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SOME COMPLEX MATRIX AND DETERMINANT FUNCTIONAL EQUATIONS

I. B. RISTESKI and V. C. COVACHEV

ABSTRACT. In this paper some linear functional equations are solved whose arguments are complex commutative matrices or determinants. The results obtained here supplement the monograph [3].

1. Introduction

First we will introduce the following notations.

In Section 2 $\mathbf{X}_i$, $\mathbf{Y}_i$, etc. are complex commutative $n \times n$ matrices. We assume that $\mathbf{O}$ is the zero matrix of appropriate dimension and $\mathbf{I}$ is the unit $n \times n$ matrix. f_i are functions of two $n \times n$-matrix arguments taking values complex $m \times s$ matrices, *i.e.*, $f_i : \mathbb{C}^{n^2} \times \mathbb{C}^{n^2} \to \mathbb{C}^{ms}$. In Section 3 $f, f_i, \dots$ are complex functions of several complex arguments.

In this paper we will use the same techniques for the solution of the functional equations considered as those developed in [**1**], [**2**] and [**4**].

2. Functional Equations with Matrix Arguments

In this section we will prove the following results.

Theorem 2.1. *The general measurable solution of the functional equation*

$$f_0(\mathbf{X}_1\mathbf{Y}_2 - \mathbf{X}_2\mathbf{Y}_1, \mathbf{X}_3\mathbf{Y}_4 - \mathbf{X}_4\mathbf{Y}_3) = f_1(\mathbf{X}_1\mathbf{Y}_3 - \mathbf{X}_3\mathbf{Y}_1, \mathbf{X}_2\mathbf{Y}_4 - \mathbf{X}_4\mathbf{Y}_2) + f_2(\mathbf{X}_1\mathbf{Y}_4 - \mathbf{X}_4\mathbf{Y}_1, \mathbf{X}_3\mathbf{Y}_2 - \mathbf{X}_2\mathbf{Y}_3), \tag{2.1}$$

where $\mathbf{X}_i, \mathbf{Y}_j$ $(1 \le i, j \le 4)$ *are complex commutative* $n \times n$ *matrices and* f_i $(0 \le i \le 2)$ *are complex* $m \times s$ *matrix functions, is given by*

$$\begin{aligned} f_0(\mathbf{X}, \mathbf{Y}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij} + \mathbf{A} + \mathbf{B}, \\ f_1(\mathbf{X}, \mathbf{Y}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij} + \mathbf{A}, \\ f_2(\mathbf{X}, \mathbf{Y}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij} + \mathbf{B}, \end{aligned} \tag{2.2}$$

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where $\mathbf{K}_{ij}$ $(1 \le i, j \le n)$, $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary complex constant* $m \times s$ *matrices, and* z_{ij} $(1 \le i, j \le n)$ *are entries of the matrix* $[z_{ij}] = \mathbf{Z} = \mathbf{X} \cdot \mathbf{Y}$.

Proof. From Equation (2.1) we easily obtain

$$f_2(\mathbf{U}, \mathbf{V}) = f_0(\mathbf{U}, \mathbf{V}) - \mathbf{A}, \tag{2.3}$$

$$f_1(\mathbf{U}, \mathbf{V}) = f_0(\mathbf{U}, \mathbf{V}) - \mathbf{B}, \tag{2.4}$$

where $\mathbf{A} = f_1(\mathbf{O}, \mathbf{O})$, $\mathbf{B} = f_2(\mathbf{O}, \mathbf{O})$.

If we put (2.3) and (2.4) into (2.1) and introduce a new matrix function g by the substitution

$$g(\mathbf{U}, \mathbf{V}) = f_0(\mathbf{U}, \mathbf{V}) - \mathbf{A} - \mathbf{B}, \tag{2.5}$$

we obtain the equation

$$\begin{aligned} &g(\mathbf{X}_1\mathbf{Y}_2 - \mathbf{X}_2\mathbf{Y}_1, \mathbf{X}_3\mathbf{Y}_4 - \mathbf{X}_4\mathbf{Y}_3) \\ &\quad = g(\mathbf{X}_1\mathbf{Y}_3 - \mathbf{X}_3\mathbf{Y}_1, \mathbf{X}_2\mathbf{Y}_4 - \mathbf{X}_4\mathbf{Y}_2) + g(\mathbf{X}_1\mathbf{Y}_4 - \mathbf{X}_4\mathbf{Y}_1, \mathbf{X}_3\mathbf{Y}_2 - \mathbf{X}_2\mathbf{Y}_3). \end{aligned} \tag{2.6}$$

For the function g we successively derive the properties

$$g(\mathbf{O}, \mathbf{O}) = \mathbf{O}, \qquad g(\mathbf{U}, \mathbf{O}) = \mathbf{O}, \qquad g(\mathbf{O}, \mathbf{U}) = \mathbf{O}$$

and, finally,

$$g(\mathbf{U}, \mathbf{V}) = -g(\mathbf{UV}, -\mathbf{I}). \tag{2.7}$$

By putting $\mathbf{X}_1 = \mathbf{Y}_2 = \mathbf{O}$, $\mathbf{X}_2 = \mathbf{X}_3 = \mathbf{X}_4 = \mathbf{Y}_1 = \mathbf{I}$, $\mathbf{Y}_3 = \mathbf{X}$, $\mathbf{Y}_4 = -\mathbf{Y}$, from Equation (2.6) we obtain

$$g(-\mathbf{I}, -\mathbf{Y} - \mathbf{X}) = g(-\mathbf{I}, -\mathbf{Y}) + g(-\mathbf{I}, -\mathbf{X})$$

or, by virtue of (2.7),

$$-g(\mathbf{X} + \mathbf{Y}, -\mathbf{I}) = -g(\mathbf{X}, -\mathbf{I}) - g(\mathbf{Y}, -\mathbf{I}).$$

By introducing a new function

$$h(\mathbf{X}) = -g(\mathbf{X}, -\mathbf{I}), \tag{2.8}$$

we obtain the Cauchy functional equation

$$h(\mathbf{X} + \mathbf{Y}) = h(\mathbf{X}) + h(\mathbf{Y}). \tag{2.9}$$

In [**5**] it is stated that the general continuous solution of the functional equation (2.9) is

$$h(\mathbf{X}) = \mathbf{A} \cdot \mathbf{X} \cdot \mathbf{B}, \tag{2.10}$$

where $\mathbf{A}$ is an arbitrary constant complex $m \times n$ matrix, and $\mathbf{B}$ is an arbitrary constant complex $n \times s$ matrix.

The function (2.10) is a solution of Equation (2.9), but it is not the general measurable solution. According to [**6**], the general measurable solution is given by

$$h(\mathbf{X}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{K}_{ij} x_{ij}, \tag{2.11}$$

where $\mathbf{K}_{ij}$ $(1 \le i,j \le n)$ are arbitrary complex constant $m \times s$ matrices, and x_{ij} $(1 \le i,j \le n)$ are entries of the matrix $[x_{ij}] = \mathbf{X}$.

On the basis of the expressions (2.7), (2.8) and (2.11) we obtain

$$g(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij}, \tag{2.12}$$

where z_{ij} $(1 \le i,j \le n)$ are entries of the matrix $[z_{ij}] = \mathbf{Z} = \mathbf{X} \cdot \mathbf{Y}$.

Now, on the basis of the expressions (2.3), (2.4), (2.5) and (2.12) we have (2.2), where $\mathbf{K}_{ij}$ and z_{ij} have the same meaning as in (2.12).

Conversely, straightforward calculations show that the functions (2.2) satisfy the functional equation (2.1). It is just here that the commutativity of the matrices is used. □

Theorem 2.2. *The general measurable solution of the functional equation*

$$\begin{aligned} &f_0(\mathbf{X}_1\mathbf{Y}_2-\mathbf{X}_2\mathbf{Y}_1, \mathbf{Z}_2\mathbf{T}_3-\mathbf{Z}_3\mathbf{T}_2)+f_1(\mathbf{X}_1\mathbf{Z}_2-\mathbf{X}_2\mathbf{Z}_1, \mathbf{T}_3\mathbf{Y}_2-\mathbf{T}_2\mathbf{Y}_3)\\ &+ f_2(\mathbf{X}_1\mathbf{T}_2-\mathbf{X}_2\mathbf{T}_1, \mathbf{Y}_3\mathbf{Z}_2-\mathbf{Y}_2\mathbf{Z}_3)+f_3(\mathbf{T}_1\mathbf{Z}_2-\mathbf{T}_2\mathbf{Z}_1, \mathbf{Y}_3\mathbf{X}_2-\mathbf{Y}_2\mathbf{X}_3)\\ &+ f_4(\mathbf{Y}_1\mathbf{T}_2-\mathbf{Y}_2\mathbf{T}_1, \mathbf{Z}_3\mathbf{X}_2-\mathbf{Z}_2\mathbf{X}_3)+f_5(\mathbf{Z}_1\mathbf{Y}_2-\mathbf{Z}_2\mathbf{Y}_1, \mathbf{T}_3\mathbf{X}_2-\mathbf{T}_2\mathbf{X}_3) = \mathbf{O}, \end{aligned} \tag{2.13}$$

where $\mathbf{X}_i, \mathbf{Y}_i, \mathbf{Z}_i, \mathbf{T}_i$ $(1 \le i \le 3)$ *are complex commutative* $n \times n$ *matrices and* f_i $(0 \le i \le 5)$ *are complex* $m \times s$ *matrix functions, is given by*

$$f_0(\mathbf{X},\mathbf{Y}) = -\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij} + \mathbf{A}_0, \tag{2.14}$$

$$f_r(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij} + \mathbf{A}_r \quad (1 \le r \le 5), \tag{2.15}$$

are arbitrary complex constant $m \times s$ *matrices,* z_{ij} $(1 \le i,j \le n)$ *are entries of the matrix* $[z_{ij}] = \mathbf{Z} = \mathbf{X} \cdot \mathbf{Y}$*, and* $\mathbf{A}_i$ $(0 \le i \le 5)$ *are constant complex* $m \times s$ *matrices such that* $\sum_{i=0}^{5} \mathbf{A}_i = \mathbf{O}$.

Proof. From Equation (2.2) by suitable substitutions we obtain

$$\sum_{i=0}^{5} \mathbf{A}_i = \mathbf{O}, \tag{2.16}$$

where $\mathbf{A}_i = f_i(\mathbf{O},\mathbf{O})$ $(0 \le i \le 5)$, and

$$\begin{aligned} f_0(\mathbf{X},\mathbf{Y}) &= -f_2(\mathbf{X},\mathbf{Y}) - \mathbf{A}_1 - \mathbf{A}_3 - \mathbf{A}_4 - \mathbf{A}_5,\\ f_1(\mathbf{X},\mathbf{Y}) &= -f_2(\mathbf{X},-\mathbf{Y}) - \mathbf{A}_0 - \mathbf{A}_3 - \mathbf{A}_4 - \mathbf{A}_5,\\ f_3(\mathbf{X},\mathbf{Y}) &= -f_2(\mathbf{X},-\mathbf{Y}) - \mathbf{A}_0 - \mathbf{A}_1 - \mathbf{A}_4 - \mathbf{A}_5,\\ f_4(\mathbf{X},\mathbf{Y}) &= -f_2(\mathbf{X},-\mathbf{Y}) - \mathbf{A}_0 - \mathbf{A}_1 - \mathbf{A}_3 - \mathbf{A}_5,\\ f_5(\mathbf{X},\mathbf{Y}) &= f_2(\mathbf{X},\mathbf{Y}) - \mathbf{A}_2 + \mathbf{A}_5. \end{aligned} \tag{2.17}$$

On the basis of the expressions (2.17) and (2.16), if we introduce a new function f by the substitution

$$f(\mathbf{X},\mathbf{Y}) = f_2(\mathbf{X},\mathbf{Y}) - \mathbf{A}_2, \tag{2.18}$$

then Equation (2.2) becomes

$$\begin{aligned} &f(\mathbf{X}_1\mathbf{Y}_2-\mathbf{X}_2\mathbf{Y}_1,\mathbf{Z}_2\mathbf{T}_3-\mathbf{Z}_3\mathbf{T}_2)+f(\mathbf{X}_1\mathbf{Z}_2-\mathbf{X}_2\mathbf{Z}_1,\mathbf{T}_2\mathbf{Y}_3-\mathbf{T}_3\mathbf{Y}_2)\\ &-f(\mathbf{X}_1\mathbf{T}_2-\mathbf{X}_2\mathbf{T}_1,\mathbf{Y}_3\mathbf{Z}_2-\mathbf{Y}_2\mathbf{Z}_3)+f(\mathbf{T}_1\mathbf{Z}_2-\mathbf{T}_2\mathbf{Z}_1,\mathbf{Y}_2\mathbf{X}_3-\mathbf{Y}_3\mathbf{X}_2)\\ &+f(\mathbf{Y}_1\mathbf{T}_2-\mathbf{Y}_2\mathbf{T}_1,\mathbf{Z}_2\mathbf{X}_3-\mathbf{Z}_3\mathbf{X}_2)-f(\mathbf{Z}_1\mathbf{Y}_2-\mathbf{Z}_2\mathbf{Y}_1,\mathbf{T}_3\mathbf{X}_2-\mathbf{T}_2\mathbf{X}_3)=\mathbf{O}. \end{aligned} \tag{2.19}$$

Clearly, we have

$$f(\mathbf{O},\mathbf{O}) = \mathbf{O} \tag{2.20}$$

and

$$f(\mathbf{X},-\mathbf{Y}) = -f(\mathbf{X},\mathbf{Y}). \tag{2.21}$$

On the basis of the above expression (2.21), Equation (2) becomes

$$\begin{aligned} &f(\mathbf{X}_1\mathbf{Y}_2-\mathbf{X}_2\mathbf{Y}_1,\mathbf{Z}_2\mathbf{T}_3-\mathbf{Z}_3\mathbf{T}_2)+f(\mathbf{X}_1\mathbf{Z}_2-\mathbf{X}_2\mathbf{Z}_1,\mathbf{T}_2\mathbf{Y}_3-\mathbf{T}_3\mathbf{Y}_2)\\ &+f(\mathbf{X}_1\mathbf{T}_2-\mathbf{X}_2\mathbf{T}_1,\mathbf{Y}_2\mathbf{Z}_3-\mathbf{Y}_3\mathbf{Z}_2)+f(\mathbf{T}_1\mathbf{Z}_2-\mathbf{T}_2\mathbf{Z}_1,\mathbf{Y}_2\mathbf{X}_3-\mathbf{Y}_3\mathbf{X}_2)\\ &+f(\mathbf{Y}_1\mathbf{T}_2-\mathbf{Y}_2\mathbf{T}_1,\mathbf{Z}_2\mathbf{X}_3-\mathbf{Z}_3\mathbf{X}_2)+f(\mathbf{Z}_1\mathbf{Y}_2-\mathbf{Z}_2\mathbf{Y}_1,\mathbf{T}_2\mathbf{X}_3-\mathbf{T}_3\mathbf{X}_2)=\mathbf{O}. \end{aligned} \tag{2.22}$$

Equation (2) has been considered by Gheorghiu [**5**] and he has proved that the function f, beside the properties (2.20) and (2.21), has the following properties

$$f(\mathbf{X},\mathbf{O}) = \mathbf{O}, \quad f(\mathbf{O},\mathbf{Y}) = \mathbf{O}, \quad f(-\mathbf{X},\mathbf{Y}) = -f(\mathbf{X},\mathbf{Y}).$$

By using the above properties of the function f and by putting $\mathbf{X}_1 = \mathbf{U}$, $\mathbf{Z}_2 = \mathbf{V}$, $\mathbf{Y}_2 = \mathbf{T}_3 = \mathbf{I}$, $\mathbf{Y}_1 = \mathbf{Z}_1 = \mathbf{T}_1 = \mathbf{X}_2 = \mathbf{T}_2 = \mathbf{X}_3 = \mathbf{Y}_3 = \mathbf{Z}_3 = \mathbf{O}$ into (2), we obtain

$$f(\mathbf{U},\mathbf{V}) = -f(\mathbf{U}\mathbf{V},-\mathbf{I}). \tag{2.23}$$

Next, we derive

$$f(\mathbf{T}_1-\mathbf{Z}_1,-\mathbf{I}) + f(\mathbf{Y}_1-\mathbf{T}_1,-\mathbf{I}) + f(\mathbf{Z}_1-\mathbf{Y}_1,-\mathbf{I}) = \mathbf{O}$$

or, taking into account the properties of the function f,

$$f(\mathbf{T}_1-\mathbf{Z}_1,-\mathbf{I}) - f(\mathbf{T}_1-\mathbf{Y}_1,-\mathbf{I}) - f(\mathbf{Y}_1-\mathbf{Z}_1,-\mathbf{I}) = \mathbf{O}.$$

By introducing the new function

$$g(\mathbf{X}) = -f(\mathbf{X},-\mathbf{I}), \tag{2.24}$$

we obtain for the function g the following matrix functional equation

$$g(\mathbf{T}_1-\mathbf{Z}_1) - g(\mathbf{Y}_1-\mathbf{Z}_1) - g(\mathbf{T}_1-\mathbf{Y}_1) = \mathbf{O}.$$

If we put $\mathbf{Y}_1 = \mathbf{O},\ \mathbf{Z}_1 = -\mathbf{Y},\ \mathbf{T}_1 = \mathbf{X}$, we obtain the Cauchy functional equation

$$g(\mathbf{X}+\mathbf{Y}) = g(\mathbf{X}) + g(\mathbf{Y}).$$

The general measurable solution of this equation is given by

$$g(\mathbf{X}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} x_{ij}, \tag{2.25}$$

where $\mathbf{K}_{ij}$ $(1 \le i, j \le n)$ are arbitrary constant complex $m \times s$ matrices, and x_{ij} $(1 \le i, j \le n)$ are entries of the matrix $[x_{ij}] = \mathbf{X}$.

Now, by using the expressions (2.25), (2.24) and (2.23), we obtain

$$f(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{K}_{ij} z_{ij}, \tag{2.26}$$

where z_{ij} $(1 \le i, j \le n)$ are entries of the matrix $[z_{ij}] = \mathbf{Z} = \mathbf{X} \cdot \mathbf{Y}$.

On the basis of the expression (2.18) we have

$$f_2(\mathbf{X}, \mathbf{Y}) = f(\mathbf{X}, \mathbf{Y}) + \mathbf{A}_2, \quad f_2(\mathbf{X}, -\mathbf{Y}) = -f(\mathbf{X}, \mathbf{Y}) + \mathbf{A}_2,$$

i.e., by virtue of the expressions (2.26), (2.16) and (2.17) we obtain (2.14), (2.15).

We can check the converse statement by a direct substitution of the functions (2.14), (2.15) into Equation (2.2).

Thus the theorem is proved. □

3. Functional Equations with Determinant Arguments

In order to formulate our next theorem we will introduce some more notations.

For $\Delta_1 = \det[x_{ij}]_{n\times n}$, $\Delta_2 = \det[y_{ij}]_{n\times n}$ and $1 \le i \le n$ we denote

$$\Delta_{1i} = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ \vdots & & & \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n} \\ y_{i1} & y_{i2} & \cdots & y_{in} \end{vmatrix}$$

$$\Delta_{2i} = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ \vdots & & & \\ y_{i-1,1} & y_{i-1,2} & \cdots & y_{i-1,n} \\ x_{n1} & x_{n2} & \cdots & x_{nn} \\ y_{i+1,1} & y_{i+1,2} & \cdots & y_{i+1,n} \\ \vdots & & & \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}.$$

For $i > n$ we assume $y_{ij} \equiv y_{i-n,j}$.

Theorem 3.1. *The general continuous solution of the equation*

$$f_0(\Delta_1, \Delta_2) = \sum_{i=1}^{n} f_i(\Delta_{1i}, \Delta_{2i}), \tag{3.1}$$

where $f_i:\ \mathbb{C}^2 \to \mathbb{C}$ $(0 \le i \le n)$*, is given by the formulae*

$$
\begin{aligned}
f_0(u,v) &= Cuv + \sum_{i=1}^{n} A_i, \\
f_r(u,v) &= Cuv + A_r \quad (1 \le r \le n),
\end{aligned}
\tag{3.2}
$$

where A_i $(1 \le i \le n)$ *and* C *are arbitrary complex constants.*

Proof. From Equation (3.1) by suitable substitutions we obtain

$$f_0(x,y) = f_i(x,y) + \sum_{\substack{j=1 \\ j\neq i}}^{n} A_j \quad (1 \le j \le n), \tag{3.3}$$

where $A_i = f_i(0,0)$ $(1 \le i \le n)$.

If we substitute the functions f_r $(1 \le r \le n)$ determined by (3.3) into Equation (3.1) and introduce a new function f by the substitution

$$f(u,v) = f_0(u,v) - \sum_{i=1}^{n} A_i, \tag{3.4}$$

then Equation (3.1) becomes

$$f(\Delta_1, \Delta_2) = \sum_{i=1}^{n} f(\Delta_{1i}, \Delta_{2i}). \tag{3.5}$$

Clearly, $f(0,0) = 0$. Next, we successively derive $f(u,0) = 0$ and $f(0,u) = 0$.

At the end, by putting $x_{ii} = 1$ $(1 \le i \le n-2)$, $y_{n-j+1,j} = 1$ $(1 \le j \le n-2)$, $x_{n-1,n-1} = x_1$, $x_{n-1,n} = y_1$, $x_{n,n-1} = x_2$, $x_{nn} = y_2$, $y_{1,n-1} = x_3$, $y_{1n} = y_3$, $y_{2,n-1} = x_4$, $y_{2n} = y_4$ and substituting all the other variables by zero, on the basis of the previous properties of the function f, Equation (3.5) takes the form

$$
\begin{aligned}
&f(x_1y_2 - x_2y_1, x_3y_4 - x_4y_3) \\
&\quad = f(x_1y_3 - x_3y_1, x_2y_4 - x_4y_2) + f(x_1y_4 - x_4y_1, x_3y_2 - x_2y_3).
\end{aligned}
\tag{3.6}
$$

From the last equation we deduce

$$f(u,v) = -f(uv,-1). \tag{3.7}$$

If we put into (3.6) $x_1 = y_2 = 0$, $x_2 = x_3 = x_4 = y_1 = 1$, $y_3 = x$, $y_4 = -y$, we derive the relation

$$f(-1,-y-x) = f(-1,-y) + f(-1,-x)$$

or, by virtue of (3.7),

$$-f(x+y,-1) = -f(x,-1) - f(y,-1).$$

If we introduce the new function

$$g(x) = -f(x,-1), \tag{3.8}$$

the last equation is reduced to the Cauchy functional equation

$$g(x+y) = g(x) + g(y),$$

whose general continuous solution is

$$g(z) = Cz, \tag{3.9}$$

where C is an arbitrary complex constant.

On the basis of the expressions (3.9), (3.8), (3.7), (3.4) and (3.3), we obtain the formulae (3.2).

Now we will consider the following identity

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 0 & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & x_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n} & 0 & 0 & \cdots & 0 \\ x_{n1} & x_{n2} & \cdots & x_{nn} & x_{n1} & x_{n2} & \cdots & x_{nn} \\ y_{11} & y_{12} & \cdots & y_{1n} & y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} & y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & & & & & & & \\ y_{n1} & y_{n2} & \cdots & y_{nn} & y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \equiv 0. \tag{3.10}$$

If we develop the determinant of the identity (3.10) by the Laplace rule and if we add to and subtract from the so obtained identity $\sum\limits_{i=1}^{n} A_i$, we conclude that the functions (3.2) are really a solution of Equation (3.1).

Thus the theorem is completely proved. □

Now, we will give the following definitions which are necessary for the next theorem.

Definition 3.2. If m and n are integers greater than one and $[x_{ij}]$ is an $mn \times n$ matrix, we denote by $\Delta(i, i+1, \dots, i+k-2, r, i+k, \dots, n+i-1)$ the determinant

$$\begin{vmatrix} x_{i1} & x_{i2} & \cdots & x_{in} \\ x_{i+1,1} & x_{i+1,2} & \cdots & x_{i+1,n} \\ \vdots & & & \\ x_{i+k-2,1} & x_{i+k-2,2} & \cdots & x_{i+k-2,n} \\ x_{r1} & x_{r2} & \cdots & x_{rn} \\ x_{i+k,1} & x_{i+k,2} & \cdots & x_{i+k,n} \\ \vdots & & & \\ x_{n+i-1,1} & x_{n+i-1,2} & \cdots & x_{n+i-1,n} \end{vmatrix}.$$

Definition 3.3. For m, n and $[x_{ij}]$ as above and $f : \mathbb{C}^m \to \mathbb{C}$ and $0 \le j < n$ we define the operator $\Xi_{n,rn+j}$ by

$$\begin{aligned} &\Xi_{n,rn+j} f\big(\Delta(1, \dots, n-1, n), \dots, \Delta(rn+1, \dots, rn+j-1, rn+j, \\ &\qquad rn+j+1, \dots, (r+1)n), \dots, \Delta((m-1)n+1, \dots, mn)\big) \\ &= f\big(\Delta(1, \dots, n-1, rn+j), \dots, \Delta(rn+1, \dots, rn+j-1, n, \\ &\qquad rn+j+1, \dots, (r+1)n), \dots, \Delta((m-1)n+1, \dots, mn)\big). \end{aligned}$$

Theorem 3.4. *The general continuous solution of the operator functional equation*

$$af\big(\Delta(1,\dots,n-1,n),\dots,\Delta((m-1)n+1,\dots,mn)\big)$$
$$= \sum_{r=n+1}^{mn} \Xi_{n,r} f\big(\Delta(1,\dots,n-1,n),\dots,\Delta((m-1)n+1,\dots,mn)\big) \tag{3.11}$$

($f:\ \mathbb{C}^m \to \mathbb{C}$, a is a complex parameter) is given by

$$\begin{aligned} &f(u_1,u_2,\dots,u_m) = C\prod_{i=1}^{m} u_i && \textit{if} \quad a = m-1,\\ &f(u_1,u_2,\dots,u_m) = \text{const} && \textit{if} \quad a = n(m-1),\\ &f(u_1,u_2,\dots,u_m) \equiv 0 && \textit{in all other cases,} \end{aligned}$$

where C is an arbitrary complex constant.

Proof. a) First we will prove the theorem for $a = m-1$. For this purpose we will need the following lemmas.

Lemma 3.5. *If at least one of the variables u_i $(1 \le i \le m-1)$ is equal to zero, then the following equality holds*

$$f(0,u_1,u_2,\dots,u_{m-1}) \equiv 0.$$

Proof of Lemma 3.5. By putting

$$x_{ij} = x \qquad (1 \le i \le mn;\ \ 1 \le j \le n),$$

Equation (3.11) becomes

$$f(0,0,\dots,0) = 0.$$

If we introduce the following substitutions

$$x_{kn+1,1} = x_k \ \ (0 \le k \le m-1), \qquad x_{kn+i,i} = 1 \ \ (0 \le k \le m-1;\ 2 \le i \le n)$$

and if we replace all the other variables by 0, then from (3.11) it follows that

$$\begin{aligned} &f(0,0,x_2,x_3,\dots,x_{m-1}) + f(0,x_1,0,x_3,\dots,x_{m-1})\\ &\qquad\qquad + \ldots + f(0,x_1,x_2,\dots,x_{m-2},0) = 0. \end{aligned} \tag{3.12}$$

Let $E_{m-1} = \{1,2,\dots,m-1\}$ and let S_r $(0 < r \le m-1)$ be a subset of the set E_{m-1} which contains r elements.

Now, we suppose that

$$f(0,v_1,v_2,\dots,v_{m-1}) = 0 \tag{3.13}$$

holds, where

$$v_i = \begin{cases} 0, & i \in S_r,\\ y_i, & i \in E_{m-1}\setminus S_r. \end{cases}$$

Under this assumption, we will show that

$$f(0,w_1,w_2,\dots,w_{m-1}) = 0,$$

where

$$w_i = \begin{cases} 0, & i \in S_{r-1}, \\ y_i, & i \in E_{m-1} \setminus S_{r-1} \end{cases}$$

if the hypothesis (3.13) is true.

By putting $x_i = w_i$ $(1 \le i \le m-1)$ into (3.12), according to the hypothesis (3.13) we obtain

$$(r-1)f(0, w_1, w_2, \dots, w_{m-1}) = 0.$$

Consequently, we proved by induction that

$$f(0, u_1, u_2, \dots, u_{m-1}) = 0$$

if exactly r $(0 < r \le m-1)$ elements among u_i $(1 \le i \le m)$ are equal to zero. $\square$

Lemma 3.6. *If at least one of the variables u_i $(1 \le i \le m-1)$ is equal to zero, then*

$$f(u_1, u_2, \dots, u_{m-1}, 0) = 0. \tag{3.14}$$

Proof of Lemma 3.6. By putting

$$x_{n,k} = 0 \;\; (1 \le k \le n), \qquad x_{(m-1)n+i,j} = 0 \;\; (1 \le i \le n-1;\ 1 \le j \le n),$$

$$x_{mn,n} = 1, \qquad x_{ni+k,k} = 1 \;\; (0 \le i \le m-2;\ 2 \le k \le n),$$

$$x_{ni+1,1} = u_{i+1} \; (0 \le i \le m-2)$$

and according to Lemma 3.5, from Equation (3.11) there follows (3.14). $\square$

Now we will continue by induction with respect m. If $m = 2$, one can argue as in Theorem 3.1. Thus we assume that $m > 2$ and that the general measurable solution of the functional equation

$$\begin{aligned} &(m-2)f\big(\Delta(1,2,\dots,n-1,n),\dots,\Delta((m-2)n+1,\dots,(m-1)n)\big) \\ &= \sum_{r=n+1}^{(m-1)n} \Xi_{n,r} f\big(\Delta(1,2,\dots,n-1,n),\dots,\Delta((m-2)n+1,\dots,(m-1)n)\big) \end{aligned} \tag{3.15}$$

is given by

$$f(u_1, u_2, \dots, u_{m-1}) = C \prod_{i=1}^{m-1} u_i, \tag{3.16}$$

where C is an arbitrary complex constant.

If we put into Equation (3.11)

$$x_{n,k} = x_{(m-1)n+1,k} \;\; (1 \le k \le n), \qquad x_{(m-1)n+i,i} = 1 \;\; (2 \le i \le n-1),$$
$$x_{mn,n} \cdot x_{(m-1)n+1,1} = 1, \qquad x_{(m-1)n+i,j} = 0 \;\; (2 \le i \le n;\ 1 \le j \le i-1),$$

$$f(x_1, x_2, \dots, x_{m-1}, 1) \equiv F(x_1, x_2, \dots, x_{m-1}),$$

then by virtue of Lemma 3.6 we obtain Equation (3).

If we take into account (3.16), then it follows that

$$f(x_1, x_2, \ldots, x_{m-1}, 1) = C \prod_{i=1}^{m-1} x_i. \tag{3.17}$$

If we substitute into Equation (3.11)

$$\begin{aligned} &x_{ni+1,1} = y_{i+1} \quad (0 \le i \le m-2), \qquad x_{mn,n} = y_m, \\ &x_{ni+k,k} = 1 \quad (0 \le i \le m-2;\ 2 \le k \le n), \quad x_{n(m-1)+k,k} = 1 \quad (1 \le k \le n-1), \end{aligned}$$

and if we replace all the other variables by zero, according to Lemma 3.5 we obtain

$$f(y_1, y_2, \ldots, y_m) = f(y_1 \cdot y_m, y_2, \ldots, y_{m-1}, 1). \tag{3.18}$$

From (3.17) and (3.18) it follows that

$$f(y_1, y_2, \ldots, y_m) = C \prod_{i=1}^{m} y_i. \tag{3.19}$$

Now we will show that, conversely, every function of the form (3.19) is really a solution of Equation (3.11). For this purpose, we will consider the following identity

$$D(j) = \begin{vmatrix} \mathcal{X} & \bigcirc_{n-1,n} \\ \tilde{\mathcal{X}} & \tilde{\mathcal{X}} \end{vmatrix} = 0, \tag{3.20}$$

where

$$\mathcal{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n-1,1} & x_{n-1,2} & \cdots & x_{n-1,n} \end{pmatrix}, \quad \tilde{\mathcal{X}} = \begin{pmatrix} x_{n1} & x_{n2} & \cdots & x_{nn} \\ x_{nj+1,1} & x_{nj+1,2} & \cdots & x_{nj+1,n} \\ \vdots & & & \\ x_{nj+n,1} & x_{nj+n,2} & \cdots & x_{nj+n,n} \end{pmatrix}$$

and $\bigcirc_{n-1,n}$ is the zero $(n-1) \times n$ matrix.

According to the expression (3.20), we conclude that the following identity holds

$$\sum_{j=1}^{m-1} D(j) \prod_{\substack{i=1 \\ i \ne j}}^{m-1} \Delta(ni+1, ni+2, \ldots, ni+n) = 0.$$

By evaluating the determinant $D(j)$ according to the Laplace rule, we show that the function given by (3.19) is really a solution of Equation (3.11). This completes the proof of Theorem 3.4 for $a = m-1$.

b) Now we will pass on to the proof of Theorem 3.4 for $a = n(m-1)$. It can be easily checked that the constant is a solution of Equation (3.11) if

$a = n(m-1)$. Now, we will prove the converse, *i.e.*, that every solution of the functional equation (3.11) for $a = n(m-1)$ is a constant.

If we put

$$x_{ni+1,1} = u_{i+1} \; (0 \le i \le m-1), \qquad x_{nk+j,j} = 1 \;\; (0 \le k \le m-1;\; 2 \le j \le n),$$

then Equation (3.11) becomes

$$n(m-1)f(u_1, u_2, \ldots, u_m)$$

$$= (m-1)f(u_1, u_2, \ldots, u_m) + (n-1)(m-1)f(0, 0, \ldots, 0),$$

so that

$$f(u_1, u_2, \ldots, u_m) = f(0, 0, \ldots, 0) = \text{const.}$$

For $x_{ij} = u$ $(1 \le i \le mn;\; 1 \le j \le n)$ from (3.11) we obtain

$$n(m-1)f(0, 0, \ldots, 0) = n(m-1)f(0, 0, \ldots, 0),$$

which means that

$$f(0, 0, \ldots, 0)$$

may be different from zero. Thus Theorem 3.4 is proved for $a = n(m-1)$.

c) At the end, we will prove Theorem 3.4 for $a \ne m-1$ and $a \ne n(m-1)$. By putting $x_{ij} = u$ $(1 \le i \le mn;\; 1 \le j \le n)$, from Equation (3.11) we obtain

$$af(0, 0, \ldots, 0) = n(m-1)f(0, 0, \ldots, 0).$$

Since $a \ne n(m-1)$, we conclude that

$$f(0, 0, \ldots, 0) = 0. \tag{3.21}$$

If we put $x_{ni+1,1} = u_{i+1}$ $(0 \le i \le m-1)$, $x_{nk+j,j} = 1$ $(0 \le k \le m-1;\; 2 \le j \le n)$, then Equation (3.11) becomes

$$[a - (m-1)]f(u_1, u_2, \ldots, u_m) = (n-1)(m-1)f(0, 0, \ldots, 0). \tag{3.22}$$

Since $a \ne m-1$, on the basis of the last two equalities (3.21) and (3.22) we obtain

$$f(u_1, u_2, \ldots, u_m) \equiv 0,$$

which means that Theorem 3.4 is completely proved. □

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I. B. Risteski, 2 Milepost Place # 606, Toronto, M4H 1C7, Ontario, Canada,
e-mail: `ice@scientist.com`

V. C. Covachev, Institute of Mathematics Bulgarian Academy of Sciences, Acad. G. Bonchev Str. 1113 Sofia, Bulgaria, *current address*: Department of Mathematics & Statistics, College of Science, Sultan Qaboos University, 123 Muscat, Sultanate of Oman,
e-mail: `vcovachev@hotmail.com, valery@squ.edu.om`