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ON THE ENDOMORPHISM RING OF A SEMI-INJECTIVE MODULE

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Abstract. Let R be a ring. A right R-module M is called **quasi-principally** (or **semi-**) **injective** if it is M-principally injective. In this paper, we show: (1) The following are equivalent for a projective module M: (a) Every M-cyclic submodule of M is projective; (b) Every factor module of an M-principally injective module is M-principally injective; (c) Every factor module of an injective R-module is M-principally injective. (2) The endomorphism ring S of a semi-injective module is regular if and only if the kernel of every endomorphism is a direct summand. (3) For a semi-injective module M, if S is semiregular, then for every $s \in S \setminus J(S)$, there exists a nonzero idempotent $\alpha \in Ss$ such that $\mathrm{Ker}(s) \subset \mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(s(1-\alpha)) \neq 0$. The converse is also considered.

1. Introduction

Let R be a ring. A right R-module M is called **principally injective** if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M. This notion was introduced by Camillo [**2**] for commutative rings. In [**7**], Nicholson and Yousif studied the struture of principally injective rings and gave some applications. In [**9**], Sanh and others extended this notion to modules. A right R-module N is called ***M*-principally injective** if every R-homomorphism from an M-cyclic submodule of M to N can be extended to M. In [**10**], Tansee and Wongwai introduced the dual notion, a right R-module N is called ***M*-principally projective** if every R-homomorphism from N to an M-cyclic submodule of M can be lifted to an R-homomorphism from N to M. A module M is called **quasi-principally** (or **semi-**) projective if it is M-principally projective. Dual to this module and following Wisbauer [**12**] we consider a semi-injective module.

Throughout this paper, R is an associative ring with identity. Let M be a right R-module, the endomorphism ring of M is denoted by $S = \mathrm{End}_R(M)$. A module N is called ***M*-generated** if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called **finitely *M*-generated**. In particular, a submodule N of M is called ***M*-cyclic** submodule of M if it is isomorphic to M/X for some submodule X of M. By the notation $N \subset^{\oplus} M$ ($N \subset^{e} M$) we mean

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that N is a direct summand (an essential submodule) of M. We denote the socle and the singular submodule of M by $\mathrm{Soc}(M)$ and $Z(M)$ respectively, and that $J(M)$ denotes the Jacobson radical of M.

2. Principal Injectivity

Definition 2.1. [**9**] Let M be a right R-module. A right R-module N is called ***M*-principally injective** if every R-homomorphism from an M-cyclic submodule of M to N can be extended to M. Equivalently, for any endomorphism s of M, every homomorphism from $s(M)$ to N can be extended to a homomorphism from M to N. N is called **principally injective** if it is R-principally injective.

Lemma 2.2. *Let M and N be R-modules. Then N is M-principally injective if and only if for each $s \in S = \mathrm{End}_R(M)$, $\mathrm{Hom}_R(M,N)s = \{f \in Hom_R(M,N) : f(\mathrm{Ker}(s)) = 0\}$.*

Proof. Clearly, $\mathrm{Hom}_R(M,N)s \subset \{f \in Hom_R(M,N) : f(\mathrm{Ker}(s)) = 0\}$. Let $f \in Hom_R(M,N)$ such that $f(\mathrm{Ker}(s)) = 0$. This leads to $\mathrm{Ker}(s) \subset \mathrm{Ker}(f)$. Then there is an R-homomorphism $\varphi : s(M) \to N$ such that $\varphi s = f$. Since N is M-principally injective, there exists an R-homomorphism $t : M \to N$ such that $t\imath = \varphi$ where $\imath : s(M) \to M$ is the inclusion map. Hence $f = ts$ and therefore $f \in Hom_R(M,N)s$.

Conversely, let $\varphi : s(M) \to N$ be an R-homomorphism. Then $\varphi s \in Hom_R(M,N)$ and $\varphi s(\mathrm{Ker}(s)) = 0$. By assumption, we have $\varphi s = us$ for some $u \in Hom_R(M,N)$. This shows that N is M-principally injective. $\square$

Example 2.3. *Let $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$ where F is a field, $M_R = \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right)$ and $N_R = \left(\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right)$. Then*

(1) *N is not M-injective.*
(2) *N is M-principally injective.*

Proof. (1) Define $\varphi : \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right)$ with $\varphi(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is clear that φ is an R-isomorphism. For any homomorphism $\alpha : \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right)$ with $\alpha(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)$ for some $x \in F$, then $\alpha(\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)) = \alpha[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)] = \left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ for every $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right)$, so that $\alpha = 0$. Therefore N is not M-injective.

(2) It follows from (1) that $\left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$ is not M-cyclic submodule of M. Hence only 0 and M are M-cyclic submodules of M, thus N is M-principally injective. $\square$

Clearly, every X-cyclic submodule of X is an M-cyclic submodule of M for every M-cyclic submodule X of M. Thus we have the following

Proposition 2.4. *N is M-principally injective if and only if N is X-principally injective for every M-cyclic submodule X of M. In particular, if X is a direct summand of M and N is M-principally injective, then N is both X-principally injective and M/X-principally injective.*

Proposition 2.5. *Let M and N be R-modules. Then M is N-principally projective and every N-cyclic submodule of N is M-principally injective if and only if N is M-principally injective and every M-cyclic submodule of M is N-principally projective.*

Proof. ($\Rightarrow$) Let f be an endomorphism of N, X an M-cyclic submodule of M and let $g : X \to f(N)$ be an R-homomorphism. Then g is extended to an R-homomorphism $h : M \to f(N)$ so h is lifted to an R-homomorphism $t : M \to N$. Thus $t\imath$ is lifted g where $\imath : X \to M$ is the inclusion map.

($\Leftarrow$) Let $\varphi : s(M) \to t(N)$ be an R-homomorphism where s, t are endomorphisms of M and N, respectively. Then φ lifts to an R-homomorphism $\widehat{\varphi} : s(M) \to N$ and so $\widehat{\varphi}$ is extended to an R-homomorphism $\alpha : M \to N$, it is clear that $t\alpha$ is an extension of φ. □

A ring R is called a left (resp. right) ***PP*-ring** if each of its principal left (resp. right) ideal is projective. This is equivalent to the fact that, for each $a \in R$ there is an idempotent e such that $\ell_R(a) = Re$ (resp. $r_R(a) = eR$).

Theorem 2.6. *The following are equivalent for a projective module M:*

(1) *Every M-cyclic submodule of M is projective;*
(2) *Every factor module of an M-principally injective module is M-principally injective;*
(3) *Every factor module of an injective R-module is M-principally injective.*

Proof. (1) $\Rightarrow$ (2) Let N be an M-principally injective module, X a submodule of N and let $\varphi : s(M) \to N/X$ be an R-homomorphism. By (1), there exists an R-homomorphism $\widehat{\varphi} : s(M) \to N$ such that $\varphi = \eta\widehat{\varphi}$ where $\eta : N \to N/X$ is the natural epimorphism. Since N is M-principally injective , there exists an R-homomorphism $t : M \to N$ which is an extension of $\widehat{\varphi}$ to M. Then ηt is an extension of φ to M.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let $t(M)$ be an M-cyclic submodule of M, $h : A \to B$ an epimorphism and let $\alpha : t(M) \to B$ be R-homomorphism. Embed A in an injective module E. Then $B \simeq A/\mathrm{Ker}(h)$ is a submodule of $E/\mathrm{Ker}(h)$; we may view $\alpha : t(M) \to E/\mathrm{Ker}(h)$, which by hypothesis we can extend to $\widehat{\alpha} : M \to E/\mathrm{Ker}(h)$. Since M is projective, $\widehat{\alpha}$ can be lifted to $g : M \to E$. It is clear that $g(t(M)) \subset A$. Therefore we have lifted α. □

Corollary 2.7.[**12**, Exercises 39.17(4)] *The following are equivalent for a ring R:*

(1) *R is a right PP-ring;*
(2) *Every factor module of a principally injective module is principally injective;*
(3) *Every factor module of an injective R-module is principally injective.*

Definition 2.8. A right R-module M is called **semi-injective** if it is M-principally injective.

In general, we have:

injective $\Longrightarrow$ quasi-injective $\Longrightarrow$ semi-injective $\Longrightarrow$ direct-injective.

Recall that an R-module M is said to be **direct-injective** if for any direct summand D of M, every monomorphism $f : D \to M$ splits. **Direct-projective** modules are defined dually. A submodule N of M is called a **fully invariant** submodule of M if $s(N) \subset N$ for every $s \in S$.

3. The Endomorphism Ring and its Jacobson Radical

Write

$$\triangle = \{s \in S : ker(s) \subset^e M\}, \quad \text{and} \quad \hat{\Diamond} = \{s \in S : \mathrm{Ker}(1+ts) = 0 \text{ for all } t \in S\}.$$

It is known that $\triangle$ is an ideal of S [**5**, Lemma 3.2]. Since $\mathrm{Ker}(s) \cap \mathrm{Ker}(1+ts) = 0$, $\triangle \subset \hat{\Diamond}$. It is well-known that, for a quasi-continuous module M, M is continuous if and only if $S/\triangle$ is regular and $J(S) = \triangle$ [**5**, Proposition 3.15]. We now investigate when $J(S) = \triangle$.

Following [**12**], an R-module M is called **π-injective** if, for all submodules U and V of M with $U \cap V = 0$, there exists $f \in S$ with $U \subset \mathrm{Ker}(f)$ and $V \subset \mathrm{Ker}(1-f)$. A module M is called a **self-generator** if it generates all its submodules.

Proposition 3.1. *Let M be semi-injective.*

(1) $J(S) = \hat{\Diamond}$.
(2) *If S is local, then* $J(S) = \{s \in S : \mathrm{Ker}(s) \neq 0\}$.
(3) *If $S/\triangle$ is regular, then $J(S) = \triangle$.*
(4) *If $S/J(S)$ is regular, then $S/\triangle$ is regular if and only if $J(S) = \triangle$.*
(5) *If* $\mathrm{Im} s \subset^e M$ *where $s \in S$, then any monomorphism $t : s(M) \to M$ can be extended to a monomorphism in S.*
(6) *If M is uniform, then S is a local ring and $J(S) = \triangle$.*
(7) *For $s \in S$, if M is uniform and s is left invertible, then s is invertible.*
(8) *M is uniform if and only if S is local and M is π-injective.*
(9) *If M is uniform, then $Z(S_S) \subset J(S)$.*

Proof. (1) For any $s \in J(S)$ and $t \in S$, $g(1+ts) = 1_M$ for some $g \in S$. Thus $\mathrm{Ker}(1+ts) = 0$, and hence $J(S) \subset \hat{\Diamond}$. On the other hand, if $\mathrm{Ker}(1+s) = 0$, then $\ell_S(\mathrm{Ker}(1+ts)) = S$. By Lemma 2.2, we have $S = S(1+ts)$ which implies $1_M = g(1+ts)$ for some $g \in S$. It follows that $s \in J(S)$.

(2) Since S is local, $Ss \neq S$ for any $s \in J(S)$. If $\mathrm{Ker}(s) = 0$, then $\alpha : s(M) \to M$ given by $\alpha(s(m)) = m$ for any $m \in M$ is an R-homomorphism. Since M is semi-injective, let $\beta \in S$ be an extension of α to M. It follows that $\beta s = 1_M$ so $Ss = S$, which is a contradiction. This shows that $J(S) \subset \{s \in S : \mathrm{Ker}(s) \neq 0\}$. The other inclusion is clear.

(3) Clearly, $\triangle \subset J(S)$. If $s \in J(S)$, then $(1 - s\alpha)s = s - s\alpha s \in \triangle$ for some $\alpha \in S$. Since $(1 - s\alpha)$ has a left inverse, $s \in \triangle$. This show that $J(S) \subset \triangle$.

(4) This follows from (3).

(5) Since M is semi-injective, there exists $g \in S$ such that $gs = ts$. Thus $\mathrm{Im}(s) \cap \mathrm{Ker}(g) = 0$. Since $\mathrm{Im}(s) \subset^e M$, $\mathrm{Ker}(g) = 0$.

(6) Since M is direct-injective, S is local provided that M is uniform [12, 41.22]. It follows that $J(S) = \triangle$ by (2).

(7) Since s has a left inverse, $\mathrm{Ker}(s) = 0$. Follows from (6) and (2), we have $s \notin J(S)$ hence s is invertible.

(8) The necessity is trivial. For the sufficiency, let U and V be submodules of M such that $U \cap V = 0$. As M is π-injective, we can choose $f \in S$ so that $U \subset \mathrm{Ker}(f)$ and $V \subset \mathrm{Ker}(1-f)$. Note that either f or $1-f$ belong to $J(S)$. If $f \in J(S)$, then $g(1-f) = 1$ for some $g \in S$. Thus $\mathrm{Ker}(1-f) = 0$, and it follows that $V = 0$. Otherwise, $U = 0$.

(9) Let $s \in Z(S_S)$. Then $\mathrm{Ker}(s) \neq 0$. For any $t \in S$ we have $\mathrm{Ker}(s) \cap \mathrm{Ker}(1+ts) = 0$, then $\mathrm{Ker}(1+ts) = 0$. Hence $s \in J(S)$ by (1). $\square$

Proposition 3.2. *Suppose M is a semi-injective and π-injective module. If S is semiperfect, then $M = \bigoplus_{i=1}^n U_i$, where U_i is uniform and semi-injective for each i.*

Proof. Since S is semiperfect and M is semi-injective, $M = U_1 \oplus \ldots \oplus U_n$, where each $\mathrm{End}_R(U_i)$ is local. Note that U_i is semi-injective. Each U_i is π-injective [12,41.20], thus by Proposition 3.1(8) we see that U_i is uniform. $\square$

The following proposition is modified from [1,Lemma 18.21]

Proposition 3.3. *If $\mathrm{Soc}(M) \subset^e M$, then*

(1) *$\triangle = \ell_S(\mathrm{Soc}(M))$, and*

(2) *$S/\triangle$ is embedded in $\mathrm{End}_R(\mathrm{Soc}(M))$ as a subring.*

Proof. (1) Let $s \in \triangle$. Then $soc(M) \subset \mathrm{Ker}(s)$, it follows that $s(\mathrm{Soc}(M)) = 0$. If, on the other hand, $s(\mathrm{Soc}(M)) = 0$, then $\mathrm{Ker}(s) \subset^e M$ and $s \in \triangle$.

(2) For each $s \in S$, let $\theta(s)$ be a map from $\mathrm{Soc}(M)$ into itself defined by $\big(\theta(s)\big)(x) = s(x)$. Since $\mathrm{Soc}(M)$ is fully invariant in M, it follows that $\theta(s) \in \mathrm{End}_R(\mathrm{Soc}(M))$ and $\theta : S \to \mathrm{End}_R(\mathrm{Soc}(M))$ is a ring homomorphism. Clearly, $\mathrm{Ker}(\theta) = \triangle$ and the proof is complete. $\square$

Corollary 3.4. *If M is semi-injective and a self-generator and if $\mathrm{Soc}(M) \subset^e M$, then*

(1) *$J(S) = \ell_S(\mathrm{Soc}(M))$, and*

(2) *$S/J(S) \simeq \mathrm{End}_R(\mathrm{Soc}(M))$.*

Proof. (1) As M is semi-injective and a self-generator, we have $J(S) = \triangle$ by [**9**, Theorem 2.13].

(2) Since M is semi-injective, every R-homomorphism in $\mathrm{End}_R(\mathrm{Soc}(M))$ can be extended to an R-homomorphism in S. By (1) and Proposition 3.3(2), it follows that $S/J(S)$ is isomorphic to $\mathrm{End}_R(\mathrm{Soc}(M))$ as rings. $\square$

Proposition 3.5. *Let M be a semi-injective module.*

(1) *If $\mathrm{Im}(s)$ is a simple right R-module, $s \in S$, then Ss is a simple left S-module.*

(2) *If* $s_1(M) \oplus \cdots \oplus s_n(M)$ *is direct,* $s_1, ..., s_n \in S$, *then* $S(s_1 + \cdots + s_n) = Ss_1 + \cdots + Ss_n$.

Proof. (1) Let A be a nonzero submodule of Ss and $0 \neq \alpha s \in A$. Then $S\alpha s \subset A$. Since $\mathrm{Im}(s)$ is simple, $\mathrm{Ker}(\alpha) \cap \mathrm{Im}(s) = 0$. Define $g : \alpha s(M) \to M$ by $g(\alpha s(m)) = s(m)$ for every $m \in M$. It it obvious that g is an R-homomorphism. Since M is semi-injective, there exists a homomorphism $h \in S$ such that $h(\alpha s) = g(\alpha s)$. Therefore $h(\alpha s) = s$ so $s \in S\alpha s$. It follows that $S\alpha s = Ss$ and hence $A = Ss$.

(2) Let $\alpha_1 s_1 + \cdots + \alpha_n s_n \in Ss_1 + \cdots + Ss_n$. For each i, define $\varphi_i : (s_1 + \cdots + s_n)(M) \to M$ by $\varphi_i((s_1 + \cdots + s_n)(m)) = s_i(m)$ for every $m \in M$. Since $s_1(M) \oplus \cdots \oplus s_n(M)$ is direct, φ_i is well-defined, so it is clear that φ_i is an R-homomorphism. Then there exists an R-homomorphism $\widehat{\varphi}_i \in S$ which is an extension of φ_i. Then $s_i = \varphi_i(s_1 + \cdots + s_n) = \widehat{\varphi}_i(s_1 + \cdots + s_n) \in S(s_1 + \cdots + s_n)$ for every $i = 1, ..., n$. Consequently, $\alpha_1 s_1 + \cdots + \alpha_n s_n \in S(s_1 + \cdots + s_n)$. Hence $Ss_1 + \cdots Ss_n \subset S(s_1 + \cdots + s_n)$. The other inclusion always holds. ☐

We call a module M a **duo module** if every submodule of M is fully invariant. M is said [**11**] to have the **summand intersection property** (SIP) if the intersection of two direct summands is again a direct summand. The module M is said [**4**] to have the **summand sum property** (SSP) if the sum of any two summands of M is again a summand.

We prove a similar result here for a semi-injective module M, with the (SIP) and (SSP). Note that every direct summand of M is of the form $s(M)$ for some $s \in S$.

Proposition 3.6. *Every duo and semi-injective module has the (SIP) and (SSP).*

Proof. Write $M = s(M) \oplus K$ and $M = t(M) \oplus L$. Since M is duo, $s(M) = s(t(M) \oplus L) = st(M) + s(L) \subset (s(M) \cap t(M)) + (s(M) \cap L) = (s(M) \cap t(M)) \oplus (s(M) \cap L) \subset s(M)$. Then $s(M) \cap t(M) \subset^{\oplus} M$. Now we write $M = s(M) \cap t(M) \oplus N$. Then $t(M) = t(M) \cap (s(M) \cap t(M) \oplus N) = s(M) \cap t(M) \oplus t(M) \cap N$ by the Modular law. So $s(M) + t(M) = s(M) + (s(M) \cap t(M) \oplus t(M) \cap N) = s(M) + t(M) \cap N = s(M) \oplus t(M) \cap N$. Since $s(M)$ and $t(M) \cap N$ are direct summands, $s(M) + t(M)$ is a direct summand of M by (C_3). ☐

Following [**6**] a ring R is called **semiregular** if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. Equivalently, R is semiregular if and only if for each element $a \in R$, there exists $e^2 = e \in Ra$ such that $a(1 - e) \in J(R)$.

Theorem 3.7. *For a semi-injective module* M, *if* S *is semiregular, then* $(*)$ *holds, where* $(*)$ *is the conditon*

$(*)$: *For every* $s \in S \setminus J(S)$, *there exists a nonzero idempotent* $\alpha \in Ss$ *such that* $\mathrm{Ker}(s) \subset \mathrm{Ker}(\alpha)$ *and* $\mathrm{Ker}(s(1 - \alpha)) \neq 0$.

If, in addition, S *is local, then the converse is true.*

Proof. Let $s \in S \setminus J(S)$. Then there exists $\alpha^2 = \alpha \in Ss$ such that $s(1-\alpha) \in J(S)$. Then $\alpha \neq 0$ and $\mathrm{Ker}(s) \subset \mathrm{Ker}(\alpha)$. If $\mathrm{Ker}(s(1-\alpha)) = 0$, then $gs(1-\alpha) = 1_M$ for some $g \in S$ by the semi-injectivity of M. It follows that $\alpha = 0$, a contradiction.

The converse follows from Proposition 3.1(2). □

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