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DISCRETE METHODS AND EXPONENTIAL DICHOTOMY OF SEMIGROUPS

A. L. SASU

Abstract. The aim of this paper is to characterize the uniform exponential dichotomy of semigroups of linear operators in terms of the solvability of discrete-time equations over $\mathbf{N}$. We give necessary and sufficient conditions for uniform exponential dichotomy of a semigroup on a Banach space X in terms of the admissibility of the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$. As an application we deduce that a C_0-semigroup is uniformly exponentially stable if and only if the pair $(C_b(\mathbf{R}_+, X), C_{00}(\mathbf{R}_+, X))$ is admissible for it and a certain subspace is closed and complemented in X.

1. Introduction

In the last decades an impressive progress has been made in the study of the exponential dichotomy of evolution equations (see [**1**]–[**5**], [**8**]–[**10**], [**12**]–[**14**], [**16**], [**18**], [**20**]–[**22**], [**24**], [**25**], [**27**]). New methods have been involved in order to study classical and new concepts of exponential dichotomy. Evolution semigroups have proved to be very interesting tools in the study of the exponential dichotomy of evolution families and of linear skew-product flows (see [**3**], [**8**]–[**10**]). Another important method is the use of the discrete-time techniques (see [**2**], [**4**], [**7**], [**8**], [**10**], [**13**], [**14**], [**25**]).

Recent results concerning the exponential dichotomy of C_0-semigroups have been proved by Phóng in [**24**], where the author gives necessary and sufficient conditions for exponential dichotomy in terms of the unique solvability of an integral equation on $BUC(\mathbf{R}, X)$ and on $APR(\mathbf{R}, X)$, respectively.

The aim of this paper is to give necessary and sufficient conditions for exponential dichotomy of semigroups in terms of the solvability of a discrete-time equation on $\mathbf{N}$. We propose a direct approach for the characterization of the uniform exponential dichotomy of an exponentially bounded semigroup in terms of the admissibility of the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$. As an application we obtain that a C_0-semigroup is uniformly exponentially dichotomic if and only if the pair $(C_b(\mathbf{R}_+, X), C_{00}(\mathbf{R}_+, X))$ is admissible for it and a certain subspace is closed and complemented in X.

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2. Main results

Let X be a real or complex Banach space. The norm on X and on $\mathcal{B}(X)$-the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$.

Definition 2.1. A family $\mathbf{T} = \{T(t)\}_{t\geq 0} \subset \mathcal{B}(X)$ is called *semigroup* if $T(0) = I$ and $T(t+s) = T(t)T(s)$, for all $t,\ s \geq 0$.

Definition 2.2. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be:

(i) *exponentially bounded* if there are $M \geq 1$ and $\omega > 0$ such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$, for all $t \geq 0$;

(ii) C_0*-semigroup* if $\lim\limits_{t\searrow 0} T(t)x = x$, for all $x \in X$.

Remark. Every C_0-semigroup is exponentially bounded (see [**23**]).

Definition 2.3. A semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is said to be *uniformly exponentially dichotomic* if there exist a projection $P \in \mathcal{B}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that:

(i) $T(t)P = PT(t)$, for all $t \geq 0$;

(ii) $T(t)_| : \operatorname{Ker} P \to \operatorname{Ker} P$ is an isomorphism, for all $t \geq 0$;

(iii) $\|T(t)x\| \leq K\mathrm{e}^{-\nu t}\|x\|$, for all $x \in \operatorname{Im} P$ and all $t \geq 0$;

(iv) $\|T(t)x\| \geq \frac{1}{K}\mathrm{e}^{\nu t}\|x\|$, for all $x \in \operatorname{Ker} P$ and all $t \geq 0$.

Definition 2.4. Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a semigroup on the Banach space X and let Y be a linear subspace of X. Y is said to be $\mathbf{T}$*-invariant* if $T(t)Y \subset Y$, for all $t \geq 0$.

Lemma 2.5. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on the Banach space X and let Y be a $\mathbf{T}$-invariant subspace. The following assertions are equivalent:*

(i) *there are $K \geq 1$ and $\nu > 0$ such that:*

$$\|T(t)x\| \leq K\mathrm{e}^{-\nu t}\|x\|, \quad \forall t \geq 0, \forall x \in Y;$$

(ii) *there are $t_0 > 0$ and $c \in (0,1)$ such that $\|T(t_0)x\| \leq c\|x\|$, for all $x \in Y$.*

Proof. It is a simple exercise. □

Lemma 2.6. *Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on the Banach space X and let Y be a $\mathbf{T}$-invariant subspace. The following assertions are equivalent:*

(i) *there are $K \geq 1$ and $\nu > 0$ such that:*

$$\|T(t)x\| \geq \frac{1}{K}\ \mathrm{e}^{\nu t}\|x\|, \quad \forall t \geq 0, \forall x \in Y;$$

(ii) *there are $t_0 > 0$ and $c > 1$ such that $\|T(t_0)x\| \geq c\|x\|$, for all $x \in Y$.*

Proof. It is a trivial exercise. □

We denote by

$$l^\infty(\mathbf{N}, X) = \{s : \mathbf{N} \to X : \sup_{n\in\mathbf{N}} \|s(n)\| < \infty\}$$

$$c_0(\mathbf{N}, X) = \{s : \mathbf{N} \to X : \lim_{n\to\infty} s(n) = 0\}$$

and by $c_{00}(\mathbf{N}, X) = \{s \in c_0(\mathbf{N}, X) : s(0) = 0\}$. With respect to the norm $|||s||| = \sup_{n\in\mathbf{N}} \|s(n)\|$, these spaces are Banach spaces.

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be an exponentially bounded semigroup on X. We consider the discrete-time equation:

$$(E_d) \qquad \gamma(n+1) = T(1)\gamma(n) + s(n+1), \quad n \in \mathbf{N}$$

with $\gamma \in l^\infty(\mathbf{N}, X)$ and $s \in c_{00}(\mathbf{N}, X)$.

Definition 2.7. We say that the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$ is *admissible* for $\mathbf{T}$ if for every $s \in c_{00}(\mathbf{N}, X)$ there is $\gamma \in l^\infty(\mathbf{N}, X)$ such that the pair (γ, s) verifies the equation (E_d).

In what follows we shall establish the connection between the uniform exponential dichotomy and the admissibility of the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$.

We consider the linear subspace

$$X_1 = \{x \in X : \sup_{t\geq 0} \|T(t)x\| < \infty\}.$$

Throughout this paper, we suppose that X_1 is a closed linear subspace which has a $\mathbf{T}$-invariant (closed) complement X_2 such that $X = X_1 \oplus X_2$. We denote by P the projection corresponding to the above decomposition, i.e. $\operatorname{Im} P = X_1$ and $\operatorname{Ker} P = X_2$.

Remark. $T(t)P = PT(t)$, for all $t \geq 0$.

Remark. If $s_1, s_2 \in c_{00}(\mathbf{N}, X)$ and $\gamma \in l^\infty(\mathbf{N}, X)$ such that the pairs (γ, s_1) and (γ, s_2) verify the equation (E_d), then $s_1 = s_2$.

Hence it makes sense to define the linear subspace $D(H) = \{\gamma \in l^\infty(\mathbf{N}, X) : \exists s \in c_{00}(\mathbf{N}, X)$ such that (γ, s) satisfies $(E_d)\}$ and the linear operator $H : D(H) \to c_{00}(\mathbf{N}, X), H\gamma = s$.

Remark. H is a closed linear operator and $\operatorname{Ker} H = \{\gamma : \gamma(n) = T(n)\gamma(0)$ and $\gamma(0) \in \operatorname{Im} P\}$.

We consider the linear subspace $\tilde{D}(H) = \{\gamma \in D(H) : \gamma(0) \in \operatorname{Ker} P\}$.

Proposition 2.8. *If the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$ is admissible for $\mathbf{T}$, then*

(i) *there is $\nu \in (0,1)$ such that $|||H\gamma||| \geq \nu|||\gamma|||$, for all $\gamma \in \tilde{D}(H)$;*
(ii) *for every $t \geq 0$, the restriction $T(t)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$ is an isomorphism.*

Proof. (i) It is easy to see that the restriction $H_| : \tilde{D}(H) \to c_{00}(\mathbf{N}, X)$ is bijective. Considering the graph norm $|||\gamma|||_H = |||\gamma||| + |||H\gamma|||$ on $\tilde{D}(H)$, we have that $(\tilde{D}(H), |||\cdot|||_H)$ is a Banach space and hence there is $\nu \in (0,1)$ such that

$$|||H\gamma||| \geq \nu |||\gamma|||_H \geq \nu |||\gamma|||, \quad \forall \gamma \in \tilde{D}(H)$$

which completes the proof of (i).

(ii) It is sufficient to show that $T(1)_| : \operatorname{Ker} P \to \operatorname{Ker} P$ is an isomorphism. Let $x \in \operatorname{Ker} P$ and $s, \gamma : \mathbf{N} \to X$ given by

$$s(n) = \begin{cases} -T(1)x, & n = 1 \\ 0 \quad\;\;\;\;\;, & n \neq 1 \end{cases} \qquad \gamma(n) = \begin{cases} x, & n = 0 \\ 0, & n \in \mathbf{N}^*. \end{cases}$$

It is easy to see that the pair (γ, s) verifies the equation (E_d). Since $\gamma(0) \in \operatorname{Ker} P$, from (i) we obtain that

$$\|T(1)x\| = |||s||| \geq \nu |||\gamma||| = \nu \|x\|. \tag{2.1}$$

Since ν does not depend on x, from (2.1) we deduce that $T(1)_|$ is injective.

Let $x \in \operatorname{Ker} P$ and

$$s : \mathbf{N} \to X, \quad s(n) = \begin{cases} -x, & n = 1 \\ 0\;\;, & n \neq 1. \end{cases}$$

From hypothesis there is $\gamma \in l^\infty(\mathbf{N}, X)$ such that the pair (γ, s) verifies the equation (E_d). Then, we have that $\gamma(n) = T(n)\gamma(1)$, for all $n \geq 2$, which shows that $\gamma(1) \in X_1 = \operatorname{Im} P$.

Let $x_1 \in \operatorname{Im} P$ and $x_2 \in \operatorname{Ker} P$ such that $\gamma(0) = x_1 + x_2$. Since $\gamma(1) = T(1)\gamma(0) - x$, we obtain that $\gamma(1) - T(1)x_1 = T(1)x_2 - x$, so $x = T(1)x_2$. This shows that $T(1)_| : \operatorname{Ker} P \to \operatorname{Ker} P$ is surjective, which completes the proof. $\square$

Theorem 2.9. *If the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$ is admissible for the semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$, then there exist $K \geq 1$ and $\nu > 0$ such that*

$$\|T(t)x\| \leq K e^{-\nu t} \|x\|, \quad \forall t \geq 0, \forall x \in \operatorname{Im} P.$$

Proof. By Proposition 2.8 (i), there is $\nu \in (0,1)$ such that

$$|||H\gamma||| \geq 2\nu |||\gamma|||, \quad \forall \gamma \in \tilde{D}(H). \tag{2.2}$$

Let $p \in \mathbf{N},\ p \geq 2$ be such that $\nu e^{\nu(p-1)} \geq \|T(1)\|$.

Let $x \in \operatorname{Im} P \setminus \{0\}$ and $\Delta_x = \{n \in \mathbf{N} : T(n)x \neq 0\}$. We have the following situations:

1. $\{0, \ldots, p\} \subset \Delta_x$. Define the sequences $s, \gamma : \mathbf{N} \to X$ by

$$s(n) = \frac{\chi_{\{1,\ldots,p\}}(n)}{\|T(n)x\|}\, T(n)x \qquad \gamma(n) = \sum_{k=0}^{n} \frac{\chi_{\{1,\ldots,p\}}(k)}{\|T(k)x\|}\, T(n)x$$

where $\chi_{\{1,\ldots,p\}}$ denotes the characteristic function of the set $\{1, \ldots, p\}$. Then $s \in c_{00}(\mathbf{N}, X)$ and since $x \in \operatorname{Im} P$, it follows that $\gamma \in l^\infty(\mathbf{N}, X)$. It is easy to see that the pair (γ, s) verifies the equation (E_d). Since $\gamma(0) = 0$ we have that $\gamma \in \tilde{D}(H)$. Then, from relation (2.2) we have that

$$1 = |||s||| = |||H\gamma||| \geq 2\nu |||\gamma|||.$$

This inequality shows that

$$2\nu \sum_{j=1}^{k} \frac{1}{\|T(j)x\|} \leq \frac{1}{\|T(k)x\|}, \quad \forall k \in \{1,\ldots,p\}. \tag{2.3}$$

Let

$$\delta(k) = \sum_{j=1}^{k} \frac{1}{\|T(j)x\|}, \quad k \in \{1,\ldots,p\}.$$

If $k \in \{2,\ldots,p\}$, then

$$\frac{1}{\|T(k)x\|} \geq 2\nu\delta(k-1) \geq (\mathrm{e}^{\nu}-1)\delta(k-1)$$

so $\delta(k) \geq \mathrm{e}^{\nu}\delta(k-1)$. It follows that

$$\frac{1}{\|T(p)x\|} \geq 2\nu\delta(p) \geq 2\nu\mathrm{e}^{\nu(p-1)}\delta(1) = \frac{2\nu\mathrm{e}^{\nu(p-1)}}{\|T(1)x\|}. \tag{2.4}$$

By relation (2.4) we obtain that

$$\|T(p)x\| \leq \frac{\|T(1)x\|}{2\nu\mathrm{e}^{\nu(p-1)}} \leq \frac{1}{2}\,\|x\|.$$

2. $p \notin \Delta_x$. Then $T(p)x = 0$.
It follows that

$$\|T(p)x\| \leq \frac{1}{2}\,\|x\|, \quad \forall x \in \mathrm{Im}\ P. \tag{2.5}$$

By relation (2.5) and Lemma 2.5 we conclude the proof. □

Theorem 2.10. *If the pair* $(l^{\infty}(\mathbf{N},X), c_{00}(\mathbf{N},X))$ *is admissible for the semigroup* $\mathbf{T} = \{T(t)\}_{t\geq 0}$*, then there are* $K \geq 1$ *and* $\nu > 0$ *such that*

$$\|T(t)x\| \geq \frac{1}{K}\,\mathrm{e}^{\nu t}\|x\|, \quad \forall t \geq 0, \forall x \in \mathrm{Ker}\,P.$$

Proof. By Proposition 2.8 (i) there exists $\nu \in (0,1)$ such that

$$|||H\gamma||| \geq \nu|||\gamma|||, \quad \forall \gamma \in \tilde{D}(H).$$

Let $x \in \mathrm{Ker}\,P \setminus \{0\}$. By Proposition 2.8 (ii) we deduce that $T(n)x \neq 0$, for all $n \in \mathbf{N}$.

For every $p \in \mathbf{N}^*$, we consider the sequences

$$\begin{aligned} s_p : \mathbf{N} \to X, \quad s_p(n) &= -\frac{\chi_{\{1,\ldots,p\}}(n)}{\|T(n)x\|}\,T(n)x \\ \gamma_p : \mathbf{N} \to X, \quad \gamma_p(n) &= \sum_{k=n+1}^{\infty} \frac{\chi_{\{1,\ldots,p\}}(k)}{\|T(k)x\|}\,T(n)x. \end{aligned}$$

Then $s_p \in c_{00}(\mathbf{N}, X)$ and $\gamma_p \in l^\infty(\mathbf{N}, X)$. Moreover, since

$$\gamma_p(0) = \left(\sum_{k=1}^{p} \frac{1}{\|T(k)x\|}\right) x \in \operatorname{Ker} P$$

we deduce that $\gamma_p \in \tilde{D}(H)$. It is easy to see that the pair (γ_p, s_p) verifies the equation (E_d), so

$$1 = |||s_p||| = |||H\gamma_p||| \geq \nu |||\gamma_p|||, \quad \forall p \in \mathbf{N}^*.$$

It follows that

$$\nu \sum_{k=n+1}^{p} \frac{1}{\|T(k)x\|} \leq \frac{1}{\|T(n)x\|}, \quad \forall n, p \in \mathbf{N}, n < p. \tag{2.6}$$

By relation (2.6) we obtain that

$$\nu \sum_{k=n+1}^{\infty} \frac{1}{\|T(k)x\|} \leq \frac{1}{\|T(n)x\|}, \quad \forall n \in \mathbf{N}. \tag{2.7}$$

From (2.7) we have that

$$\sum_{k=n}^{\infty} \frac{1}{\|T(k)x\|} \geq (\nu + 1) \sum_{k=n+1}^{\infty} \frac{1}{\|T(k)x\|}, \quad \forall n \in \mathbf{N}. \tag{2.8}$$

Let $n \in \mathbf{N}^*$ such that $c = \nu(1+\nu)^n > 1$. By relations (2.7) and (2.8) we deduce that

$$\frac{1}{\|x\|} \geq \nu \sum_{k=1}^{\infty} \frac{1}{\|T(k)x\|} \geq \nu(1+\nu)^n \sum_{k=n+1}^{\infty} \frac{1}{\|T(k)x\|} \geq \frac{c}{\|T(n+1)x\|}.$$

It follows that $\|T(n+1)x\| \geq c\|x\|$. Taking into account that n and c do not depend on x, we obtain that

$$\|T(n+1)x\| \geq c\|x\|, \quad \forall x \in \operatorname{Ker} P.$$

Then, from Lemma 2.6 we deduce the conclusion. □

Lemma 2.11. *Let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a semigroup on the Banach space X. If $\mathbf{T}$ is uniformly exponentially dichotomic relative to the projection P, then $X_1 = \operatorname{Im} P$.*

Proof. Obviously $\operatorname{Im} P \subset X_1$. Let K, ν be given by Definition 2.2. If $x \in X_1$, then from

$$\begin{aligned} \|x - Px\| &\leq K e^{-\nu t} \|T(t)(I-P)x\| \\ &\leq K e^{-\nu t} (\|T(t)x\| + K e^{-\nu t} \|Px\|), \quad \forall t \geq 0 \end{aligned}$$

we obtain that $x \in \operatorname{Im} P$. So $\operatorname{Im} P = X_1$. □

The main result of this section is given by:

Theorem 2.12. *An exponentially bounded semigroup $\mathbf{T} = \{T(t)\}_{t \geq 0}$ is uniformly exponentially dichotomic if and only if the following statements hold:*

(i) *the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$ is admissible for $\mathbf{T}$;*
(ii) *the subspace X_1 is closed and it has a $\mathbf{T}$-invariant complement.*

Proof. Necessity. Let P be given by Definition 2.2. If $s \in c_{00}(\mathbf{N}, X)$, consider the sequence $\gamma : \mathbf{N} \to X$ defined by

$$\gamma(n) = \sum_{k=0}^{n} T(n-k)Ps(k) - \sum_{k=n+1}^{\infty} T(k-n)_{|}^{-1}(I-P)s(k)$$

where $T(k)_{|}^{-1}$ denotes the inverse of the operator $T(k)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$. Then $\gamma \in l^{\infty}(\mathbf{N}, X)$ and the pair (γ, s) verifies the equation (E_d). It follows that the pair $(l^{\infty}(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$ is admissible for $\mathbf{T}$.

From Lemma 2.11 we deduce that $X_1 = \operatorname{Im} P$. It follows that X_1 is closed and it has a complement – $\operatorname{Ker} P$ – which is $\mathbf{T}$-invariant.

Sufficiency. It results from Proposition 2.8, Theorem 2.9 and Theorem 2.10. □

3. Applications for the case of C_0-semigroups

Let X be a Banach space. We denote by $C_b(\mathbf{R}_+, X)$ the space of all bounded continuous functions $v : \mathbf{R}_+ \to X$ and by $C_{00}(\mathbf{R}_+, X) = \{v \in C_b(\mathbf{R}_+, X) : v(0) = \lim_{t\to\infty} v(t) = 0\}$.

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on X. We consider the integral equation

$$(E_c) \qquad f(t) = T(t-s)f(s) + \int_s^t T(t-\tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0$$

with $f \in C_b(\mathbf{R}_+, X)$ and $v \in C_{00}(\mathbf{R}_+, X)$.

Definition 3.1. The pair $(C_b(\mathbf{R}_+, X), C_{00}(\mathbf{R}_+, X))$ is said to be *admissible* for $\mathbf{T}$ if for every $v \in C_{00}(\mathbf{R}_+, X)$ there is $f \in C_b(\mathbf{R}_+, X)$ such that the pair (f, v) verifies the equation (E_c).

The central result of this section is:

Theorem 3.2. *The C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is uniformly exponentially dichotomic if and only if*

(i) *the pair $(C_b(\mathbf{R}_+, X), C_{00}(\mathbf{R}_+, X))$ is admissible for $\mathbf{T}$;*
(ii) *the subspace X_1 is closed and it has a $\mathbf{T}$-invariant complement.*

Proof. Necessity. For $v \in C_{00}(\mathbf{R}_+, X)$, we consider the function $f : \mathbf{R}_+ \to X$ defined by

$$f(t) = \int_0^t T(t-s)Pv(s)\, ds - \int_t^{\infty} T(s-t)_{|}^{-1}(I-P)v(s)\, ds$$

where $T(s)_{|}^{-1}$ denotes the inverse of the operator $T(s)_{|} : \operatorname{Ker} P \to \operatorname{Ker} P$. It is easy to see that $f \in C_b(\mathbf{R}_+, X)$ and the pair (f, v) verifies the equation (E_c), so the pair $(C_b(\mathbf{R}_+, X), C_{00}(\mathbf{R}_+, X))$ is admissible for $\mathbf{T}$. From Lemma 2.11 we deduce that $X_1 = \operatorname{Im} P$, so it is closed and it has a complement – $\operatorname{Ker} P$ – which is $\mathbf{T}$-invariant.

Sufficiency. Let $\alpha : [0,1] \to [0,2]$ be a continuous function with the support contained in $(0,1)$ and $\int_0^1 \alpha(\tau)\, d\tau = 1$. For $s \in c_{00}(\mathbf{N}, X)$ we consider the function

$$v : \mathbf{R}_+ \to X, \quad v(t) = T(t-[t])s([t])\alpha(t-[t]).$$

Then v is continuous and $v(0) = 0$. Moreover, if $M \geq 1$ and $\omega > 0$ are chosen such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$, for all $t \geq 0$, then we have $\|v(t)\| \leq 2M\mathrm{e}^{\omega}\|s([t])\|$, for all $t \geq 0$, so $v \in C_{00}(\mathbf{R}_+, X)$. By hypothesis, there is $f \in C_b(\mathbf{R}_+, X)$ such that

$$f(t) = T(t-s)f(s) + \int_s^t T(t-\tau)v(\tau)\, d\tau, \quad \forall t \geq s \geq 0.$$

Then, for every $n \in \mathbf{N}$, we obtain that

$$\begin{aligned} f(n+1) &= T(1)f(n) + \int_n^{n+1} T(n+1-\tau)v(\tau)\, d\tau \\ &= T(1)f(n) + T(1)s(n). \end{aligned} \tag{3.1}$$

Denoting by $\gamma(n) = f(n) + s(n)$, for all $n \in \mathbf{N}$, from (3.1) we deduce that

$$\gamma(n+1) = T(1)\gamma(n) + s(n+1), \quad \forall n \in \mathbf{N}$$

so the pair (γ, s) verifies the equation (E_d). Since $s \in c_{00}(\mathbf{N}, X)$ and $f \in C_b(\mathbf{R}_+, X)$, it follows that $\gamma \in l^\infty(\mathbf{N}, X)$.

So the pair $(l^\infty(\mathbf{N}, X), c_{00}(\mathbf{N}, X))$ is admissible for $\mathbf{T}$. By Theorem 2.12 we obtain the conclusion. □

References

1. Ben-Artzi A. and Gohberg I., *Dichotomies of perturbed time-varying systems and the power method*, Indiana Univ. Math. J. **42** (1993), 699–720.

2. Ben-Artzi A., Gohberg I. and Kaashoek M. A., *Invertibility and dichotomy of differential operators on the half-line*, J. Dynam. Differential Equations **5** (1993), 1–36.

3. Chicone C. and Latushkin Y., *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys and Monographs. Amer. Math. Soc. **70**, Providence, RI, (1999).

4. Chow S. N. and Leiva H., *Existence and roughness of the exponential dichotomy for linear skew-product semiflows in Banach space*, J. Differential Equations **120** (1995), 429–477.

5. ———, *Two definitions of exponential dichotomy for skew-product semiflow in Banach spaces*, Proc. Amer. Math. Soc. **124** (1996), 1071–1081.

6. Daleckii J. and Krein M., *Stability of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, (1974).

7. Henry D., *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, New York, (1981).

8. Latushkin Y. and Randolph T., *Dichotomy of differential equations on Banach spaces and an algebra of weighted translation operators*, Integral Equations Operator Theory **23** (1995), 472–500.

9. Latushkin Y., Randolph T. and Schnaubelt R., *Exponential dichotomy and mild solutions of nonautonomous equations in Banach spaces*, J. Dynam. Differential Equations **10** (1998), 489–510.

10. Latushkin Y. and Schnaubelt R., *Evolution semigroups, translation algebras and exponential dichotomy of cocycles*, J. Differential Equations **159** (1999), 321–369.

11. Massera J. L. and Schäffer J. J., *Linear Differential Equations and Function Spaces*, Academic Press, New York, (1966).

12. Megan M., Sasu B. and Sasu A. L., *On nonuniform exponential dichotomy of evolution operators in Banach spaces*, Integral Equations Operator Theory **44** (2002), 71–78.

13. Megan M., Sasu A. L. and Sasu B., *Discrete admissibility and exponential dichotomy for evolution families*, Discrete Contin. Dynam. Systems **9** (2003), 383–397.

14. ______, *Theorems of Perron type for uniform exponential dichotomy of linear skew-product semiflows*, Bull. Belg. Mat. Soc. Simon Stevin **10** (2003), 1–21.

15. ______, *Perron conditions for uniform exponential expansiveness of linear skew-product flows*, Monatsh. Math. **138** (2003), 145-157.

16. ______, *Perron conditions for pointwise and global exponential dichotomy of linear skew-product flows*, accepted for publication in Integral Equations Operator Theory.

17. ______, *Theorems of Perron type for uniform exponential stability of linear skew-product semiflows*, accepted for publication in Dynam. Contin. Discrete Impulsive Systems.

18. Van Minh N., Räbiger F. and Schnaubelt R., *Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half-line*, Integral Equations Operator Theory **32** (1998), 332–353.

19. Nagel R. (Ed.), *One parameter semigroups of positive operators*, Lect. Notes Math. **1184**, Springer-Verlag, Berlin, (1984).

20. Palmer K. J., *Exponential dichotomies for almost periodic equations*, Proc. Amer. Math. Soc. **101** (1987), 293–298.

21. ______, *Exponential dichotomies, the shadowing lemma and transversal homoclinic points*, Dynamics reported **1** (1988), 265–306.

22. ______, *Exponential dichotomy and Fredholm operators*, Proc. Amer. Math. Soc. **104** (1988), 149–156.

23. Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, (1983).

24. Phóng V. Q., *On the exponential stability and dichotomy of C_0-semigroups*, Studia Math. **132** (1999), 141–149.

25. Pliss V. A. and Sell G. R., *Robustness of the exponential dichotomy in infinite-dimensional dynamical systems*, J. Dynam. Differential Equations **3** (1999), 471–513.

26. Sasu A. L. and Sasu B., *A lower bound for the stability radius of time-varying systems*, Proc. Amer. Math. Soc. **132** (2004), 3653–3659.

27. Sacker R. J. and Sell G. R., *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations **113** (1994), 17–67.

A. L. Sasu, Faculty of Mathematics and Computer Science, West University of Timişoara, Romania, *e-mail*: `sasu@math.uvt.ro, lbsasu@yahoo.com`,
`http://www.math.uvt.ro/alsasu`