_n, t_{n+1})$. Using a variational technique similar to the way the estimate (16) is established in the steady state case, the following *a priori* estimates on $u_{\mathcal{D}}$ may be obtained:

$$\|u_{\mathcal{D}}\|_{L^\infty(0,T;\,L^2(\Omega))} \le C, \qquad k \sum_{n=1}^{M} \|u_{\mathcal{D}}(\cdot, t_n)\|_{1,\mathcal{D}}^2 \le C. \tag{25}$$

where C only depends on the initial condition. As in the steady state case, the second relation above yields an estimate on the space translates:

$$\|u_{\mathcal{D}}(\cdot + \eta, \cdot) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^2(0,T;\,L^2(\Omega))} \le C\big(|\eta|(|\eta| + h_{\mathcal{D}})\big)^{\frac{1}{2}}, \qquad \forall \eta \in \mathbb{R}^d.$$

Using equation (24), we are then able to derive an estimate on the time translates:

$$\|u_{\mathcal{D}}(\cdot, \cdot + \tau) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^2(0,T;\,L^2(\Omega))} \le C|\tau|^{\frac{1}{2}}, \qquad \forall \tau \in \mathbb{R}.$$

By a discrete Rellich theorem, we deduce as in the steady state case the convergence in $L^2(0,T;\,L^2(\Omega))$ of $u_{\mathcal{D}}$ to some function $\bar{u} \in L^2(0,T;\,H_0^1(\Omega))$. As in the elliptic case, a passage to the limit in the scheme yields that $\bar{u} = u$, weak solution of (23). This analysis may be generalized to the case of non-homogeneous Dirichlet boundary conditions, see [**9**].

5. The Stokes problem

A huge amount of literature is devoted to the numerical solution of the Stokes and Navier–Stokes equations. Among the proposed methods is the wellknown finite element method [**54, 55, 58**] and finite volume method [**73, 74**]; finite difference schemes on staggered grids were also studied [**70, 71**]. This type of staggered scheme was also generalized to non–cartesian finite volume grids [**41, 42**]. However, staggered grids are not easy to handle in the computational practice, and several industrial and commercial codes are based on co-located finite volume method, that is a method where the primitive variables (velocity and pressure) are used, and all located within a discretization cell; in this section we shall give an example of a co–located finite volume scheme for which a convergence theory was developped for both the Stokes and Navier–Stokes equations.

5.1. The continuous problem

The centred finite volume scheme may also be used to discretize the Navier–Stokes equations. For reasons of simiplicity, let us start with the steady state Stokes equations. The aim is to find $u : \Omega \to \mathbb{R}^d$ and $p : \Omega \to \mathbb{R}$ such that:

$$\left\{ \begin{aligned} -\nu \Delta u + \nabla p &= f && \text{in } \Omega, \\ \mathrm{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ \int_\Omega p \,\mathrm{d}x &= 0. \end{aligned} \right. \tag{26}$$

Let $E(\Omega) := \{v \in (H^1_0(\Omega))^d, \mathrm{div} v = 0 \text{ a.e. in } \Omega\}$, and assume that $f \in L^2(\Omega)^d$. A weak formulation of (26) is:

$$
\begin{aligned}
&u = (u^{(1)}, \dots, u^{(d)})^t \in E(\Omega),\\
&\nu \int_\Omega \nabla u : \nabla v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x, \qquad \forall v \in E(\Omega),
\end{aligned} \tag{27}
$$

with $\displaystyle\int_\Omega \nabla u : \nabla v \,\mathrm{d}x = \sum_{i=1,d} \int_\Omega \nabla u^{(i)} \cdot \nabla v^{(i)} \,\mathrm{d}x.$

5.2. Discrete gradient and divergence

As in the preceding sections, we consider the discrete space $H_{\mathcal{T}}(\Omega) \subset L^2(\Omega)$ of piecewise constant functions on the control volumes. In order to construct a finite volume scheme, we need to discretize the divergence operator. Let us remark that for $u \in H^1(\Omega)^d$, one has:

$$
\int_K \mathrm{div} u \,\mathrm{d}x = \sum_{L \in \mathcal{N}_K} \int_{\sigma_{KL}} u \cdot \mathbf{n}_{K,\sigma_{KL}} \,\mathrm{d}\gamma(x).
$$

Adopting a centred discretization of $u \cdot \mathbf{n}$ on σ_{KL} leads to the following definition of a discrete divergence operator:

$$
\text{for } u \in H_{\mathcal{T}}(\Omega)^d, \quad (\mathrm{div}_{\mathcal{T}} u)_K = \frac{1}{|K|} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \, \frac{(u_K + u_L)}{2} \cdot \mathbf{n}_{K,\sigma_{KL}},
$$

so that $\mathrm{div}_{\mathcal{T}} u$ is a linear operator from $H_{\mathcal{T}}(\Omega)^d$ to $H_{\mathcal{T}}(\Omega)$. Note that one could also choose a more precise interpolation of the values u_K and u_L than their mean value, see [**44**].

Remark 1. Note that since $\mathbf{n}_{K,\sigma_{KL}} = -\mathbf{n}_{L,\sigma_{KL}}$, we have:

$$
\int_\Omega \mathrm{div}_{\mathcal{T}} u(x) \,\mathrm{d}x = \sum_{K \in \mathcal{T}} |K| (\mathrm{div}_{\mathcal{T}} u)_K = 0, \qquad \forall u \in H_{\mathcal{T}}(\Omega). \tag{28}
$$

Now we define the discrete gradient as the adjoint of the discrete divergence, that is a linear operator $\nabla_{\mathcal{T}}$ from $H_{\mathcal{T}}(\Omega)$ to $H_{\mathcal{T}}(\Omega)^d$ such that:

$$
\int_\Omega \mathrm{div}_{\mathcal{T}} u \; p \,\mathrm{d}x = -\int_\Omega u \cdot \nabla_{\mathcal{T}} p \,\mathrm{d}x, \quad \forall u \in H_{\mathcal{T}}(\Omega)^d, \ \forall p \in H_{\mathcal{T}}(\Omega).
$$

An easy calculation leads to:

$$
(\nabla_{\mathcal{T}} p)_K = \frac{1}{|K|} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \, \frac{(p_L - p_K)}{2} \, \mathbf{n}_{K,\sigma_{KL}}. \tag{29}
$$

Since $\displaystyle\sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \, \mathbf{n}_{K,\sigma_{KL}} = 0$, one may also write the discrete gradient as:

$$
(\nabla_{\mathcal{T}} p)_K = \frac{1}{|K|} \sum_{L \in \mathcal{N}_K} |\sigma_{KL}| \, \frac{(p_L + p_K)}{2} \, \mathbf{n}_{K,\sigma_{KL}}, \tag{30}
$$

this latter form being conservative.

Let us then give some convergence properties of the discrete gradient.

Theorem 2 (Weak convergence of the gradient)**.** *Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of admissible meshes of Ω with vanishing mesh size, and $(u_n)_{n\in\mathbb{N}} \subset L^2(\Omega)$ such that $u_n \in H_{\mathcal{T}_n}(\Omega)$ and $\|u^{(n)}\|_{1,\mathcal{T}_n} \le C$ for all $n \in \mathbb{N}$. Then there exists $\bar{u} \in H_0^1(\Omega)$ and a subsequence of $(u^{(n)})_{n\in\mathbb{N}}$ (still denoted $(u^{(n)})_{n\in\mathbb{N}}$) such that $u^{(n)} \to \bar{u}$ as $n \to +\infty$ in $L^2(\Omega)$, and such that, for any $\varphi \in C_c^\infty(\Omega)$,*

1. $\displaystyle\lim_{n\to+\infty} [u^{(n)}, P_{\mathcal{T}_n}\varphi]_{\mathcal{T}_n} = \int_\Omega \nabla\bar{u}\cdot\nabla\varphi \,\mathrm{d}x.$
2. $\nabla_{\mathcal{T}_n} u^{(n)}$ *weakly converges to* $\nabla\bar{u}$ *in* $L^2(\Omega)^d$ *as* $n \to +\infty$.

Item 1 is already known from the study in the elliptic case. Item 2 follows from the following lemma.

Lemma 2 (Consistency of the discrete derivatives)**.** *Let $\mathcal{T}$ be a finite volume mesh satisfying the orthogonality condition. With the notations introduced in Lemma 1, let $\varphi \in C_c^\infty(\Omega)$, let us define the consistency error $R_{\partial_i,\mathcal{T}}(\varphi) \in H_\mathcal{T}(\Omega)$ on the discrete derivative by:*

$$R_{\partial_i,\mathcal{T}}(\varphi) = \partial_\mathcal{T}^{(i)} P_\mathcal{T}\varphi - \Pi_\mathcal{T}(\partial^{(i)}\varphi).$$

where $\partial_\mathcal{T}^{(i)} P_\mathcal{T}\varphi$ stands for the component $^{(i)}$ of the above defined discrete gradient. Then:

$$\|R_{\partial_i,\mathcal{T}}(\varphi)\|_{-1,\mathcal{T}} \le C_\varphi h_\mathcal{T}.$$

The proof of this lemma uses the consistency of the approximation of the normal flux $u\cdot\mathbf{n}$ (see [**43**] for details).

5.3. A stabilized finite volume scheme for the Stokes equations

Let $E_\mathcal{T}(\Omega) = \{u \in (H_\mathcal{T}(\Omega))^d, \mathrm{div}_\mathcal{T}(u) = 0\}$, then a natural finite volume discretization of problem (27) is:

$$u \in E_\mathcal{T}(\Omega), \qquad \nu[u,v]_\mathcal{T} = \int_\Omega f\cdot v \,\mathrm{d}x, \qquad \forall v \in E_\mathcal{T}(\Omega),$$

where $[u,v]_\mathcal{T}$ stands for $\sum_{i=1,d}[u^{(i)},v^{(i)}]_\mathcal{T}$. However, this is not a very useful form since the "direct" construction of the space $E_\mathcal{T}(\Omega)$ is far from being an easy task. The standard way to proceed is then to write the condition $\mathrm{div}_\mathcal{T}(u) = 0$ as a constraint, but it is well known that such a scheme suffers from some stability problems, related to the fact that no inf-sup condition is not satisfied for colocated schemes. A cure for this problem which has become classical in the finite element framework, is then to use a modified divergence constraint including a stabilization term, which yields a scheme of the following form:

$$(31)\qquad \begin{cases} (u,p) \in H_\mathcal{T}(\Omega)^d \times H_\mathcal{T}(\Omega), \\ \nu[u,v]_\mathcal{T} - \displaystyle\int_\Omega p\,\mathrm{div}_\mathcal{T}(v)\,\mathrm{d}x = \int_\Omega f\cdot v\,\mathrm{d}x, & \forall v \in H_\mathcal{T}(\Omega)^d, \\ \displaystyle\int_\Omega \mathrm{div}_\mathcal{D}(u)\,q\,\mathrm{d}x = -\langle p,q\rangle_{\mathcal{T},\lambda}, & \forall q \in H_\mathcal{T}(\Omega), \end{cases}$$

where

$$\langle p, q\rangle_{\mathcal{T},\lambda} = \sum_{\sigma_{KL}\in\mathcal{E}_{\rm int}} \lambda_{K|L}\frac{|\sigma_{KL}|}{d_{KL}}(p_L - p_K)(q_L - q_K), \tag{32}$$

and the coefficients $\lambda_{K|L}$ are determined according to the choice of stabilization. A first possible choice [**43**], inspired by the well known Brezzi–Pitkäranta [**14**] scheme in the finite element framework, is to take $\lambda_{K|L} = \beta {h_{\mathcal{D}}}^{\alpha}$, $\alpha \in (0, 2)$. A stabilization by "clusters" was recently introduced [**22, 45**], which yields a scheme the accuracy of which is less affected by the size of the stabilization coefficient [**21**]. The idea is to introduce a partition of the mesh into clusters, each cluster containing some control volumes of the mesh. It is assumed that the maximum diameter of each cluster is bounded by a constant times the mesh size, and therefore, it tends to zero with the mesh size. For any control volume K we denote by $\mathcal{C}_K$ the cluster which contains K; let $\gamma \geq 0$, we define the cluster stabilization by:

$$\lambda_{K|L} = \begin{cases} 0, & \mathcal{C}_K \neq \mathcal{C}_L, \\ \gamma, & \mathcal{C}_K = \mathcal{C}_L. \end{cases}$$

Note that one could also consider a stabilization term γ on each cluster which would depend on h, and would lessen the weight of the stabilization within each cluster. The pros and cons of the various choices are currently being investigated.

Stabilizations by penalization of the pressure jumps across either all the internal edges of the mesh or only the internal edges of macro-elements have already been proposed in the finite element context for the stabilization of the so-called $Q_1 - Q_0$ element [**62**]; besides an extension to the finite volume framework, the above scheme considerably generalizes the notion of macro-element. Under some simple geometrical assumptions for the clusters, we are able to prove that the pair of spaces associating $H_{\mathcal{T}}(\Omega)^d$ for the velocity and constant by cluster pressures is "inf-sup stable" [**46**]. The cluster stabilization can then be interpreted as a minimal stabilization procedure, as defined by Brezzi and Fortin [**13**]; this interpretation suggests a variation of γ as the square of the mesh size [**46**].

The finite volume scheme (31) may also be written in its more classical flux form:

$$\begin{cases} -\nu \displaystyle\sum_{L\in\mathcal{N}_K} \frac{|\sigma_{KL}|}{d_{KL}}(u_L - u_K) - \nu \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\rm ext}} \frac{|\sigma|}{d_{K,\sigma}}(-u_K) & \\ \qquad\qquad + \displaystyle\sum_{L\in\mathcal{N}_K} |\sigma_{KL}|\,\frac{(p_L - p_K)}{2}\,\mathbf{n}_{K,\sigma_{KL}} = \int_K f \ {\rm d}x, & \forall K \in \mathcal{T}, \\ \displaystyle\sum_{L\in\mathcal{N}_K} G_{K,L}(u_{\mathcal{T}}, p_{\mathcal{T}}) = 0, & \forall K \in \mathcal{T}, \end{cases}$$

where

$$G_{K,L}(u_{\mathcal{T}}, p_{\mathcal{T}}) = |\sigma_{KL}|\frac{(u_K + u_L)}{2}\cdot \mathbf{n}_{K,\sigma_{KL}} - \lambda_{KL}\frac{|\sigma_{KL}|}{d_{KL}}(p_L - p_K). \tag{33}$$

This finite volume scheme must be supplemented by the condition $\displaystyle\int p_{\mathcal{T}} \ {\rm d}x = 0$.

As in the elliptic case, the convergence analysis for this scheme is based on *a priori* estimates. First, taking $v = u_{\mathcal{T}}$ and $q = p_{\mathcal{T}}$ in (31) yields:

$$\nu^2 \|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 + 2\nu\, |p_{\mathcal{T}}|_{\mathcal{T},\lambda}^2 \le \|f\|_{-1,\mathcal{T}}^2,$$

where $\|\cdot\|_{1,\mathcal{T}}$ and $\|\cdot\|_{-1,\mathcal{T}}$ are now the discrete H^1 and H^{-1} norms on $H_{\mathcal{T}}(\Omega)^d$, easily deduced from their scalar counterparts, and $|\cdot|_{\mathcal{T},\lambda}$ is the semi–norm associated with the inner product defined by (32). Note that for both considered stabilizations, the above estimate on the pressure is mesh dependent, and therefore does not yield a uniform estimate.

The second step is then to prove an L^2 estimate on the pressure. To this purpose, we take benefit of the fact that the inf-sup condition is verified at the continuous level, so there exists $\bar v \in H_0^1(\Omega)^d$ such that $\mathrm{div}\bar v = p_{\mathcal{T}}$ and $\|\bar v\|_{H_0^1(\Omega)^d} \le C\|p_{\mathcal{T}}\|_{L^2(\Omega)}$ [**69**]; taking $\Pi_{\mathcal{T}}\bar v$ as test function in the first relation of the scheme (31) then yields and estimate of $\|p_{\mathcal{T}}\|_{L^2(\Omega)} - |p_{\mathcal{T}}|_{\mathcal{T},\lambda}$ which, combined with the preceding bound, yields the result.

From these estimates, we then obtain existence and uniqueness of u and p solution to (31), which implies the weak convergence of both velocities and pressure in $L^2(\Omega)$. As in the elliptic case, the compactness on the velocities, and the regularity of the limit, are obtained by estimates on the translates. We thus obtain the strong convergence in $L^2(\Omega)$ of a subsequence of the approximate velocities to some $\tilde u \in H_0^1(\Omega)$, and the convergence of a subsequence of approximate pressures to some $\tilde p$ weakly in $L^2(\Omega)$. In order to conclude the convergence proof, we then consider $\varphi \in C_c^\infty(\Omega)^d$, and $v = P_{\mathcal{T}}\varphi$ in (31). A passage to the limit as the mesh size tends to 0, using the weak convergence of the divergence and of the gradient (Theorem 2) yields that $(\tilde u, \tilde p)$ is the solution to (27).

If we assume that the weak solution $(\bar u, \bar p)$ to (27) belongs to $H^2(\Omega)^d \times H^1(\Omega)$, we may also obtain an error estimate, we refer to [**43, 44, 45, 46**] for both theoretical and numerical results.

6. Transient isothermal incompressible Navier Stokes

Let us now consider the (adimensionalised) isothermal incompressible Navier Stokes; we seek $u : \Omega \times [0,T] \to \mathbb{R}^d$ and $p : \Omega \times [0,T] \to \mathbb{R}$ such that:

$$\text{(34)} \qquad \begin{cases} u_t - \nu\Delta u + \mathrm{div}(u \otimes u) + \nabla p = f & \text{in } \Omega \times (0,T), \\ \mathrm{div} u = 0, & \text{in } \Omega \times (0,T), \\ u = 0 & \text{in } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where u_0 is a divergence free vector field of $L^\infty(\Omega)^d$, $u \otimes u$ is the tensor such that $(u \otimes u)^{(i,j)} = u^{(i)}u^{(j)}$ and $\mathrm{div}(u \otimes u)^{(i)} = \sum_{j=1}^d \partial_j (u \otimes u)^{(i,j)}$, so that if $\mathrm{div} u = 0$, then $\mathrm{div}(u \otimes u) = \sum_{i=1}^d u_i \partial_i u = (u \cdot \nabla)u$.

Let us then consider a convenient weak formulation of (34), in the sense that it is the formulation obtained when passing to the limit in the finite volume scheme which we shall introduce in the sequel (see e.g. [**77**] or [**10**] for other weak formulations). Let $E(\Omega) = \{v \in H_0^1(\Omega)^d; \mathrm{div} v = 0 \text{ a.e. in } \Omega\}$; we seek a function u of time and space such that:

$$
\begin{aligned}
& u \in L^2(0,T;\, E(\Omega)) \cap L^\infty(0,T;\, L^2(\Omega)^d),\\
& -\int_0^T \int_\Omega u \cdot \partial_t \varphi \;\mathrm{d}x\;\mathrm{d}t - \int_\Omega u_0 \cdot \varphi(\cdot,0)\;\mathrm{d}x\\
& + \nu \int_0^T \int_\Omega \nabla u : \nabla \varphi \;\mathrm{d}x\;\mathrm{d}t + \int_0^T \int_\Omega (u\cdot\nabla)u \cdot \varphi \;\mathrm{d}x\;\mathrm{d}t\\
= & \int_0^T \int_\Omega f(x)\cdot \varphi \;\mathrm{d}x\;\mathrm{d}t,\\
& \forall \varphi \in L^2(0,T;E(\Omega)) \cap C_c^\infty(\Omega\times[0,T))^d.
\end{aligned}
\tag{35}
$$

In order to define the finite volume scheme, we need to discretize the nonlinear convection term, which is integrated over a control volume K in the following way:

$$
\int_K (u\cdot\nabla)u \;\mathrm{d}x = \int_{\partial K} (u\cdot \mathbf{n}_K) u \;\mathrm{d}\gamma(x) = \sum_{\sigma_{KL}\in\mathcal{E}_K} \int_{\sigma_{KL}} (u\cdot\mathbf{n}_{K,\sigma})u\;\mathrm{d}\gamma(x),
$$

which is then naturally discretized as:

$$
\sum_{\sigma_{KL}\in\mathcal{E}_K} G_{K,L}(u_\mathcal{T},p_\mathcal{T})\;\frac{u_K+u_L}{2},
$$

where $G_{K,L}(u_\mathcal{T},p)$ is the discretisation of the mass flux through the edge separating K and L which was introduced in (33). We then obtain the following discrete approximation of the nonlinear form $b(u,v,w) = \displaystyle\int_\Omega (u\cdot\nabla)v\cdot w\;\mathrm{d}x$:

$$
b_\mathcal{T}(u_\mathcal{T},v_\mathcal{T},w_\mathcal{T}) = \sum_{\sigma_{KL}\in\mathcal{E}_K} G_{K,L}(u_\mathcal{T},p_\mathcal{T})\;\frac{v_K+v_L}{2}\cdot w_K.
$$

We perform a time discretisation of the system of equations (34) by the well known Crank-Nicolson scheme:

$$
\left\{
\begin{aligned}
& \frac{u^{n+1}-u^n}{\delta t} - \nu\Delta u^{n+\frac12} + (u^{n+\frac12}\cdot\nabla)u^{n+\frac12} + \nabla p^{n+\frac12} = f^{n+\frac12}\\
& \mathrm{div} u^{n+\frac12} = 0,
\end{aligned}
\right.
$$

with $u^{n+\frac12} = \frac12(u^n+u^{n+1})$ and $p^{n+\frac12} = \frac12(p^n+p^{n+1})$. With the same definition of $H_\mathcal{D}(\Omega\times(0,T))$ as in the parabolic case (space and time piecewise constant

functions), the finite volume scheme for (35) may then be written:
(36)

$$\begin{cases} (u_{\mathcal{D}}, p_{\mathcal{D}}) \in H_{\mathcal{D}}(\Omega \times (0,T))^d \times H_{\mathcal{D}}(\Omega \times (0,T)), \\ \displaystyle\int_\Omega \frac{u_{\mathcal{D}}^{n+1} - u_{\mathcal{D}}^n}{\delta t} v \ \mathrm{d}x + \nu[u_{\mathcal{D}}^{n+\frac12}, v]_{\mathcal{D}} + b_{\mathcal{D}}(u_{\mathcal{D}}^{n+\frac12}, u_{\mathcal{D}}^{n+\frac12}, v) \\ \qquad\qquad - \displaystyle\int_\Omega p_{\mathcal{D}}^{n+\frac12} \mathrm{div}_{\mathcal{D}}(v) \ \mathrm{d}x = \int_\Omega f \cdot v \ \mathrm{d}x, \quad \forall v \in H_{\mathcal{T}}(\Omega)^d, \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u_{\mathcal{D}}^{n+\frac12}) \ q \ \mathrm{d}x = -\langle p_{\mathcal{D}}^{n+\frac12}, q\rangle_{\mathcal{T},\lambda}, \qquad \forall q \in H_{\mathcal{T}}(\Omega), \end{cases}$$

with ${u_{\mathcal{D}}}^{n+\frac12} = \frac12({u_{\mathcal{D}}}^n + {u_{\mathcal{D}}}^{n+1})$ and ${p_{\mathcal{D}}}^{n+\frac12} = \frac12({p_{\mathcal{D}}}^n + {p_{\mathcal{D}}}^{n+1})$. As in the previous sections, the convergence of the scheme is obtained by first deriving a compactness property for a family of approximate solutions, thanks to some estimates on the translates, which are a bit more difficult to obtain in the present case. Let us for instance study the three-dimensional case and have a glance at the estimates on the translates which may be obtained for the continuous problem. Let u be a solution to (35). First, since $u \in L^2(0,T;\, E(\Omega))$, we get that:

$$\|u(\cdot + \eta, \cdot) - u(\cdot, \cdot)\|_{L^2(0,T;\, L^2(\Omega)^3)} \le C|\eta|, \qquad \forall \eta \in \mathbb{R}^3. \tag{37}$$

Next, since $u \in L^2(0,T;\, E(\Omega))$ and $u_t \in L^{\frac43}(0,T;\, E(\Omega)')$, we have that:

$$\|u(\cdot, \cdot + \tau) - u(\cdot, \cdot)\|_{L^{\frac43}(0,T;\, L^2(\Omega)^3)} \le C|\tau|^{\frac12}, \qquad \forall \tau \in \mathbb{R}. \tag{38}$$

In fact, we may also remark that we have the simpler estimates $u \in L^2(0,T;\, E(\Omega))$ and $u_t \in L^1(0,T;\, E(\Omega)')$ which yield:

$$\|u(\cdot, \cdot + \tau) - u(\cdot, \cdot)\|_{L^1(0,T;\, L^2(\Omega)^3)} \le C|\tau|^{\frac12}, \qquad \forall \tau \in \mathbb{R}, \tag{39}$$

but note that, contrary to the parabolic case, we have no $L^2(0,T;\, L^2(\Omega)^3)$ estimate on the time translates. We thus derive corresponding discrete estimates to (37) and (39) for the discrete problem. Let $u_{\mathcal{D}} \in H_{\mathcal{D}}(\Omega \times (0,T))$ be a solution to (36). Then there exists $C \in \mathbb{R}_+$ depending only on Ω, ν, u_0, f, T such that [**43**]:

$$\|u_{\mathcal{D}}\|_{L^\infty(0,T;\, L^2(\Omega)^3)} \le C \text{ and } \|u_{\mathcal{D}}\|_{L^2(0,T;\, H_{\mathcal{D}}(\Omega))} \le C.$$

Furthermore, if one assumes some reasonable regularity assumptions on the mesh, see [**43**], then there exists $C \in \mathbb{R}_+$ depending only on Ω, ν, u_0, f, T and on the regularity of the mesh such that the following estimates on the space and time translates hold:
(40)

$$\begin{aligned} &\|u_{\mathcal{D}}(\cdot + \eta, \cdot) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^2(0,T;\, L^2(\Omega)^3)} \le C\big(|\eta|(|\eta| + h_{\mathcal{D}})\big)^{\frac12}, && \forall\, \eta \in \mathbb{R}^3, \\ &\|u_{\mathcal{D}}(\cdot, \cdot + \tau) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^1(0,T;\, L^2(\Omega)^3)} \le C\tau^{\frac12}, && \forall \tau \in \mathbb{R}_+. \end{aligned}$$

The estimate on the space translates is identical to the parabolic case; the proof on the time translates, however, is much more technical, in particular because we have to deal with L^1 and not L^2, we refer to [**43**] for details. The proof of the convergence of the discrete approximation $u_{\mathcal{D}}$ to the solution of (35) may be found in [**43**] in the case where the stabilisation pressure term is not taken into account in the nonlinear convective term. The proof in the case presented here is somewhat similar. Using the above estimates and the Kolmogorov theorem, we get the convergence of a subsequence of the approximate solutions to $\bar{u} \in L^2(0,T;\, E(\Omega))$ in $L^1(0,T;\, L^2(\Omega)^3)$ as the mesh size tends to 0. Finally, a passage to the limit in the scheme yields that $\bar{u}$ is indeed a solution of (35).

7. Hyperbolic equations

Let us finally briefly mention the wide use of finite volume schemes for nonlinear hyperbolic equations. We refer to [**56, 57, 38, 64, 7**] for more on this subject. Here we only consider the following nonlinear hyperbolic equation:

$$\left\{\begin{array}{l} u_t + \mathrm{div}(\mathbf{v} f(u)) = 0 \quad \text{ in } \mathbb{R}^d \times (0,T), \\ u(\cdot,0) = u_0, \end{array}\right. \tag{41}$$

$u_0 \in L^\infty(\Omega), \mathbf{v} \in \mathbb{R}^d,\ f \in C^1(\mathbb{R},\mathbb{R}),\ f' \geq 0$. It is well known that the above problem is well–posed, in the sense that it admits a unique weak entropy solution, that is a function u satisfying:

$$\left\{\begin{array}{l} u \in L^\infty(\mathbb{R}^d \times (0,T)), \\ \displaystyle\int_0^T \int_{\mathbb{R}^d} (\eta(u)\varphi_t + \Phi(u)\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^d} \eta(u_0(x))\varphi(x)\,\mathrm{d}x \geq 0, \\ \forall\, \eta \in C^2(\mathbb{R}), \Phi\ ;\ \Phi' = f'\eta', \forall\, \varphi \in C_c^\infty(\mathbb{R}^d \times [0,T), \mathbb{R}_+). \end{array}\right. \tag{42}$$

With the same notations as in the previous sections, let $\mathcal{T}$ be a finite volume mesh of Ω. A finite volume scheme with an upwind choice for the convection flux can be written:

$$\left\{\begin{array}{l} \displaystyle |K|\frac{u_K^{n+1} - u_K^n}{\delta t} + \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}^{n+1} = 0,\ n \geq 0, \\ \displaystyle u_K^0 = \frac{1}{|K|}\int_K u_0(x)\,\mathrm{d}x, \end{array}\right.$$

with: $F_{K,\sigma}^{n+1} = v_{K,\sigma}^+ f(u_K^{n+1}) - v_{K,\sigma}^- f(u_L^{n+1})$. Note that this flux is consistent without any condition on the mesh, since there is no more diffusion flux. Multiplying the scheme by u_K and summing on K yields an L^∞ estimate on $u_{\mathcal{D}}$: there exists C only depending on $u_0, T, \mathbf{v}$ such that:

$$\|u_{\mathcal{D}}\|_{L^\infty(\mathbb{R}^d\times(0,T))} \leq C. \tag{43}$$

Hence if we consider a family of meshes with vanishing size, we get the weak $\star$ convergence (up to a subsequence) to $\bar{u}$ in $L^\infty(\mathbb{R}^d \times (0,T))$. This estimate is not sufficient to pass to the limit in the scheme even in the linear case (except in the case of uniform meshes). In order to obtain convergence we use the so-called weak-BV inequality, first used in the linear case in [**19**] and nonlinear case in [**20**], and named *BV* because it involves the jumps of the discrete function at the interfaces:

$$\sum_{\sigma_{KL}\in\mathcal{E}_{\mathrm{int}}} |v_{K,\sigma_{KL}}|(f(u_K^n) - f(u_L^n))^2 \leq C. \tag{44}$$

This estimate is obtained thanks to the diffusion term added by the upwinding on $f(u)$. Roughly speaking, this diffusion term may be seen as the discretisation of the continuous diffusion term $h_{\mathcal{D}} \sum_{i=1}^{d} \partial^{(i)}(|\mathbf{v}^{(i)} f'(u)|\, \partial^{(i)}u)$, so that the scheme may be seen as the discretisation of the following parabolic equation:

$$u_t + \operatorname{div}(\mathbf{v} f(u)) - h_{\mathcal{D}} \sum_{i=1}^{d} \partial^{(i)}(|\mathbf{v}^{(i)} f'(u)|\, \partial^{(i)}u) = 0 \tag{45}$$

Along the same lines, we may remark that the BV inequality (44) is related to the following weak H^1 inequality obtained from Equation (45):

$$\sum_{i=1}^{d} \|\mathbf{v}^{(i)} f'(u)\, \partial^{(i)}u)\|_{L^2(K)} \leq \frac{1}{\sqrt{h_{\mathcal{D}}}}, \text{ for any compact subset } K \text{ of } \mathbb{R}^d \times (0,T).$$

Even though this inequality is sufficient to pass to the limit in the linear case, it does not yield strong compactness, so that one needs yet another tool in the nonlinear case. Indeed, from the L^∞ estimate, we only obtain a weak $\star$ converging subsequence of approximate solutions, and the question is how to pass to the limit in the nonlinearity. The key to this point is the nonlinear weak $\star$ convergence [**34**] or [**38**, page 965], which is equivalent to the notion of Young measure [**76**]. The notion of nonlinear weak $\star$ convergence may be stated as follows:

Theorem 3 (Non linear weak $\star$ convergence)**.** *Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence of $L^\infty(\mathbb{R}^d \times (0,T))$. There exist $\bar{u} \in L^\infty(\Omega \times (0,T) \times (0,1))$ and a subsequence of $(u_n)_{n\in\mathbb{N}}$, still denoted $(u_n)_{n\in\mathbb{N}}$, such that $g(u_n)$ tends to $\int_0^1 g(\bar{u}(\cdot,\alpha))\mathrm{d}\alpha$ in $L^\infty(\Omega \times (0,T))$ weak $\star$, as $n \to +\infty$, that is:*

$$\int_\Omega g(u_n(x))\varphi(x)\,\mathrm{d}x \to \int_0^1 \int_\Omega g(\bar{u}(x,\alpha))\varphi(x)\,\mathrm{d}x\mathrm{d}\alpha,$$

for all $\varphi \in L^1(\Omega \times (0,T))$ and all $g \in C(\mathbb{R},\mathbb{R})$. We shall say that u_n converges (up to a subsequence) in the nonlinear weak $\star$ sense. Note that $\int_0^1 g(\bar{u}(x,\alpha))\,\mathrm{d}\alpha = \int_\mathbb{R} g(s)\,\mathrm{d}\nu_x(s)$, and that ν_x is a probability on $\mathbb{R}$.

Using the nonlinear $\star$ convergence, we get that a subsequence of approximate solutions converges to an entropy weak process solution of (41), that is a function $\bar{u}$ such that:

$$\left\{\begin{array}{l} u \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+ \times (0,1)), \\ \displaystyle\int_0^1 \int_{\mathbb{R}_+} \int_\Omega (\eta(u)\varphi_t + \Phi(u)\cdot\nabla\varphi\ \mathrm{d}x\ \mathrm{d}t\mathrm{d}\alpha + \int_{\mathbb{R}_+} \eta(u_0(x))\ \varphi(x)\,\mathrm{d}x \geq 0, \\ \forall\ \eta \in C^2(\mathbb{R}), \Phi\ ;\ \Phi' = f'\eta', \forall\ \varphi \in C_c^\infty(\mathbb{R}^d \times [0,T), \mathbb{R}_+). \end{array}\right. \tag{46}$$

The following uniqueness theorem then allows to conclude to the convergence of the scheme towards the entropy weak solution.

Theorem 4. *If $\bar{u} \in L^\infty(\Omega \times (0,T) \times (0,1))$ is an entropy weak process solution then:*

- $\bar{u}(x,\alpha)$ *does not depends on* α.
- $\bar{u}$ *is the unique entropy weak solution* u.

The proof uses the doubling variables method of Krushkov, [**52, 34**] or [**38**].

Hence, if we consider a family of approximate solutions on meshes with mesh size tending to 0, we get that there exists a subsequence of this solution tending to a weak entropy process solution, which is, by the above theorem, the unique entropy weak solution of (41). The convergence holds in $L^p(\mathbb{R}^d \times (0,T))$ for all $p < \infty$. Note that (non optimal) error estimates may also be obtained, see e.g. [**34, 17, 78, 25**].

8. Conclusions and perspectives

In this paper, we presented an outline of the analysis of the cell centred finite volume method for elliptic, parabolic equations, for the incompressible Navier–Stokes equations and for scalar hyperbolic conservation laws. Numerous works now exist for the analysis of the cell centred scheme for a number of problems and applications; to cite only a few on elliptic or parabolic problems, let us mention the works on general boundary conditions [**53, 11**], non coercive problems with H^{-1} or measure right hand side [**30, 31**]; other topics include nonlinear reaction diffusion equations and degenerate equations, see [**36, 65, 79, 40**] and references therein, variational inequalities [**61**], hyperbolic equations with boundary conditions and discontinuous fluxes, see [**48**] and references therein. Similar tools were also used for *a posteriori estimates* and mesh adaptation [**63, 72**], domain decomposition [**1, 16, 75**], numerical homogeneisation [**35**] or image processing [**66, 67**]. It is quite impossible to give a full review on the ongoing works on finite volumes; let us however mention the difficulty of anisotropic diffusion problems or diffusion problems on distorted meshes [**2, 39, 28, 29**], which give rise to a number of methods for the construction of discrete gradients and divergence operators, raising the issue of the discrete maximum principle [**8**]. Some techniques are also being developed for coupled systems leading to irregular right hand sides [**12, 18**], and for diffusion problems in the presence of singularities in the domain [**4, 27**]. Two phase flow in porous media was maybe one of the major incentive for the development of the analysis of cell centred finite volume schemes, and has been and still is often addressed [**47, 32**]. Boundary conditions for hyperbolic problems [**79, 5**] and the difficult problem of efficient solvers for hyperbolic systems [**49, 50, 51**] are also being intensively studied.

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