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A NOTE ON UPPER BOUND FOR CHROMATIC NUMBER OF A GRAPH

L. STACHO

Abstract. Let G be a graph and let s be the maximum number of vertices of the same degree, each at least $(\Delta(G)+2)/2$, where $\Delta(G)$ is the maximum degree in G. We show that the chromatic number $\chi(G) \le \left\lceil \frac{s}{s+1}(\Delta(G)+2)\right\rceil$.

A simple graph G is said to be k-**colorable** if its vertices can be colored with at most k colors such that no two adjacent vertices have the same color. The smallest k for which G is k-colorable is called the **chromatic number** $\chi(G)$ of the graph G. Determining $\chi(G)$ is an old and very hard problem. A classical result of Brooks [**2**] says that $\chi(G) \le \Delta(G)+1$, where $\Delta(G)$ denotes the maximum degree in G. In addition, Brooks shoved that the complete graphs and odd cycles are the only graphs for which the upper bound is attained. Excluding the existence of smaller complete subgraphs can further improve the upper bounds for $\chi(G)$, as it can be seen from the following result obtained independently by Borodin and Kostochka [**1**], Catlin [**3**], and Lawrence [**4**]:

Theorem 1. *If $K_r \not\subseteq G$, where $4 \le r \le \Delta(G)+1$, then $\chi(G) \le \frac{r-1}{r}(\Delta(G)+2)$.*

The result is in fact a nice application of Brooks theorem and the following result, observed by Lovász [**5**]: If $d_1 + d_2 + \cdots + d_q \ge \Delta(G) - q + 1$, then $V(G)$ can be decomposed into classes $V_1, V_2, \ldots, V_q$, such that the subgraph G_i induced by V_i has $\Delta(G_i) \le d_i$. Letting $q = \lfloor(\Delta(G)+1)/r\rfloor$, $d_1 = d_2 = \cdots = d_{q-1} = r-1$, and $d_q \ge r-1$ so that $\sum d_i = \Delta(G) - q + 1$ give the upper bound.

If a graph has only small complete subgraphs, then Theorem 1 substantially improves Brooks upper bound. However, if the graph is dense, then it usually has large complete subgraphs and hence, the upper bound from Theorem 1 is almost the same (if not worse) as the original Brooks upper bound. In what follows, we give another relaxation of Brooks theorem based on the following invariant. Let V_i denote the set of vertices of degree i in the graph G. Now, we define $s = \max_{i \ge (\Delta(G)+2)/2} |V_i|$, i.e. s is the maximum number of vertices of the same degree, each at least $(\Delta(G)+2)/2$. Our upper bound is:

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At present, the author is PIMS postdoc fellow at School of Computing Science, Simon Fraser University, Burnaby BC, V5A 1S6 Canada.

Theorem 2. *For any graph G, $\chi(G) \le \left\lceil \frac{s}{s+1}(\Delta(G)+2) \right\rceil$.*

Proof. Let $d_1 \ge d_2 \ge \cdots \ge d_n$ be the degree sequence of G. We let $k = \left\lceil \frac{s}{s+1}(\Delta(G)+2) \right\rceil$. We claim that $d_k < k$. If $d_k < \frac{\Delta(G)+2}{2}$, then since $s \ge 1$, the claim is true. Otherwise, $d_k \ge \frac{\Delta(G)+2}{2}$. Now, for $i = 1, 2, \ldots, k$, $d_i \le \Delta(G) - \left\lceil \frac{i}{s} \right\rceil + 1 < \Delta(G) - \frac{i}{s} + 2$. In particular, $d_k < \Delta(G) - \frac{k}{s} + 2 \le k$, as claimed.

It follows that G is k-degenerate, i.e. any subgraph of G contains a vertex of degree $< k$. It is well-known that vertices of any k-degenerate graph can be properly colored with at most k colors. Thus we have $\chi(G) \le \left\lceil \frac{s}{s+1}(\Delta(G)+2) \right\rceil$. $\square$

The following examples demonstrate that in some cases Theorem 2 gives much better upper bound for $\chi(G)$ as Theorem 1 does. Let H_n be any graph on the vertex set $\{u_1, u_2, \ldots, u_n\}$, in which $d(u_i) \le n - i$. Let G_n be the graph obtained from the union of the complete graph K_n and H_n by connecting u_i to $v_1, v_2, \ldots, v_i$ for $i = 1, 2, \ldots, n$. It is not difficult to observe that G_n contains K_{n+1}, $\Delta(G_n) = 2n - 1$, $s = 1$, and $\chi(G_n) = n + 1$. By Theorem 1, $\chi(G_n) \le (1 - \frac{1}{n+2})(2n+1)$, however by Theorem 2, $\chi(G_n) \le n + 1$, which is, in fact, the exact chromatic number of G_n. Finally, note that Theorem 2 does not use Brooks theorem.

References

1. Borodin O.V. and Kostochka A.V., *On an upper bound of a graph's chromatic number, depending on the graph's degree and density*, J. Combin Theory B **23** (1977), 247–250.

2. Brooks R.L., *On colouring the nodes of a network*, Proc. Cambridge Phil. Soc. **37** (1941), 194–197.

3. Catlin P.A., *A bound on the chromatic number of a graph*, Discrete Math. **22** (1978), 81–83.

4. Lawrence J., *Covering the vertex set of a graph with subgraphs of smaller degree*, Discrete Math. **2** (1978), 61–68.

5. Lovász L., *On decomposition of graphs*, Studia Sci. Math. Hungar. **1** (1966), 237–238.

L. Stacho, Department of Informatics, Slovak Academy of Sciences, Dúbravská 9, P.O. Box 56, 840 00 Bratislava 4, Slovakia, *e-mail*: `stacho@savba.sk`