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ON INJEKTIVITY, P-INJEKTIVITY AND YJ-INJEKTIVITY

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Abstract. A sufficient condition is given for a ring to be either strongly regular or left self-injective regular with non-zero socle. If A is a left self-injective ring such that the left annihilator of each element is a cyclic flat left A-module, then A is left self-injective regular. Quasi-Frobenius rings are characterized. A right non-singular, right YJ-injective right FPF ring is left and right self-injective regular of bounded index. Right YJ-injective strongly π-regular rings have nil Jacobson radical. P.I.-rings whose essential right ideals are idempotent must be strongly π-regular. If every essential left ideal of A is an essential right ideal and every singular right A-module is injective, then A is von Neumann regular, right hereditary.

1. Introduction

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. A is called semi-primitive (resp. 1. left non-singular, 2. right non-singular) if $J = 0$ (resp. 1. $Z = 0$, 2. $Y = 0$). For any left A-module M, $Z({}_AM) = \{y \in M \mid l(y) \text{is an essential left ideal of } A\}$ is the singular submodule of M. ${}_AM$ is called singular (resp. non-singular) if $Z({}_AM) = M$ (resp. $Z({}_AM) = 0$). Right singular submodules are similarly defined. Thus $Z = Z({}_AA)$ and $Y = Z(A_A)$.

It is well-known that A is von Neumann regular iff A is absolutely flat (in the sense that all left (right) A-modules are flat). Similarly, we may call A absolutely p-injective [**16**] or absolutely YJ-injective [**27**].

Recall that a left A-module M is

(a) p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M ([**7**], [**13**], [**16**]),
(b) YJ-injective if, for any $0 \neq a \in A$, there exist a positive integer n with $a^n \neq 0$ such that every left A-homomorphism of Aa^n into M extends to one of A into M ([**14**], [**22**], [**23**], [**27**]). A is called a left p-injective (resp. YJ-injective) ring if ${}_AA$ is p-injective (resp. YJ-injective). P-injectivity and YJ-injectivity are similarly defined on the right side.

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Following [**7**], we shall write "A is VNR" whenever A is a von Neumann regular ring. The vast amount of papers on flat modules over non-VNR rings motivates the study of p-injective and YJ-injective modules. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. An ideal of A will always mean a two-sided ideal of A. A is called fully (resp. 1. fully right, 2. fully left) idempotent if every ideal (resp. 1. right ideal, 2. left ideal) of A is idempotent.

Recall that A is left GQ-injective [**21**] if, for any left ideal I isomorphic to a complement left ideal of A, every left A-homomorphism of I into A extends to an endomorphism of ${}_AA$. It is clear that left GQ-injective rings generalize UTUMI's left continuous rings. We know that A/J is VNR and $J = Z$ if A is left GQ-injective [**21**, Proposition 1]. Note that if A is right YJ-injective, then $J = Y$ [**22**, Proposition 1] and [**23**, Lemma 3]) but A/J needs not be VNR even if A is a P.I.-ring (cf. [**2**, p. 853]).

Proposition 1. *Let A be a semi-prime ring such that for any essential left ideal L of A which is an ideal of A, A/L_A is flat. Then A is semi-primitive. If each maximal left ideal of A is either injective or an ideal of A, then A is either strongly regular or a left self-injective regular ring with non-zero socle.*

Proof. For any $b \in J$, set $L = AbA + r(AbA)$. If K is a complement left ideal of A such that $E = L \oplus K$ is an essential left ideal of A, then $AbAK \subseteq AbA \cap K \subseteq L \cap K = 0$ which implies that $K \subseteq r(AbA)$, whence $K = K \cap r(AbA) \subseteq K \cap L = 0$, showing that $E = L$ is an essential left ideal of A. Since L is an ideal of A, A/L_A is flat by hypothesis. Now $b \in L$ implies that $b = db$ for some $d \in L$ [**4**, p. 458]. If $d = u + v$, where $u \in AbA$, $v \in r(AbA)$, then $b = ub + vb$. Since A is semi-prime, $v \in r(AbA) = l(AbA)$ which implies that $vb = 0$. Therefore $(1 - u)b = 0$ and since $u \in J$, $1 - u$ is left invertible in A which yields $b = 0$. This proves that $J = 0$.

Now suppose that every maximal left ideal of A is an ideal of A. Since A is semi-primitive, then A is reduced (cf. the proof of [**22**, Lemma 4.1]. For any $a \in A$, $r(AaA) = r(aA) = l(aA) = l(a)$ and $T = AaA + l(a)$ is an essential left ideal of A which is an ideal of A. Therefore A/T_A is flat and since $a \in T$, $a = ta$ for some $t \in T$ [**4**, p. 458]. Then $1 - t \in l(a) \subseteq T$ implies that $1 \in T$, whence $A = T = AaA + l(a)$. If $1 = w + s$, $w \in AaA$, $s \in l(a)$, $a = wa \in (Aa)^2$. We have just shown that A is fully left idempotent. Therefore A is strongly regular by [**1**, Theorem 3.1]. Next suppose there exist a maximal left ideal M of A which is not an ideal of A. By hypothesis, ${}_AM$ is injective. Then A is left self-injective by [**26**, Lemma 4]. Since $J = 0$, A is VNR with non-zero socle. □

The first part of Proposition 1 shows the validity of the next result.

Proposition 2. *Let A be a semi-prime left GQ-injective ring such that for any essential left ideal L of A which is an ideal of A, A/L_A is flat. Then A is VNR.*

(Apply [**21**, Proposition 1]).

If "left GQ-injective" is replaced by "right GQ-injective", Proposition 2 remains valid.

It is now known that right p-injective left p.p. rings need not be VNR [**5**, p. 271]. In other words, if every principal left ideal of A is a projective left annihilator, then A needs not be VNR.

However, we may have

Proposition 3. *The following conditions are equivalent:*

(1) *A is VNR,*

(2) *Every principal left ideal of A is the flat left annihilator of an element of A.*

Proof. (1) implies (2) evidently.

Assume (2). For any $b \in A$, $Ab = l(c)$, $c \in A$, and ${}_AAb$ is flat. Since $Ac = l(d)$, $d \in A$, ${}_AAc$ being flat, then $A/Ab = A/l(c) \approx Ac$ which implies that A/Ab is a finitely related flat left A-module which is therefore projective [**4**, p. 459]. It follows that ${}_AAb$ is a direct summand of ${}_AA$ and hence (2) implies (1). □

Remark 1. A right p-injective, right Noetherian, left semi-hereditary ring is semi-simple Artinian [**6**, Lemma 20.27].

Proposition 4. *The following conditions are equivalent:*

(1) *A is a left continuous VNR ring,*

(2) *Every principal left ideal and every complement left ideal of A are flat left annihilators of elements of A.*

Proof. (1) implies (2) evidently.

Assume (2). By Proposition 3, A is VNR which implies that every left ideal isomorphic to a direct summand of ${}_AA$ is a direct summand of ${}_AA$. For each element $b \in A$, $l(b)$ is a direct summand of ${}_AA$. Therefore every complement left ideal is a direct summand of ${}_AA$. Thus A is left continuous and (2) implies (1). □

Proposition 5. *The following conditions are equivalent:*

(1) *A is a left self-injective regular ring,*

(2) *A is left self-injective ring such that the left annihilator of each element of A is a cyclic flat left A-module.*

Proof. It is clear that (1) implies (2).

Assume (2). Since A is left self-injective, it is well-known that every finitely generated right ideal of A is a right annihilator. For any $a \in A$, $l(a) = Ac$, $c \in A$, is a cyclic flat left A-module. Now $l(c) = Au$, $u \in A$, and $A/Au = A/l(c) \approx Ac$ is a finitely related flat left A-module. Then ${}_AA/Au$ is projective [**4**, p. 459] which implies that ${}_AAu$ is a direct summand of ${}_AA$. Since cA is a right annihilator, $cA = r(l(cA)) = r(l(c)) = r(Au)$ which is therefore a direct summand of A_A. Now $cA \approx A/r(c)$ implies that $r(c)$ is a direct summand of A_A, whence $aA = r(l(aA)) = r(l(a)) = r(Ac) = r(c)$ is a direct summand of A_A. A is therefore VNR and (2) implies (1). □

Quasi-Frobeniusean rings are left and right Artinian, self-injective rings whose one-sided ideals are annihilators.

Proposition 6. *If A is a commutative ring whose p-injective modules are injective and flat, then A is quasi-Frobenius.*

Proof. Since every direct sum of p-injective A-modules is p-injective, and every p-injective A-module is, by hypothesis, injective, then any direct sum of injective A-modules is injective which implies that A is a Noetherian ring [**7**, Theorem 20.1]. Since every injective A-module is flat, then A must be a p-injective ring by [**8**, Theorem 3.3]. Therefore A is quasi-Frobenius by a result of H. H. Storrer [**11**, Proposition 2]. □

Corollary 6.1. A commutative ring A is a principal ideal quasi-Frobenius ring iff every finitely generated ideal of A is principal and every p-injective A-module is injective and flat.

QUESTION: Are commutative quasi-Frobenius rings characterized by the hypothesis of Proposition 6?

Proposition 7. *Let A be a ring such that the injective hull of ${}_AA$ is a generator of A-mod. If every p-injective left A-module is injective, then A is quasi-Frobenius.*

Proof. Let E be the injective hull of ${}_AA$. For any projective left A-module P, there exist Q, a direct sum of copies of E, and an epimorphism $g :_A E \to E_A P$. We know that any direct sum of p-injective left A-modules is p-injective. Therefore ${}_AQ$ is p-injective. Now $E/\ker g \approx P$ implies that $\ker g$ is a direct summand of ${}_AE$. Then $E \approx \ker g \oplus (E/\ker g)$ implies that ${}_AE/\ker g$ is p-injective. By hypothesis, ${}_AE/\ker g$ is injective which yields ${}_AP$ injective. By [**7**, Theorem 3.5C], A is quasi-Frobenius. □

Remark 2. Indeed, quasi-Frobenius rings may be characterized as follows: A is quasi-Frobenius iff the injective hull of ${}_AA$ is a generator of A-Mod and every p-injective projective left A-module is injective.

Recall that (a) A is right PF if every faithful right A-module is a generator of Mod-A; (b) A is right FPF if every finitely generated faithful right A-module is a generator of Mod-A.

Note that if A is right PF, then any projective right (or left) A-module is p-injective. Also, PF-rings may be decomposed as follows: If A is right PF, then $A = E(J_A) \oplus B$, where $E(J_A)$ is the injective hull of J_A and B is a semi-simple Artinian ring (cf. [**24**, p. 103]).
(Any non-singular right A-module is completely reducible and injective).

Proposition 8. *Let A be a right non-singular, right YJ-injective, right FPF ring. Then A is a right self injective regular ring of bounded index. Moreover, every essential right ideal of A contains a non-zero ideal which is an essential right ideal of A.*

Proof. Since A is right non-singular, then Q, the maximal right quotient ring of A, is VNR and Q is a right self-injective ring. Since A is right YJ-injective, $J = Y$

by [**22**, Proposition 1] and [**23**, Lemma 3]. By [**25**, Lemma 6], every non-zero-divisor is invertible in A and consequently, A coincides with its classical right (and left) quotient ring. By [**3**, Theorem 1.3], A coincides with Q. By [**3**, Theorem 1.8], A is a right self-injective regular ring of bounded index and every essential right ideal of A contains a non-zero ideal which is an essential right ideal. □

The following corollary follows from a theorem of S. Page [**7**, Theorem 5.49].

Corollary 8.1. The following conditions are equivalent:

(1) A is a right self-injective VNR ring of bounded index;
(2) A is a left self-injective VNR ring of bounded index;
(3) A is a right non-singular right YJ-injective right FPF ring;
(4) A is a left non-singular left YJ-injective left FPF ring.

A is called a right bounded ring if every essential right ideal of A contains a non-zero ideal of A [**7**, p. 117]. Any right FPF ring is right bounded. A special case of right bounded rings is an ERT ring. (A is called ERT if every essential right ideal of A is an ideal of A).

Remark 3. If A is a semi-prime ERT ring containing an injective maximal right ideal, then A is a right and left self-injective regular, right and left FPF, right and left V-ring of bounded index (cf. [**7**, Theorem 5.49] and [**20**, Lemma 1.1]).
(In that case, A contains an injective maximal left ideal).

Note that if A is a prime right bounded ring, then A is right non-singular.

Proposition 9. *Let A be a left or right YJ-injective ring whose divisible singular left modules are injective and flat. Then A is a VNR, left hereditary ring.*

Proof. Let M be an injective left A-module, N a submodule of M. If $E(N)$ denotes the injective hull of ${}_AN$, then $M = E(N) \oplus P$ for some submodule P of M. Now $E(N)/N$ is a singular left A-module which is divisible (since any quotient module of a divisible left A-module is divisible) and by hypothesis, $E(N)/N$ is injective. Also, $(P + N)/N \approx P$ and therefore $M/N = E(N)/N \oplus (P + N)/N$ is injective. It is well-known that A is then left hereditary. Since A is either left or right YJ-injective, by [**25**, Lemma 6], every non-zero-divisor is invertible in A. For any $a \in A$, let C be a complement left ideal of Asuch that $L = Aa \oplus C$ is an essential left ideal of A. Then ${}_AA/L$ is singular, divisible which is therefore flat. Now $a \in L$ implies that $a = au$ for some $u \in L$ [**4**, p. 458]. If $u = ba + c$, $b \in A$, $c \in C$, then $a = aba + ac$ which yields $a - aba = ac \in Aa \cap C = 0$, proving that A is VNR. □

The next remark is motivated by [**10**, Theorem 3.4].

Remark 4. *Let A have a two-sided classical quotient ring which is semi-simple, Artinian. Then A is left hereditary iff every divisible singular left A-module is injective.*

We now give a sufficient condition for the Jacobson radical to be nil.

Proposition 10. *Let A be a right YJ-injective, strongly Π-regular ring. Then J, the Jacobson radical of A, is a nil ideal.*

Proof. By [**22**, Proposition 1] and [**23**, Lemma 3], $J = Y$, the right singular ideal of A. Suppose that J is not nil. Then there exist $y \in J = Y$ such that $y^m \neq 0$ for all positive integers m. Since A is strongly Π-regular, there exist a positive integer n such that $y^n = dy^{n+1}$ for some $d \in A$. Now $r(dy) \cap y^n A = 0$ and since $dy \in Y$, then $y^n = 0$, a contradiction! □

This proves that J is a nil ideal of A.

Theorem 11. *If A is a P.I.-ring whose essential right ideals are idempotent, then A is strongly Π-regular.*

Proof. Let B be a prime factor ring of A. Since every essential right ideal of A is idempotent, then this property is inherited by B. Let T be a non-zero ideal of B, $t \in T$. Let K be a complement right subideal of T such that $R = tB \oplus K$ is an essential right subideal of T. Since T is an essential right ideal of B, then so is R. Therefore $R = R^2$. Now $t \in R^2$ implies that

$$t = \sum (ta_i + k_i)(tb_i + c_i), \quad a_i, b_i \in B, \quad k_i, c_i \in K,$$

whence

$$t - \sum ta_i(tb_i + c_i) = \sum k_i(tb_i + c_i) \in K \cap tB = 0.$$

Therefore

$$t = \sum ta_i(tb_i + c_i) \in tT.$$

We have proved that for any $b \in B$, $b \in (bB)^2$ which yields $bB = (bB)^2$. B is therefore a fully right idempotent ring. Since A is P.I.-ring whose prime factor rings are fully right idempotent, by [**9**, Lemma 5], A is strongly Π-regular. □

Combining Proposition 10 with Theorem 11, we get

Proposition 12. *If A is a right YJ-injective, P.I.-ring whose essential right ideals are idempotent, then A is a strongly Π-regular ring with nil Jacobson radical.*

If A is a right YJ-injective ring, then any minimal left ideal of A is a left annihilator [**22**, Lemma 3].

Proposition 13. *Let A be a left and right YJ-injective, right Noetherian, semi-perfect, strongly Π-regular ring. Then A is QF.*

Proof. By [**23**, Lemma 3], every minimal one-sided ideal of $A$is an annihilator. By Proposition 10, J is a nil ideal. Since A is right Noetherian, J is nilpotent. Since A is semi-perfect, then A/J is Artinian. Consequently, A is semi-primary. It follows that A is right Artinian. By [**11**, Proposition 1], A is quasi-Frobenius. □

The following proposition connects p-injectivity, YJ-injectivity with PF and FPF rings.

Proposition 14.

(1) *A semi-perfect right YJ-injective right FPF ring is right self-injective;*
(2) *A is right PF iff A is a left and right p-injective, semi-perfect, right FPF, left Kasch ring;*
(3) *(a) If A is right YJ-injective, left perfect, right FPF, then A is right PF;*
(b) If, further, A contains an injective maximal left ideal, then A is a quasi-Frobenius ring.

Proof. (1) Apply [**7**, Theorem 5.43] to [**22**, Proposition 1] and [**23**, Lemma 3].
(2) Apply [**24**, Proposition 6(3)] to (1).
(3) (a) Apply [**7**, Corollary 4.21] to (1).
(b) If A is a right Kasch ring containing an injective maximal left ideal, then A is left self-injective [**26**, p. 14]. Then (b) follows from [**7**, Theorem 4.22 and Theorem 4.23A] and (a).

□

A ring whose singular right modules are p-injective needs not be VNR. Indeed, the 2×2 upper triangular matrix ring over a field is a P.I., left and right Artinian, hereditary ring whose singular right (and left) modules are injective but is not VNR (the Jacobson radical is non zero).

However, A is VNR iff all projective and singular left A-modules are p-injective (cf. [**17**, Theorem 9]).

If every injective right A-module is flat, then every projective left A-module is p-injective. But if every singular right A-module is flat, then A must be VNR [**18**, Theorem 5].

Note that [**27**, Theorem 9] implies the following:

A is VNR iff for each $0 \neq a \in A$, there exist a positive integer n such that Aa^n is a non-zero direct summand of ${}_AA$. Also, A is Π-regular iff every left A-module M has the following property: for each $a \in A$, there exist a positive integer n such that every left A-homomorphism of Aa^n into M extends to one of A into M [**27**, Theorem 3].

Concerning p-injectivity, some authors prefer the whole expression “principal injectivity” (cf. for example, T. Y. Lam: Lectures on modules and ring. Graduate texts in Math. Springer 189(1998)) but the term “p-injectivity” is used in numerous papers since several years and, in particular, in the books of C. Faith [**7**] and R. Wisbauer [**13**].

A commutative ring whose singular modules are injective is VNR, hereditary [**7**, Theorem 4.1E]. Our last theorem gives a non-commutative version of that result. If A is semi-prime, it is well-known that any essential left ideal of A which is an ideal of A must be an essential right ideal of A. But the converse is obviously not true.

Proposition 15. *Let A be a ring such that any essential left ideal which is an ideal of A is an essential right ideal of A. If every cyclic singular right A-module is p-injective, then A is fully right idempotent.*

Proof. For any $b \in A$, set $E = AbA + r(AbA)$. Then E is an essential left ideal of A as in Proposition 1. Since E is an ideal of A, by hypothesis, E_A is essential in A_A. Since $E \subseteq AbA + r(b)$, then $R = AbA + r(b)$ is an essential right ideal of A. A/R_A is cyclic singular which is therefore p-injective. Define the right A-homomorphism $f : bA \to A/R$ by $f(ba) = a + R$ for all $a \in A$. Then there exist $d \in A$ such that $1 + R = f(b) = db + R$. Now $1 - db \in R$ which implies that $1 \in R$ (in as much as $db \in AbA \subseteq R$), leading to $A = R = AbA + r(b)$. Therefore $b \in (bA)^2$ which proves that A is fully right idempotent. $\square$

Applying [**18**, Proposition 9] to Proposition 15, we get

Proposition 16. *If every essential left ideal of A is an essential right ideal of A and every singular right A-module is injective, then A is VNR, right hereditary.*

We add a last remark motivated by Corollary 8.1.

Remark 5. A is simple Artinian if A is a prime ring having an injective maximal left ideal and of bounded index.

Since its introduction by R. E. Johnson (1957), the concept of the singular submodule of a module has motivated a tremendous amount of research in that area (a standard reference is K. R. Goodearl's book: Ring Theory, Non singular rings and modules, Marcel Dekker (1976)).

Rings whose singular modules are injective were introduced and developed by K. R. Goodearl (1972). In that direction, we may note that if A is a left non-singular ring, the singular submodule of every injective left A-module is injective [**15**, Theorem 4]. This result leads to a negative answer to a question raised by F. L. Sandomierski [**15**, p. 339]. A. Zak's comment in MR40(1970)#5664 and the paper of A. K. Tiwary and S. A. Paramhans: On closures of submodules. Indian J. Pure and Appl. Math. 8 (1977), 1415–1419(MR 80i#16041)). Non-singular rings include VNR rings, hereditary rings, semi-prime Noetherian rings and prime rings with non-zero socle. For VNR and associated rings, one may consult K. R. Goodearl's classic: Von Neumann regular rings, Pitman (1979).

Quoting C. Faith [**7**, p. 180], the most significant and imaginative departure from the structure theory of N. Jacobson is R. E. Jonhnson's concept of the maximal quotient ring of a non-singular ring (1951). A is a left non-singular ring iff A has a VNR maximal left quotient ring Q. In that case, ${}_AQ$ is the injective hull of ${}_AA$ and Q is a left self-injective ring. In general, an arbitrary ring A is not always embeddable in a self-injective ring [**7**, p. 309]. But Menal-Vamos' theorem [**7**, Theorem 6.1] guarantees that any ring may be embedded in a left (and right) p-injective ring.

Finally, a last result on singular submodules due to J. Zelmanovitz (Canad. J. Math. 23(1971), 1094–1101, Corollary 10): If A is VNR left self-injective, any essentially finitely generated left A-module contains its singular submodule as a direct summand.

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