Acta Math. Univ. Comenianae
Vol. LXXIII, 2(2004), pp. 187–196

COMPONENTS AND LOCAL PREHOMOGENEITY

S. AL GHOUR

Abstract. Prehomogeneous components are introduced. Then the concept of local prehomogeneity as a generalization of both prehomogeneity and local homogeneity is introduced. As a generalization of local prehomogeneity, the concept of prelocal prehomogeneity is also introduced. Many results concerning these concepts are obtained. Several counter examples regarding the relations obtained in this paper are given. Many open questions are also proposed.

1. Introduction

Let A be a subset of a space (X,τ). We denote the complement of A in X by $X-A$, the closure and the interior of A respectively by $\overline{A}$ and $\mathrm{Int}(A)$, the relative topology on A by $\tau|A$, A is *preopen* [**6**] if $A \subseteq \mathrm{Int}(\overline{A})$. $\mathrm{PO}(X,\tau)$ is the family of all preopen sets in X. The topology on X with the subbase $\mathrm{PO}(X,\tau)$ will be denoted by τ^* and is called *the topology generated by preopen sets* [**7**]. The union of all preopen subsets of X contained in A is called the *preinterior* of A and is denoted by $\mathrm{pInt}(A)$. The complement of a preopen set is called *preclosed*. A is called *preclopen* if it is both preopen and preclosed. (X,τ) is called *preconnected* [**2**] if X can not be written as union of two non empty disjoint preopen sets. A is *α-set* [**8**] if $A \subseteq \mathrm{Int}(\overline{\mathrm{Int}(A)})$. The family of all α-sets in a space (X,τ), denoted by τ^α is again a topology on X satisfying $\tau \subseteq \tau^\alpha$. The complement of an α-set is called *α-closed*. A is called *α-clopen* if it is both α-set and α-closed. A function $f:(X,\tau)\to(X,\tau)$ is *preirresolute* [**2**] if $f^{-1}(A)\in \mathrm{PO}(X,\tau_1)$ for all $A\in \mathrm{PO}\,(Y,\tau_2)$. f is *a prehomeomorphism* [**7**] if f is bijective and $A\in \mathrm{PO}\,(X,\tau_1)$ iff $f\,(A)\in \mathrm{PO}\,(Y,\tau_2)$, i.e., f is a bijection and both f and f^{-1} are preirresolute. f is an *α-homeomorphism* [**7**] if f is bijective and $A\in\tau^\alpha$ iff $f(A)\in\tau^\alpha$. If (X,τ) is a space, then $\mathrm{PH}\,(X,\tau)$ will denote the group of all prehomeomorphisms from (X,τ) onto itself.

Sierpinski [**9**], introduced the notion of homogeneity as follows:

Received December 12, 2003.
2000 *Mathematics Subject Classification.* Primary 54B, 54C.
Key words and phrases. Homogeneous and prehomogeneous components, minimal preopen sets, homogeneity and local homogeneity, prehomogeneity and local prehomogeneity, generated topologies.

Definition 1.1. A space (X, τ) is homogeneous if for any two points $x, y \in X$ there exists an autohomeomorphism f on (X, τ) such that $f(x) = y$.

Afterwards, many modifications of homogeneity were introduced.

Definition 1.2. [**3**] A space (X, τ) is called LH (locally homogeneous) at x in X provided that there exists an open set U in X containing x such that for any $y \in U$ there is a homeomorphism $f : (X, \tau) \to (X, \tau)$ such that $f(x) = y$. A space (X, τ) is called LH if it is LH at each $x \in X$.

Let $\backsim$ be a relation defined on X by $x \backsim y$ if there is an autohomeomorphism f on (X, τ) such that $f(x) = y$. This relation is an equivalence relation on X. The equivalence class $C(X, \tau, x) = \{y \in X : x \backsim y\}$ is called *homogeneous component of* X at x. Homogeneous components are invariants under homeomorphisms and indeed homogeneous subspaces of X. It is clear that (X, τ) is homogeneous iff it has only one homogeneous component.

Homogeneous components have played a vital role in homogeneity research. The author in [**3**] has used homogeneous components in characterizing LH spaces. In [**1**], Al Ghour defines prehomogeneity as a generalization of homogeneity as follows:

Definition 1.3. A space (X, τ) is prehomogeneous if for any two points $x, y \in X$ there exists $f \in PH(X, \tau)$ such that $f(x) = y$.

He obtained many results concerning prehomogeneity.

In the present paper, prehomogeneous components are introduced and studied. Then the concept of local prehomogeneity as a generalization of both prehomogeneity and local homogeneity is introduced. As a generalization of local prehomogeneity, prelocal prehomogeneity is introduced. Homogeneous components and prehomogeneous components will characterize local prehomogeneity and prelocal prehomogeneity respectively. Some open questions regarding the concepts of this paper are proposed.

Definition 1.4. [**1**] Let (X, τ) be a space. A non empty preopen set A of X is called a minimal preopen set in X if any preopen set in X which is contained in A is $\emptyset$ or A.

Definition 1.5. [**2**] A space (X, τ) is locally indiscrete if every open subset of X is closed.

The following Lemmas will be used in the sequel.

Lemma 1.1. [**5**] *Every homeomorphism is a prehomeomorphism but not conversely.*

Lemma 1.2. [**1**] *Let* (X, τ) *be a space and* $A \subseteq X$, *then* A *is a minimal preopen set in* X *iff* $A \in \mathrm{PO}(X, \tau)$ *and* A *is a singleton.*

Lemma 1.3. [**2**] *Let* (X, τ) *be a space. If* $A \subseteq B \subseteq X$ *and* $A \in \mathrm{PO}(X, \tau)$, *then* $A \in \mathrm{PO}(B, \tau|B)$.

Lemma 1.4. [**2**] *For a space (X,τ) the following are equivalent.*

a) *(X,τ) is locally indiscrete.*
b) *Every singleton in X is preopen.*

Lemma 1.5. [**2**] *Let (X,τ) be a space and $A, B \subseteq X$ such that $A \in \mathrm{PO}\,(B, \tau\,|B\,)$ and $B \in \mathrm{PO}\,(X,\tau)$. Then $A \in \mathrm{PO}\,(X,\tau)$.*

Lemma 1.6. *If $f : (X,\tau_1) \to (Y,\tau_2)$ is a prehomeomorphism, $A \in \mathrm{PO}\,(X,\tau_1)$ and $f\,(A) \in \mathrm{PO}\,(Y,\tau_2)$, then the restriction function of f on A $f_{|A} : (A,\tau_1\,|A\,) \to (f\,(A)\,, \tau_2\,|f\,(A)\,)$ is a prehomeomorphism.*

Proof. Lemmas 1.3 and 1.5. □

Lemma 1.7. [**1**] *Let (X,τ) be a space which contains a minimal preopen set. Then the following are equivalent.*

a) *(X,τ) is prehomogeneous.*
b) *(X,τ) is locally indiscrete.*

Lemma 1.8. [**5**] *If X, Y are T_1 spaces, then the classes of prehomeomorphisms and α-homeomorphisms from X onto Y coincide.*

Lemma 1.9. [**3**] *A space (X,τ) is locally homogeneous at $x \in X$ iff $C\,(X,\tau,x)$ is open in X.*

Lemma 1.10. [**3**] *A space (X,τ) is locally homogeneous iff $C\,(X,\tau,x)$ is clopen in X for all $x \in X$.*

Lemma 1.11. [**3**] *Every connected* LH *space is homogeneous.*

Lemma 1.12. [**2**] *Let (X,τ) be a space and $A, B \subseteq X$. If $A \in \tau^{\alpha}$ and $B \in \mathrm{PO}\,(X,\tau)$, then $A \cap B \in \mathrm{PO}\,(X,\tau)$.*

Lemma 1.13. *Let (X,τ) be a space and let A be an α-clopen subset of X. Suppose that $f_1 \in \mathrm{PH}\,(A,\tau\,|A\,)$ and $f_2 \in \mathrm{PH}\,(X-A,\tau\,|X-A\,)$, and define $f : (X,\tau) \to (X,\tau)$ by*

$$f\,(x) = \begin{cases} f_1\,(x) & if\ x \in A \\ f_2\,(x) & if\ x \in X-A \end{cases}$$

Then $f \in \mathrm{PH}\,(X,\tau)$.

Proof. Follows from the definition and Lemma 1.12. □

Lemma 1.14. [**4**] *Let (X,τ) be a space. Then $\mathrm{PO}\,(X,\tau) = \mathrm{PO}\,(X,\tau^{\alpha})$.*

2. Prehomogeneous Components

Definition 2.1. Let (X,τ) be a space. We define the equivalence relation $\widetilde{p}$ on X as follows. For $x_1, x_2 \in X$, we say $x_1\ \widetilde{p}\ x_2$ iff there exists $f \in \mathrm{PH}\,(X,\tau)$ such that $f\,(x_1) = x_2$.

Definition 2.2. *A subset of a space (X,τ) which has the form $\mathrm{pC}(X,\tau,x) = \{y \in X : x\ \widetilde{p}\ y\}$ is called the prehomogeneous component of X at x.*

The following result follows easily.

Proposition 2.1. *A space (X,τ) is prehomogeneous iff it has exactly one prehomogeneous component.*

Proposition 2.2. *Let (X,τ) be a space and $x \in X$. Then $C(X,\tau,x) \subseteq \mathrm{pC}(X,\tau,x)$ for all $x \in X$.*

Proof. Lemma 1.1. □

The following proposition will be needed in the sequel.

Proposition 2.3. *If $f \in \mathrm{PH}(X,\tau)$, then $f(\mathrm{pC}(X,\tau,x)) = \mathrm{pC}(X,\tau,x)$ for any prehomogeneous component $\mathrm{pC}(X,\tau,x)$.*

Proof. If $y \in f(\mathrm{pC}(X,\tau,x))$, then $y = f(s)$ for some $s \in \mathrm{pC}(X,\tau,x)$. Therefore, $y \,\widetilde{p}\, s$ and $s \,\widetilde{p}\, x$ and hence $y \,\widetilde{p}\, x$. Thus, $y \in \mathrm{pC}(X,\tau,x)$. Conversely, let $y \in \mathrm{pC}(X,\tau,x)$ and choose $s \in X$ such that $f(s) = y$, then $s \in \mathrm{pC}(X,\tau,x)$ and so $y \in f(\mathrm{pC}(X,\tau,x))$. □

Theorem 2.1. *Let (X,τ) be a space, $x \in X$ and*

$$M = \{y : \{y\} \text{ is a minimal preopen set in } (X,\tau)\}.$$

Then the following are equivalent.

(a) *$\{x\}$ is a minimal preopen set in (X,τ).*
(b) *$\mathrm{pC}(X,\tau,x) \subseteq M$.*
(c) *$\mathrm{pC}(X,\tau,x) \in \mathrm{PO}(X,\tau)$ and the subspace $(\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$ is locally indiscrete.*

Proof. (a) $\Rightarrow$ (b) Let $y \in \mathrm{pC}(X,\tau,x)$, then there exists $f \in \mathrm{PH}(X,\tau)$ such that $f(x) = y$ and so $f(\{x\}) = \{y\}$. Thus, $\{y\} \in \mathrm{PO}(X,\tau)$ and so by Lemma 1.2, it follows that $\{y\}$ is a minimal preopen set. Therefore, $y \in M$.

(b) $\Rightarrow$ (c) Since $\mathrm{pC}(X,\tau,x) \subseteq M$, then for each $y \in \mathrm{pC}(X,\tau,x), \{y\} \in \mathrm{PO}(X,\tau)$ and by Lemma 1.3, $\{y\} \in PO(\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$. Therefore, $\mathrm{pC}(X,\tau,x) \in \mathrm{PO}(X,\tau)$ and by Lemma 1.4, it follows that the subspace $(\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$ is locally indiscrete.

(c) $\Rightarrow$ (a) Since $(\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$ is locally indiscrete, then by Lemma 1.4, it follows that $\{x\}$ is preopen in $(\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$. Since $\mathrm{pC}(X,\tau,x) \in \mathrm{PO}(X,\tau)$, it follows by Lemma 1.5 that $\{x\} \in \mathrm{PO}(X,\tau)$, and so by Lemma 1.2, it follows that $\{x\}$ is a minimal preopen set in (X,τ). □

In Theorem 2.1, the equality in (b) does not hold in general as the following example shows.

Example 2.1. Let $X = \{a,b,c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$. Then the set M defined in Theorem 2.1 is $M = \{a,b\}$. However, $\mathrm{pC}(X,\tau,a) = \{a\}$.

The following known result shows that homogenous components are homogeneous subspaces.

Proposition 2.4. *Let (X,τ) be a space and $x \in X$. Then the subspace $(C(X,\tau,x), \tau|C(X,\tau,x))$ is homogeneous.*

Question 2.1. Is every prehomogeneous component of a space (X,τ) a prehomogeneous subspace?

The following theorem answers Question 2.1 partially.

Theorem 2.2. *Let (X,τ) be a space and $x \in X$. If $\mathrm{pC}(X,\tau,x) \in \mathrm{PO}(X,\tau)$, then the subspace $(\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$ is prehomogeneous.*

Proof. Let $x_1, x_2 \in \mathrm{pC}(X,\tau,x)$, then there are $f_1, f_2 \in \mathrm{PH}(X,\tau)$ such that $f_1(x_1) = x$ and $f_2(x) = x_2$. Define $f : (X,\tau) \to (X,\tau)$ by $f = f_2 \circ f_1$, then $f \in \mathrm{PH}(X,\tau)$ and $f(x_1) = x_2$. Now by Proposition 2.3, $f(\mathrm{pC}(X,\tau,x)) = \mathrm{pC}(X,\tau,x)$. Therefore, since $\mathrm{pC}(X,\tau,x) \in \mathrm{PO}(X,\tau)$, it follows by Lemma 1.6, that the restriction function

$$f_{|\mathrm{pC}(X,\tau,x)} : (\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x)) \to (\mathrm{pC}(X,\tau,x), \tau|\mathrm{pC}(X,\tau,x))$$

is a prehomeomorphism takes x_1 to x_2. □

The following theorem improves Theorem 3.11 of [1]. Their proofs are similar.

Theorem 2.3. *Let (X,τ) be a space and $x \in X$. Then $\mathrm{pC}(X,\tau,x) = \mathrm{pC}(X,\tau^\alpha,x)$.*

Corollary 2.1. *Let (X,τ) be a space and $x \in X$. Then $C(X,\tau^\alpha,x) \subseteq \mathrm{pC}(X,\tau,x)$.*

In Example 2.2 below we shall see that the equality in Corollary 2.1 does not hold in general. However, in T_1 spaces the following result improves Theorem 3.13 of [1].

Theorem 2.4. *If (X,τ) is a T_1 space and $x \in X$, then $C(X,\tau^\alpha,x) = \mathrm{pC}(X,\tau,x)$.*

Proof. Lemma 1.8. □

Lemma 2.1. *Let (X,τ) be a space. Then $\mathrm{PH}(X,\tau) \subseteq H(X,\tau^*)$ where $H(X,\tau^*)$ is the group of homeomorphisms from (X,τ^*) onto itself.*

Proof. The proof is similar to that used in Theorem 3.14 of [1]. □

Theorem 2.5. *Let (X,τ) be a space and $x \in X$. Then $\mathrm{pC}(X,\tau,x) \subseteq C(X,\tau^*,x)$.*

Proof. Lemma 2.1. □

Example 3.15 of [1] shows that the equality in Theorem 2.5 does not hold in general. However, we have the following easy to prove result.

Theorem 2.6. *Let (X,τ) be a space and $x \in X$. Then*

$$\bigcup_{y \in C(X,\tau^*,x)} \mathrm{pC}(X,\tau,y) = C(X,\tau^*,x).$$

Throughout Examples 2.2, 2.3 and 2.4 we are going to calculate homogeneous and prehomogeneous components for some spaces. These examples will be also used in the sequel.

Example 2.2. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then by Lemma 1.7, it is easy to see that (X, τ) is prehomogeneous and so by Proposition 2.1, it follows that $\mathrm{pC}(X, \tau, a) = X$. On the other hand, it is not difficult to see that $C(X, \tau, a) = C(X, \tau^{\alpha}, a) = \{a\}$ and $C(X, \tau, b) = C(X, \tau^{\alpha}, b) = \{b, c\}$.

Example 2.3. Consider the usual Euclidean space $X = (0, 2) \cup \{3\}$ with the usual topology. Then $C(X, \tau, 1) = \mathrm{pC}(X, \tau, 1) = (0, 2)$ and $C(X, \tau, 3) = \mathrm{pC}(X, \tau, 3) = \{3\}$.

Proof. It is not difficult to see that $C(X, \tau, 1) = (0, 2)$ and $C(X, \tau, 3) = \{3\}$. Now by Proposition 2.2, and the fact that prehomogeneous components form a partition on X, we conclude that $\mathrm{pC}(X, \tau, 1) = (0, 2)$ or $\mathrm{pC}(X, \tau, 1) = X$. If $\mathrm{pC}(X, \tau, 1) = X$, then by Proposition 2.1, it follows that X is prehomogeneous. Since $\{3\}$ is a minimal preopen set, it follows using Lemma 1.7 that X is locally indiscrete which is not true. Therefore, $\mathrm{pC}(X, \tau, 1) = (0, 2)$ and $\mathrm{pC}(X, \tau, 3) = \{3\}$. $\square$

Example 2.4. Let R be the set of real numbers with the topology τ having the family $\{[-a, a] : a \in R \text{ and } a > 1\}$ as a base. Then

(a) $[-1, 1] \subseteq C(X, \tau, 0)$.
(b) $(1, \infty) \subseteq C(X, \tau, 2)$.
(c) $(-\infty, -1) \subseteq C(X, \tau, 2)$.
(d) $C(X, \tau, 0) = \mathrm{pC}(X, \tau, 0) = [-1, 1]$ and $C(X, \tau, 2) = \mathrm{pC}(X, \tau, 2) = R - [-1, 1]$.

Proof. (a) Let $x \in [-1, 1]$. Define $f : (X, \tau) \to (X, \tau)$ by $f(x) = 0$, $f(0) = x$, and $f(t) = t$ elsewhere. Then for each $a > 1$, $f^{-1}([-a, a]) = [-a, a]$ and so f and $f^{-1} = f$ are both continuous. Therefore, since f is a bijection, it follows that f is a homeomorphism and so $x \in C(X, \tau, 0)$.

(b) Let $x > 1$. Define $f : (X, \tau) \to (X, \tau)$ by

$$f(t) = \begin{cases} (x-1)(t+1) - 1 & \text{if } \; t < -1 \\ t & \text{if } \; -1 \leq t \leq 1 \\ (x-1)(t-1) + 1 & \text{if } \; t > 1 \end{cases}$$

Then for each $a > 1$, $f^{-1}([-a, a]) = [-\alpha, \alpha]$ where $\alpha = \frac{a+x-2}{x-1}$. Since $a > 1$, $a + x - 2 > x - 1$ and so $\alpha > 1$. Therefore, $f^{-1}([-a, a]) \in \tau$ and hence f is continuous. Also if $b > 1$, then $f([-b, b]) = [-\beta, \beta]$ where $\beta = (x-1)(b-1) + 1$. Since $x > 1$ and $b > 1$, $(x-1)(b-1) > 0$ and so $\beta > 1$. Therefore, $f([-b, b]) \in \tau$ and hence f is open. It is not difficult to see that f is a bijection. Therefore, f is a homeomorphism with $f(2) = x$ and so $x \in C(X, \tau, 2)$.

(c) Let $x < -1$. Define $f : (X, \tau) \to (X, \tau)$ by

$$f(t) = \begin{cases} (x+1)(t+1)+1 & \text{if} \quad t < -1 \\ -t & \text{if} \quad -1 \le t \le 1 \\ (x+1)(t-1)-1 & \text{if} \quad t > 1 \end{cases}$$

By a similar proof to that used in (c), we can show that $f : (X, \tau) \to (X, \tau)$ is a homeomorphism with $f(2) = x$. Therefore, $x \in C(X, \tau, 2)$.

(d) Let $M = \{y : \{y\}$ is a minimal preopen set in $(X, \tau)\}$, then it is not difficult to see that $M = [-1, 1]$ and so by Theorem 2.1, it follows that $\mathrm{pC}(X, \tau, 0) \subseteq [-1, 1]$. Therefore, by Proposition 2.2 and (a) we conclude that $C(X, \tau, 0) = \mathrm{pC}(X, \tau, 0) = [-1, 1]$. On the other hand, by (b), (c) and the fact that $\mathrm{pC}(X, \tau, 0) = [-1, 1]$, it follows that $R - [-1, 1] \subseteq C(X, \tau, 2) \subseteq \mathrm{pC}(X, \tau, 2) \subseteq R - [-1, 1]$ and so $C(X, \tau, 2) = \mathrm{pC}(X, \tau, 2) = R - [-1, 1]$. □

3. Local Prehomogeneity

Definition 3.1. A space (X, τ) is called LPH (locally prehomogeneous) at x in X provided that there exists an open set U in X containing x such that for any $y \in U$ there is $f \in \mathrm{PH}(X, \tau)$ such that $f(x) = y$. A space (X, τ) is called LPH if it is LPH at each $x \in X$.

Remark 3.1. Let (X, τ) be a space and $x \in X$. Then (X, τ) is LPH at x iff $x \in \mathrm{Int}(\mathrm{pC}(X, \tau, x))$.

The proof of each of the following two theorems follows from the definitions.

Theorem 3.1. *Every locally homogeneous space is locally prehomogeneous.*

Theorem 3.2. *Every prehomogeneous space is locally prehomogeneous.*

The converse of Theorem 3.1 is not true in general as the following example shows.

Example 3.1. Consider the set of natural numbers N with the topology $\tau = \{\emptyset\} \cup \{\{n, n+1, n+2, \ldots\} : n \in N\}$. Then (N, τ) is LPH but not LH.

Proof. We proved in [**1**] that (N, τ) is a prehomogeneous space and so by Theorem 3.2, it is an LPH space. On the other hand, since (N, τ) is a connected non homogeneous space, it follows by Lemma 1.11, that (N, τ) is not LH. □

In Example 2.3, the homogeneous components are clopen, so by Lemma 1.10, it follows that the space is an LH. Therefore, by Theorem 3.1 it must be LPH. However, it is not prehomogeneous because of Proposition 2.1. This shows that the converse of Theorem 3.2 is not true in general.

As in the case of local homogeneity, Example 2.1 shows that local prehomogeneity at some point does not imply local prehomogeneity in general.

Proposition 3.1. *Let (X, τ) be a space and let $x, y \in X$ for which $x \sim y$. Then (X, τ) is locally prehomogeneous at x iff (X, τ) is locally prehomogeneous at y.*

Proof. Suppose that $x \sim y$ and (X,τ) is locally prehomogeneous at x. Let $f:(X,\tau)\to(X,\tau)$ be a homeomorphism for which $f(x)=y$ and let U_x be an open set for which $x\in U_x\subseteq \mathrm{pC}(X,\tau,x)$. Then $y\in f(U_x)\subseteq f(\mathrm{pC}(X,\tau,x))$, but by Proposition 2.3, we have $f(\mathrm{pC}(X,\tau,x))=\mathrm{pC}(X,\tau,x)$. Therefore, (X,τ) is locally prehomogeneous at y. The other direction is similar to the above one. $\square$

By a similar method to that used in the proof of Proposition 3.1 one can obtain the following result.

Proposition 3.2. *Let (X,τ) be a space and let $x,y\in X$ for which $x\sim y$. Then (X,τ) is locally homogeneous at x iff (X,τ) is locally homogeneous at y.*

Question 3.1. Let (X,τ) be a space and let $x,y\in X$ for which (X,τ) is LPH at x and $x\widetilde{p}y$. Is it true that (X,τ) is LPH at y?

Theorem 3.3. *Let (X,τ) be a space, then the following are equivalent.*

(a) *(X,τ) is LPH.*
(b) *$\mathrm{pC}(X,\tau,x)$ is clopen in X for all $x\in X$.*
(c) *For each $x\in X$, there exists an open set V containing x such that for any $y\in V$, there is $f\in \mathrm{PH}(X,\tau)$ such that $f(x)=y$ and $f(t)=t$ for all $t\in X-V$.*

Proof. (a) $\Rightarrow$ (b) Since $\{\mathrm{pC}(X,\tau,x):x\in X\}$ forms a partition on X, it is sufficient to show that $\mathrm{pC}(X,\tau,x)$ is open in X for each $x\in X$. Let $x\in X$ and let $y\in \mathrm{pC}(X,\tau,x)$. Since (X,τ) LPH at y, there exists an open set U in X such that $y\in U\subseteq \mathrm{pC}(X,\tau,y)$. Since $y\in \mathrm{pC}(X,\tau,x)\cap \mathrm{pC}(X,\tau,y)$, $\mathrm{pC}(X,\tau,x)=\mathrm{pC}(X,\tau,y)$. Therefore, $y\in U\subseteq \mathrm{pC}(X,\tau,x)$ and hence $\mathrm{pC}(X,\tau,x)$ is an open set in X.

(b) $\Rightarrow$ (c) Let $x\in X$, then by (b), $\mathrm{pC}(X,\tau,x)$ is clopen in X. Take $V=\mathrm{pC}(X,\tau,x)$ and let $y\in V$, then there exists $g\in \mathrm{PH}(X,\tau)$ such that $g(x)=y$. Now according to Proposition 2.3, $g(V)=V$. Define $f:(X,\tau)\to(X,\tau)$ by $f(x)=g(x)$ if $x\in V$ and $f(x)=x$ if $x\in X-V$. Then by Lemma 1.13, it follows that $f\in \mathrm{PH}(X,\tau)$. Moreover, $f(x)=y$ and $f(t)=t$ for all $t\in X-V$.

(c) $\Rightarrow$ (a) Obvious. $\square$

The following corollary says that the converse of Theorem 3.2 is true for connected spaces.

Corollary 3.1. *Every connected LPH space is prehomogeneous.*

Theorem 3.4. *If (X,τ) is an LPH space, then (X,τ^α) is LPH.*

Proof. Theorems 2.3 and 3.3, and the fact that $\tau\subseteq\tau^\alpha$. $\square$

Question 3.2. Let (X,τ) be a space for which (X,τ^α) is either LPH or LH. Is it true that (X,τ) is LPH?

Question 3.3. Let (X,τ) be an LPH space. Is it true that (X,τ^α) is LH?

Question 3.4. If we add the assumption that (X,τ) is a T_1 space, which of the implications in Questions 3.2 and 3.3 is true?

Theorem 3.5. *If (X,τ) is an LPH space then (X,τ^*) is LH.*

Proof. Let $x \in X$, then by Theorem 2.6,

$$C(X,\tau^*,x) = \bigcup_{y \in C(X,\tau^*,x)} \mathrm{pC}(X,\tau,y).$$

Since (X,τ) is LPH, then by Theorem 3.3 it follows that $\mathrm{pC}(X,\tau,y) \in \tau$ for each $y \in C(X,\tau^*,x)$. Thus, $C(X,\tau^*,x) \in \tau \subseteq \tau^*$. Therefore, the homogeneous components for (X,τ^*) are clopen in (X,τ^*) and hence by Lemma 1.10, it follows that (X,τ^*) is an LH space. $\square$

In Example 3.15 of [**1**] it is not difficult to see that $\mathrm{pC}(X,\tau,c) = \{c,d\}$ and so by Theorem 3.3 it follows that the space is not LPH. However, we had seen that the space (X,τ^*) is homogeneous and so LH. This shows that the converse of Theorem 3.5 is not true in general.

4. Prelocal Prehomogeneity

Definition 4.1. A space (X,τ) is called PLPH (prelocally prehomogeneous) at x in X provided that there exists a preopen set A in X containing x such that for any $y \in A$ there is $f \in \mathrm{PH}(X,\tau)$ such that $f(x) = y$. A space (X,τ) is called PLPH if it is PLPH at each $x \in X$.

Remark 4.1. Let (X,τ) be a space and $x \in X$. Then (X,τ) is PLPH at x iff $x \in \mathrm{pInt}(\mathrm{pC}(X,\tau,x))$.

The following result shows that PLPH concept is a natural generalization for the LPH concept.

Theorem 4.1. *Every LPH space is PLPH.*

Proposition 4.1. *Let (X,τ) be a space and let $x,y \in X$ for which $x\widetilde{p}y$. Then (X,τ) is PLPH at x iff (X,τ) is PLPH at y.*

Proof. Mimic the proof of Proposition 3.1. $\square$

Theorem 4.2. *Let (X,τ) be a space and $x \in X$, then the following are equivalent.*

(a) *(X,τ) is PLPH at x.*
(b) *pInt*$(\mathrm{pC}(X,\tau,x)) \neq \emptyset$.
(c) $\mathrm{pC}(X,\tau,x)$ *is preopen.*
(d) *(X,τ) is PLPH at y for any $y \in \mathrm{pC}(X,\tau,x)$.*

Proof. (a) $\Rightarrow$ (b) Remark 4.1.

(b) $\Rightarrow$ (c) Suppose that $\mathrm{pInt}(\mathrm{pC}(X,\tau,x)) \neq \emptyset$ and let $y \in \mathrm{pC}(X,\tau,x)$. Choose $t \in \mathrm{pInt}(\mathrm{pC}(X,\tau,x))$. Since $y,t \in \mathrm{pC}(X,\tau,x)$, then there exists $f \in \mathrm{PH}(X,\tau)$ such that $f(t) = y$. Therefore, $y \in f(\mathrm{pInt}(\mathrm{pC}(X,\tau,x))) \subseteq f(\mathrm{pC}(X,\tau,x))$. Since by Proposition 2.3 $f(\mathrm{pC}(X,\tau,x)) = \mathrm{pC}(X,\tau,x)$, it follows that $\mathrm{pC}(X,\tau,x)$ is preopen.

(c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a) are obvious. $\square$

Corollary 4.1. *Every prehomogeneous component is either preopen or has an empty preinterior.*

Corollary 4.2. *Let (X, τ) be a space, then the following are equivalent.*
(a) *(X, τ) is PLPH.*
(b) $p\mathrm{Int}\,(\mathrm{pC}(X, \tau, x)) \neq \emptyset$ *for all $x \in X$.*
(c) $\mathrm{pC}(X, \tau, x)$ *is preclopen for all $x \in X$.*

Corollary 4.3. *Every preconnected PLPH space is prehomogeneous.*

Now by Theorem 3.3 and Corollary 4.2, it follows that the space in Example 2.4 is PLPH but not LPH. This shows that the converse of Theorem 4.1 is not true in general. It is also an example of a connected PLPH space that is not prehomogeneous.

Theorem 4.3. *Let (X, τ) be a space. Then (X, τ) is PLPH iff (X, τ^α) is PLPH.*

Proof. Corollary 4.1, Theorem 2.3 and Lemma 1.14. □

Corollary 4.4. *Let (X, τ) be a space. If (X, τ^α) is LPH, then (X, τ) is PLPH.*

Note that Corollary 4.4 improves Theorem 4.1.

By Corollary 2.1 and the fact that $C\,(X, \tau, x) \subseteq C\,(X, \tau^\alpha, x)$ we can easily conclude that the space (X, τ^α) in Example 2.4 is not LPH. Therefore, the converse of Corollary 4.4 is not true in general.

References

1. Al Ghour S., *Minimality and prehomogeneity*, Acta Math. Univ. Comenianae **72**(2) (2003), 237–244.

2. Dontchev J., *Survey on preopen sets*, The Proceedings of the 1998 Yatsushiro Topological Conference, 1–18.

3. Fora A. A., *New and old types of homogeneity*, Turkish J. Math. **24** (2000), 335–344.

4. Janković D. S., *A note on mappings of extremally disconnected spaces*, Acta Math. Hungar. **46**(1–2) (1985), 83–92.

5. Kupka I., *On some classes of sets related to generalized continuity*, Acta Math. Univ. Comenianae, **56/57** (1989), 55–61.

6. Mashhour A. S., Abd El-Monsef M. E. and El-Deeb S. N., *On pre-continuous and weak pre-continuous mappings*, Proc. Math. Phys. Soc. Egypt., **53** (1982), 47–53.

7. Mashhour A. S., Abd El-Monsef M. E. and Hasanein I. A., *On pretopological spaces*, Bull. Math. Soc. Sci. Math. R. S. Roumanie, (N.S.) **28**(1) (1984), 39–45.

8. Njåstad O., *On some classes of nearly open sets*, Pacific J. Math. **15** (1965), 961–970.

9. Sierpiński W., *Sur une propriété topologique des ensembles dénombrable dense en soi*, Fund. Math., **1** (1920), 11–28.

S. al Ghour, Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid, 22110, Jordan., *e-mail*: `algore@just.edu.jo`