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ON k-ABELIAN p-FILIFORM LIE ALGEBRAS I

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Abstract. We classify the $(n-5)$-filiform Lie algebras which have the additional property of a non-abelian derived subalgebra. We show that this property is strongly related with the structure of the Lie algebra of derivations; explicitly we show that if a $(n-5)$-filiform Lie algebra is characteristically nilpotent, then it must be 2-abelian. We also give applications to the construction of solvable rigid laws whose nilradical is k-abelian with mixed characteristic sequence, as well as applications to the theory of nilalgebras of parabolic subalgebras of the exceptional simple Lie algebra E_6.

1. Generalities

Definition 1. A Lie algebra law over $\mathbb{C}^n$ is a bilinear alternated mapping $\mu \in Hom(\mathbb{C}^n \times \mathbb{C}^n, \mathbb{C}^n)$ which satisfies the conditions

1. $\mu(X,X) = 0,\ \forall\, X \in \mathbb{C}^n$
2. $\mu(X,\mu(Y,Z)) + \mu(Z,\mu(X,Y)) + \mu(Y,\mu(Z,X)) = 0,\ \forall\, X,Y,Z \in \mathbb{C}^n$ (Jacobi identity).

If μ is a Lie algebra law, the pair $\mathfrak{g} = (\mathbb{C}^n, \mu)$ is called Lie algebra. From now on we identify the Lie algebra with its law μ.

Remark 2. *We say that μ is the law of $\mathfrak{g}$, and where necessary we use the bracket notation to describe the law :*

$$[X,Y] = \mu(X,Y),\ \forall\, X,Y \in \mathfrak{g}\,.$$

The nondefined brackets are zero or obtained by antisymmetry.

Let $\mathfrak{g}_n = (\mathbb{C}^n, \mu)$ be a nilpotent Lie algebra. For any nonzero vector $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ let $c(X)$ be the ordered sequence of similitude invariants for the nilpotent operator $ad_\mu(X)$; i.e., the ordered sequence of dimensions of Jordan blocks of this operator. The set of these sequences is ordered lexicographically.

Definition 3. The characteristic sequence of $\mathfrak{g}_n$ is an isomorphism invariant $c(\mathfrak{g}_n)$ defined by

$$c(\mathfrak{g}_n) = \max_{X \in \mathfrak{g}_n - C^1\mathfrak{g}_n} \{c(X)\}\ .$$

A nonzero vector $X \in \mathfrak{g}_n - C^1\mathfrak{g}_n$ for which $c(X) = c(\mathfrak{g}_n)$ is called characteristic vector.

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Remark 4. *In particular, the algebras with maximal characteristic sequence $(n-1,1)$ correspond to the filiform algebras introduced by Vergne* [**12**]*. Thus it is natural to generalize this concept to lower sequences.*

Definition 5. A nilpotent Lie algebra $\mathfrak{g}_n$ is called p-filiform if its characteristic sequence is $(n-p,1,..^{p}..,1)$.

Remark 6. *This definition was first given in* [**6**]*. The $(n-1)$-filiform Lie algebras are abelian, while the $(n-2)$-filiform are the direct sum of an Heisenberg algebra $\mathfrak{h}_{2p+1}$ and abelian algebras. A classification of the $(n-3)$-filiform can also be found in* [**6**]*. These and any $(n-4)$-filiform Lie algebra have non trivial diagonalizable derivations* [**2**]*. This fact is important, for it is telling us that their structure is relatively simple. To search for nilpotent algebras with rank zero (i.e, with no nonzero diagonalizable derivations) we must start with the $(n-5)$-filiform Lie algebras. As the difficulty of distinguishing isomorphism classes increases considerably for bigger indexes, it seems reasonable to consider additional assumptions made on the algebras to be classified. For example, the filtration given by the central descending sequence can be used to impose additional conditions on the p-filiformness.*

Remark 7. *For indexes $p \geq (n-4)$ the number of isomorphism classes is finite. The index $p=(n-5)$ is the first for which an infinity of isomorphism classes exists.*

Definition 8. Let $\mathfrak{g}_n$ be a nilpotent Lie algebra. The smallest integer k such that the ideal $C^k\mathfrak{g}_n$ is abelian is called commutativity index of $\mathfrak{g}_n$.

Definition 9. A Lie nilpotent algebra $\mathfrak{g}_n$ is called k-abelian if k is the smallest positive integer such that

$$C_{\mathfrak{g}}\left(C^k\mathfrak{g}\right) \supset C^k\mathfrak{g} \quad \text{and} \quad C_{\mathfrak{g}}\left(C^{k-1}\mathfrak{g}\right) \not\supset C^{k-1}\mathfrak{g}$$

where $C_{\mathfrak{g}}(I)$ denotes the centralizer of I in $\mathfrak{g}$.

Remark 10. *The preceding definition is equivalent to impose that the commutativity index of $\mathfrak{g}_n$ is exactly k. In* [**8**] *a less restrictive definition of k-abelianity is considered. The purpose there is to study certain topological properties of the variety of filiform laws $\mathfrak{F}^m$. Our definition is more restrictive: the k-abelian Lie algebras do **not** contain the $(k-1)$-abelian algebras; the reason is justified by the important structural difference between algebras having its ideal $C^k\mathfrak{g}_n$ abelian and those having it not. On the other side we avoid unnecessary repetitions.*

As we are considering here the $(n-5)$-filiform Lie algebras, we have to determine which abelianity indexes are admissible. Only the nonsplit algebras are of interest for us, thus from now on we will understand nonsplit Lie algebra when we say Lie algebra.

Lemma 11. *Let $\mathfrak{g}_n$ be an $(n-5)$-filiform Lie algebra. Then there exists a basis $\{\omega_1, .., \omega_6, \theta_1, .., \theta_{n-6}\}$ of $(\mathbb{C}^n)^*$ such that the law is expressible as*

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0\\
d\omega_3 &= \omega_1 \wedge \omega_2\\
d\omega_4 &= \omega_1 \wedge \omega_3 + \sum_{i=1}^{n-6} \alpha_i^2 \theta_i \wedge \omega_2\\
d\omega_5 &= \omega_1 \wedge \omega_4 + \sum_{i=1}^{n-6} \left(\alpha_i^2 \theta_i \wedge \omega_3 + \alpha_i^3 \theta_i \wedge \omega_2\right) + \beta_2 \omega_3 \wedge \omega_2\\
d\omega_6 &= \omega_1 \wedge \omega_5 + \sum_{i=1}^{n-6} \left(\alpha_i^2 \theta_i \wedge \omega_4 + \alpha_i^3 \theta_i \wedge \omega_3 + \alpha_i^4 \theta_i \wedge \omega_2\right)\\
&\quad + \sum_{1 \le i,j \le n-6} a_{ij}^1 \theta_i \wedge \theta_j + \beta_1 \left(\omega_5 \wedge \omega_2 - \omega_3 \wedge \omega_4\right) + \beta_2 \omega_4 \wedge \omega_2 + \beta_3 \omega_3 \wedge \omega_2\\
d\theta_j &= \varepsilon_i^j \theta_i \wedge \omega_2 - \beta_{1,j} \left(\omega_5 \wedge \omega_2 - \omega_3 \wedge \omega_4\right) + \beta_{3,j} \omega_3 \wedge \omega_2,\ 1 \le j \le n-6
\end{aligned}$$

The proof is trivial.

Lemma 12. *Any $(n-5)$-filiform Lie algebra $\mathfrak{g}_n$ is either 1 or 2-abelian.*

Proof. If the algebra is 1-abelian, it is simply an algebra whose derived algebra is abelian [**5**]. If it is 2-abelian, then there exist $X, Y \in C^1\mathfrak{g}_n$ such that $0 \neq [X, Y]$. From the above equations it is immediate to derive the possibilities:

1. dim $C^1\mathfrak{g}_n = 6$
2. dim $C^1\mathfrak{g}_n = 5$ and $\exists X, Y \in C^1\mathfrak{g}_n$ such that $[X, Y] \neq 0$
3. dim $C^1\mathfrak{g}_n = 4$ and $\exists X, Y \in C^1\mathfrak{g}_n$ such that $[X, Y] \neq 0$.

□

Remark 13. *From this lemma we see how important is to consider our stronger version of the k-abelianity. In particular we will see its connection with the characteristic nilpotence.*

Notation 14. For $n \geq 7$ let $\mathfrak{g}_0^n$ be the Lie algebra whose Maurer-Cartan equations are

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0\\
d\omega_j &= \omega_1 \wedge \omega_{j-1},\ 3 \le j \le 6\\
d\theta_j &= 0,\ 1 \le j \le n-6
\end{aligned}$$

Let $\{X_1, .., X_6, Y_1, .., Y_{n-6}\}$ be a dual basis of $\{\omega_1, .., \omega_6, \theta_1, .., \theta_6\}$. Let $V_1 = \langle X_1, .., X_6 \rangle_{\mathbb{C}}$ and $V_2 = \langle Y_1, .., Y_{n-6} \rangle_{\mathbb{C}}$. We write $B(V_i, V_j)$ to denote the space of bilinear alternated mappings from V_i to V_j.
Let us consider the following applications :

1. $\psi_{i,j}^{1}\in B\left(V_2,V_1\right),\ 1\leq i,j\leq n-6:$

$$\psi_{i,j}^{1}\left(Y_i,Y_l\right)=\begin{cases} X_6 & \text{if } i=k\ ,j=l\\ 0 & \text{otherwise}\end{cases}$$

2. $\psi_{i}^{j}\in Hom\left(V_2\times V_1,V_1\right),\ j=2,3,4$

$$\psi_{i}^{j}\left(Y_k,X_l\right)=\begin{cases} X_{l+j} & \text{if } i=k,\ 2\leq l\leq 6-j\\ 0 & \text{otherwise}\end{cases}$$

3. $\varphi_{1,k}\in B\left(V_1,V_2\right),\ 1\leq k\leq n-6:$

$$\varphi_{1,k}\left(X_5,X_2\right)=\varphi_{1,k}\left(X_3,X_4\right)=Y_k$$

4. $\varphi_{3,k}\in B\left(V_1,V_2\right),\ 1\leq k\leq n-6:$

$$\varphi_{3,k}\left(X_3,X_2\right)=Y_k$$

5. $\varphi_1\in B\left(V_1,V_1\right):$

$$\varphi_1\left(X_5,X_2\right)=\varphi_1\left(X_3,X_4\right)=X_6$$

6. $\varphi_2\in B\left(V_1,V_1\right):$

$$\varphi_{\uparrow}\left(X_k,X_2\right)=X_{k+2},\ k=3,4$$

7. $\varphi_3\in B\left(V_1,V_1\right):$

$$\varphi_3\left(X_3,X_2\right)=X_6$$

where the undefined brackets are zero or obtained by antisymmetry.

Lemma 15. *For $n\geq 7$, $1\leq k\leq n-6$ and $l=2,3,4$ the mappings $\psi_{1,k},\psi_{3,k},\psi_i^l$, $\varphi_{1,k},\varphi_{3,k},\varphi_1,\varphi_2,\varphi_3$ are 2-cocycles of the subspace $Z^2\left(\mathfrak{g}_0^n,\mathfrak{g}_0^n\right)$.*

Notation 16. For convenience we introduce the following notation:

$$\sum_{\substack{t=i\\ m>k}}^{j}\psi_{f(t),f(t)+1}$$

where the sum is only defined whenever $m\geq k+1$.

Proposition 17. *Any nonsplit $(n-5)$-filiform Lie algebra with* $\dim C^1\mathfrak{g}_n=6$ *is isomorphic to one of the following laws:*

1. $\mathfrak{g}_{2m}^{1}\,(m\geq 4):$

$$\mathfrak{g}_0^{2m}+\varphi_{1,1}+\varphi_{3,2}+\psi_2^3+\sum_{\substack{t=2\\ m>4}}^{m-3}\psi_{2t-1,2t}^{1}$$

2. $\mathfrak{g}_{2m}^{2}\,(m\geq 5):$

$$\mathfrak{g}_0^{2m}+\varphi_{1,1}+\varphi_{3,2}+\psi_3^3+\sum_{t=2}^{m-3}\psi_{2t-1,2t}^{1}$$

3. $\mathfrak{g}_{2m+1}^3\,(m\geq 4)$:

$$\mathfrak{g}_0^{2m+1}+\varphi_{1,1}+\varphi_{3,2}+\psi_3^3+\sum_{\substack{t=2\\ m>4}}^{m-3}\psi_{2t,2t+1}^1$$

4. $\mathfrak{g}_{2m}^4\,(m\geq 4)$:

$$\mathfrak{g}_0^{2m}+\varphi_{1,1}+\varphi_{3,2}+\sum_{\substack{t=2\\ m>4}}^{m-3}\psi_{2t-1,2t}^1$$

5. $\mathfrak{g}_{2m}^5\,(m\geq 4)$:

$$\mathfrak{g}_0^{2m}+\varphi_{1,1}+\varphi_2+\varphi_{3,2}+\psi_2^3+\sum_{\substack{t=2\\ m>4}}^{m-3}\psi_{2t-1,2t}^1$$

Proof. Suppose $\beta_{1,k}\neq 0$ for $k\neq 1$. We can take $\beta_{1,1}=1$ and $\beta_{1,i}=0$ for $i\geq 2$, as well as $\beta_1=0$. The Jacobi conditions imply

$$\alpha_1^2=\alpha_1^3=\alpha_1^1=0,\ \alpha_1^4=\sum_{k\geq 2}\beta_{3,k}\alpha_{k,j}^4,\ \sum\beta_{3,k}\alpha_{i,k}^1=0$$

Let $\beta_{3,k}\neq 0$ for $k\neq 1$, so that we can choose $\beta_{3,2}=1,\beta_{3,k}=0$ for $k\neq 2$. A change of basis allows to take $\beta_3=0$. From the conditions above we deduce $\alpha_1^4=\alpha_i^1=0$. Consider the change $\omega_2'=\alpha\omega_1+\omega_2$ with $\alpha\neq 0$. Then we have

$$\begin{cases}\alpha_{j,i}^4=0,\ j\geq 2\\ \alpha_j^3\beta_2=0,\ j\geq 2\end{cases}$$

There are two cases :

1. If $\alpha_2^3\neq 0$ we suppose $\alpha_2^3=1$ and $\alpha_i^3=0$, $\forall i\neq 2$ through a linear change. Reordering the forms θ_i we can suppose $\alpha_{2t-1,2t}^1=1$ for $2\leq t\leq\frac{n-6}{2}$; $\alpha_{i,j}^1$ for the remaining. We obtain a unique class of nonsplit Lie algebras in even dimension and isomorphic to $\mathfrak{g}_{2m}^1$.
2. If $\alpha_2^3=0$
   (a) If $\alpha_i^3\neq 0$ with $i\geq 3$ we can suppose $\alpha^3{}_3=1$ and $\alpha_j^3=0$ for $j\neq 3$. Reordering the θ_i we obtain one algebra in even and one algebra in odd dimension, which are respectively isomorphic to $\mathfrak{g}_{2m}^2$ and $\mathfrak{g}_{2m+1}^3$.
   (b) If $\alpha_i^3=0$ for $i\geq 3$ we obtain in an analogous way two even dimensional algebras, respectively isomorphic to $\mathfrak{g}_{2m}^4$ and $\mathfrak{g}_{2m}^5$.

□

Remark 18. *It is very easy to see that the obtained algebras are pairwise non isomorphic, as their infinitesimal deformations are not cohomologous cocycles in the cohomology space $H^2\left(\mathfrak{g}_0^{2m},\mathfrak{g}_0^{2m}\right)$. This calculations are routine and will be ommited in future.*

Remark 19. *From the linear system associated (for the elementary properties of these systems see* [**1**] *) to the algebras above it follows the existence of nonzero eigenvectors for diagonalizable derivations, so that the rank is at least one. Then the algebra of derivations has nonzero semi-simple derivations.*

Notation 20. We define the set

$$\mathfrak{h}_2 = \{\mathfrak{g} \mid \mathfrak{g} \text{ is nonsplit, 2-abelian and } (n-5)\text{-filiform } \}$$

We now express conditions making reference to the reduced system of forms given before :

We say that $\mathfrak{g} \in \mathfrak{h}_2$ satisfies property $(P1)$ if

$$\dim\ C^1\mathfrak{g} = 5$$
$$\beta_{1,1} = 1$$

Remark 21. *The general condition would be* $\beta_{1,k} \neq 0$ *for a* $k \geq 1$ *and* $\beta_{3,k} = 0$ *for any* k*. Now a elementary change of basis allows to reduce it to the preceding form.*

Proposition 22. *Let* $\mathfrak{g}_n$ *be an* $(n-5)$*-filiform Lie algebra satisfying the property* $(P1)$ *. The* $\mathfrak{g}_n$ *is isomorphic to one of the following laws:*

1. $\mathfrak{g}_{2m+1}^{6}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{1,1} + \psi_2^3 + \sum_{t=2}^{m-3} \psi_{2t,2t+1}^1$$

2. $\mathfrak{g}_{2m}^{7}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{1,1} + \psi_2^3 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

3. $\mathfrak{g}_{2m+1}^{8}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{1,1} + \varphi_2 + \sum_{\substack{t=2\\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

4. $\mathfrak{g}_{2m+1}^{9}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{1,1} + \varphi_3 + \sum_{\substack{t=1\\ m>3}}^{m-3} \psi_{2t,2t+1}$$

5. $\mathfrak{g}_{2m+1}^{10}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{1,1} + \sum_{\substack{t=1\\ m>3}}^{m-1} \psi_{2t,2t+1}^1$$

Proof. $\beta_{3,k} = 0$ for all k. The characteristic sequence implies $\alpha_i^4 = \alpha_i^3\beta_3 = 0$ for all i. Moreover, a change of basis of the type $\omega_2' = \omega_4 - \frac{j^1}{2}\omega_2$ allows to suppose $\alpha_{1,j}^4 = 0$.

1. If $\exists\, \alpha_i^3 \neq 0$ we suppose $\alpha_2^3 = 1, \alpha_i^3 = 0, \forall i \neq 2$. A change of basis allows $\beta_3 = 0$. There are two possibilities: an even dimensional algebra isomorphic to $\mathfrak{g}_{2m}^7$ and an odd dimensional one isomorphic to $\mathfrak{g}_{2m+1}^6$.
2. $\alpha_i^3 = 0, \forall i$.
   (a) If $\beta_2 \neq 0$ we put $\beta_2 = 1$ and $\beta_3 = 0$ with a linear change of basis. We obtain a unique algebra in odd dimension isomorphic to $\mathfrak{g}_{2m+1}^8$.
   (b) If $\beta_2 = 0$ there are two possibilities, depending on β_3 : we obtain two odd dimensional algebras which are respectively isomorphic to $\mathfrak{g}_{2m+1}^9$ and $\mathfrak{g}_{2m+1}^{10}$.

□

A Lie algebra $\mathfrak{g} \in \mathfrak{h}_2$ satisfies property $(P2)$ if

$$\dim C^1\mathfrak{g}_n = 5$$
$$\beta_{3,t} \neq 0 \text{ for t} \geq 1$$

Proposition 23. *Let $\mathfrak{g}_n$ be an algebra with property $(P2)$. Then $\mathfrak{g}_n$ is isomorphic to one of the following laws:*

1. $\mathfrak{g}_{2m+1}^{11}\,(m \geq 3)$:
$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \psi_1^3 + \psi_1^4 + \sum_{\substack{t=1\\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$
2. $\mathfrak{g}_{2m}^{12}\,(m \geq 4)$:
$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \psi_1^3 + \psi_1^4 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$
3. $\mathfrak{g}_{2m}^{13}\,(m \geq 4)$:
$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \psi_1^3 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$
4. $\mathfrak{g}_{2m+1}^{14}\,(m \geq 4)$:
$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \psi_1^3 + \sum_{t=1}^{m-3} \psi_{2t,2t+1}^1$$
5. $\mathfrak{g}_{2m+1}^{15}\,(m \geq 4)$:
$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \psi_1^4 + \psi_2^3 + \sum_{t=1}^{m-3} \psi_{2t,2t+1}^1$$

6. $\mathfrak{g}_{2m}^{16}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \psi_1^4 + \psi_2^3 + \sum_{\substack{t=2 \\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

7. $\mathfrak{g}_{2m+1}^{17}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \psi_1^4 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

8. $\mathfrak{g}_{2m+1}^{18}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \psi_3^4 + \psi_2^3 + \sum_{\substack{t=2 \\ m>4}}^{m-3} \psi_{2t,2t+1}^1$$

9. $\mathfrak{g}_{2m}^{19}\,(m \geq 5)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \psi_{24}^1 + \psi_2^3 + \psi_3^4 + \sum_{\substack{t=2 \\ m>5}}^{m-4} \psi_{2t+1,2t+2}^1$$

10. $\mathfrak{g}_{2m}^{20}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \psi_2^3 + \sum_{\substack{t=2 \\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

11. $\mathfrak{g}_{2m+1}^{21}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \psi_2^3 + \sum_{t=1}^{m-3} \psi_{2t,2t+1}^1$$

12. $\mathfrak{g}_{2m}^{22}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \psi_2^4 + \sum_{\substack{t=2 \\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

13. $\mathfrak{g}_{2m+1}^{23}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \sum_{t=1}^{m-3} \psi_{2t,2t+1}^1$$

14. $\mathfrak{g}_{2m+1}^{24,\alpha}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \alpha\varphi_2 + \psi_1^3 + \psi_1^4 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

15. $\mathfrak{g}_{2m}^{25,\alpha}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \alpha\varphi_2 + \psi_1^3 + \psi_1^4 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

16. $\mathfrak{g}_{2m}^{26}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_2 + \psi_1^3 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

17. $\mathfrak{g}_{2m+1}^{27}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_2 + \psi_1^3 + \sum_{\substack{t=1\\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

18. $\mathfrak{g}_{2m+1}^{28}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \varphi_2 + \psi_2^3 + \psi_3^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t,2t+1}^1$$

19. $\mathfrak{g}_{2m}^{29}\,(m \geq 5)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \varphi_2 + \psi_2^3 + \psi_3^4 + \psi_{24}^1 + \sum_{\substack{t=2\\ m>5}}^{m-4} \psi_{2t+1,2t+2}^1$$

20. $\mathfrak{g}_{2m}^{30}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \varphi_2 + \psi_2^3 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

21. $\mathfrak{g}_{2m+1}^{31}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_{3,1} + \varphi_1 + \varphi_2 + \psi_2^3 + \sum_{t=1}^{m-3} \psi_{2t,2t+1}^1$$

22. $\mathfrak{g}_{2m}^{32}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_{3,1} + \varphi_1 + \varphi_2 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

Proof. The starting assumptions are $\beta_{1,k} = 0$ for all k and $\beta_{3,k} \neq 0$ for some $k \geq 1$. We can suppose $\beta_{3,1} = 1$, and from the Jacobi conditions we obtain

$$\alpha_1^2 = 0,\ \alpha_{i1}^1 = 2\alpha_i^2\beta_1,\ i \geq 2$$

From the characteristic sequence we deduce the nullity of α_i^3 for all i, so that $\alpha_{i1}^1 = 0$ $\forall i$ as well.

1. $\alpha_1^3 = 1$: a combination of the linear changes of basis allow to take $\beta_1 = 0$.
   (a) $\alpha_1^4 = 1$:.
      (i) If $\alpha_i^4 = 0, \forall i \geq 2$ we reorder the $\{\theta_2, .., \theta_{n-6}\}$ such that $\alpha_{2t-1,2t}^1 = 1$ for $1 \leq t \leq \frac{n-6}{2}$ and the remaining brackets zero. The decisive structure constant is β_2. If it is zero we obtain an odd dimensional Lie algebra isomorphic to $\mathfrak{g}_{2m+1}^{11}$. If not, $\beta_2 = \alpha$ is an essential parameter. So we obtain an infinite family of odd dimensional Lie algebras isomorphic to the family $\mathfrak{g}_{2m+1}^{24,\alpha}$.
      (ii) $\exists\, \alpha_i^4 \neq 0, i \geq 2$. Without loss of generality we can choose $\alpha_2^4 \neq 0$ and the remaining zero for $i \geq 3$. It is easy to deduce $\alpha_{2j}^1 = 0, \forall j$. Reordering $\{\theta_3, .., \theta_{n-6}\}$ in the previous manner we obtain an even dimensional Lie algebra and an infinite family of even dimensional algebras, which are respectively isomorphic to $\mathfrak{g}_{2m}^{12}$ and $\mathfrak{g}_{2m}^{25,\alpha}$.
   (b) Now take $\alpha_1^4 = 0$
      (i) If there is an index $i \geq 2$ such that $\alpha_i^4 \neq 0$ we can suppose $\alpha_2^4 = 1$ and the remaining zero. Reordering the $\{\theta_3, .., \theta_{n-6}\}$ as before we obtain two even dimensional Lie algebras, respectively isomorphic to $\mathfrak{g}_{2m}^{13}$ and $\mathfrak{g}_{2m}^{26}$.
      (ii) $\alpha_i^4 = 0, \forall i$. A similar reordering of the θ_i gives two Lie algebras in odd dimension isomorphic to $\mathfrak{g}_{2m+1}^{14}$ and $\mathfrak{g}_{2m+1}^{27}$.
2. Suppose now $\alpha_1^3 = 0$.
   (a) $\alpha_1^4 = 1, \alpha_i^4 = 0\ \forall i \geq 2$. A linear change allows to annihilate β_2.
      (i) $\alpha_2^3 = 1$, and the remaining zero.
      There are two possible cases, depending on α_{23}^1 : if it is nonzero we obtain an odd dimensional algebra isomorphic to $\mathfrak{g}_{2m+1}^{15}$ and if it is zero an algebra in even dimension isomorphic to $\mathfrak{g}_{2m}^{16}$.
      (ii) $\alpha_i^3 = 0$ for any $i \geq 2$: we obtain an odd dimensional Lie algebra isomorphic to $\mathfrak{g}_{2m+1}^{17}$.
   (b) $\alpha_1^4 = 0$
      (i) $\alpha_2^3 \neq 0$ and $\alpha_i^3 = 0, \forall i \geq 3$. With a linear change we can suppose $\alpha_2^4 = 0$.
      A-1) $\alpha_3^4 = 1$ and $\alpha_i^4 = 0$ for $i \geq 4$. A linear change allows to suppose $\alpha_{3j}^1 = 0$ for all j. If $\alpha_{2j}^1 = 0$ for all j we obtain the algebras $\mathfrak{g}_{2m+1}^{18}$ and $\mathfrak{g}_{2m+1}^{28}$. If not, reorder $\{\theta_4, .., \theta_{n-6}\}$ such that $\alpha_{24}^1 = 1$. We obtain the algebras $\mathfrak{g}_{2m}^{9}, \mathfrak{g}_{2m}^{29}$.
      A-2) $\alpha_i^4 = 0\ \ \forall i$. If $\alpha_{23}^1 = 0$ we obtain the Lie algebras $\mathfrak{g}_{2m}^{20}, \mathfrak{g}_{2m}^{30}$, and if $\alpha_{23}^1 \neq 0$ we obtain the algebras $\mathfrak{g}_{2m+1}^{21}$ and $\mathfrak{g}_{2m+1}^{31}$.
      (ii) $\alpha_i^3 = 0\ \forall i$.
      B-1) $\alpha_2^4 = 1, \alpha_{2j}^1 = 0$. We obtain two Lie algebras in even dimension isomorphic to $\mathfrak{g}_{2m}^{22}$ and $\mathfrak{g}_{2m}^{32}$.
      B-2) $\alpha_i^4 = 0$ for $i \geq 2$: there is only one algebra in odd dimension which is isomorphic to $\mathfrak{g}_{2m+1}^{23}$.

$\square$

There is only one remaining case, namely the corresponding to the $(n-5)$-filiform 2-abelian Lie algebras with minimal dimension of its derived algebra.

Proposition 24. *Let* $\mathfrak{g}_n$ *be an* 2*-abelian algebra with* $\dim\, C^1\mathfrak{g}_n = 4$. *Then* $\mathfrak{g}_n$ *is isomorphic to one of the following laws:*

1. $\mathfrak{g}_{2m}^{33}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_1 + \psi_1^3 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

2. $\mathfrak{g}_{2m}^{34}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_1 + \varphi_2 + \psi_1^3 + \psi_2^4 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t-1,2t}^1$$

3. $\mathfrak{g}_{2m+1}^{35}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \psi_1^3 + \psi_2^4 + \psi_{13}^1 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t,2t+1}^1$$

4. $\mathfrak{g}_{2m+1}^{36}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \varphi_2 + \psi_1^3 + \psi_2^4 + \psi_{13}^1 + \sum_{\substack{t=2\\ m>4}}^{m-3} \psi_{2t,2t+1}^1$$

5. $\mathfrak{g}_{2m+1}^{37}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \psi_1^3 + \psi_1^4 + \sum_{\substack{t=1\\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

6. $\mathfrak{g}_{2m}^{38}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_1 + \psi_1^3 + \sum_{t=1}^{m-3} \psi_{2t-1,2t}^1$$

7. $\mathfrak{g}_{2m}^{39}\,(m \geq 4)$:

$$\mathfrak{g}_0^{2m} + \varphi_1 + \varphi_2 + \psi_1^3 + \sum_{t=1}^{m-3} \psi_{2t-1,2t}^1$$

8. $\mathfrak{g}_{2m+1}^{40}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \psi_1^3 + \sum_{t=1}^{m-3} \psi_{2t,2t+1}^1$$

9. $\mathfrak{g}_{2m+1}^{41}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \varphi_2 + \psi_1^3 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

10. $\mathfrak{g}_{2m+1}^{42}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \psi_1^4 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

11. $\mathfrak{g}_{2m+1}^{43}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m+1} + \varphi_1 + \varphi_2 + \psi_1^4 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t,2t+1}^1$$

12. $\mathfrak{g}_{2m}^{44}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m} + \varphi_1 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t-1,2t}^1$$

13. $\mathfrak{g}_{2m}^{45}\,(m \geq 3)$:

$$\mathfrak{g}_0^{2m} + \varphi_1 + \varphi_2 + \sum_{\substack{t=1 \\ m>3}}^{m-3} \psi_{2t-1,2t}^1$$

Proof. The starting assumptions for this case are

$$\beta_1 \neq 0 \text{ and } \beta_{3,k} = 0, k \geq 1$$

In particular the Jacobi condition forces $\alpha_i^2 = 0\ \forall i$.

1. Suppose β_1 and $\alpha_1^3 \neq 0$ $\left(\text{so } \alpha_j^3 = 0 \text{ for } i \geq 2\right)$.
We observe that if there exists an $\alpha_{ij}^1 \neq 0$ then a linear change of basis allows to suppose $\alpha_i^4 = \alpha_j^4 = 0$. So we have the conditions

$$d_i a_{ij} = d_j a_{ij} = 0,\ 1 \leq i, j \tag{1}$$

   (a) $\exists\, \alpha_i^4 \neq 0$ with $i \geq 2$ .We can suppose $\alpha_2^4 = 1$ $\left(\text{so } \alpha_{2j}^1 = 0 \text{ by } (1)\right)$ and $\alpha_i^4 = 0, \forall i \geq 2$.
      (i) If $\alpha_{1j}^1 = 0$ for all j we obtain two even dimensional algebras isomorphic to $\mathfrak{g}_{2m}^{33}$ and $\mathfrak{g}_{2m}^{34}$.
      (ii) If $\alpha_{1j}^1 \neq 0$ for an index j we can suppose $\alpha_{13}^1 = 1$. We obtain two odd dimensional algebras isomorphic respectively to $\mathfrak{g}_{2m+1}^{35}$ and $\mathfrak{g}_{2m+1}^{36}$.
   (b) $\alpha_i^4 = 0,\ \forall i \geq 2$
      (i) If $\alpha_1^4 \neq 0$, then $\alpha_{1j}^1 = 0$ by (1). Reordering $\{\theta_2, .., \theta_{n-6}\}$ we obtain an algebra isomorphic to $\mathfrak{g}_{2m+1}^{37}$.
      (ii) If $\alpha_1^4 = 0$ and $\alpha_{12}^1 \neq 0$ we obtain two algebras isomorphic to $\mathfrak{g}_{2m}^{38}$ and $\mathfrak{g}_{2m}^{39}$.

(iii) If $\alpha_1^4 = \alpha_{1j}^1 = 0$, $\forall j$ we obtain two algebras in odd dimension isomorphic to $\mathfrak{g}_{2m+1}^{40}$ and $\mathfrak{g}_{2m+1}^{41}$.

2. Suppose $\beta_1 \neq 0$ and $\alpha_i^3 = 0$, $\forall i$. Additionally we can suppose $\beta_3 = 0$.
   (a) If $\alpha_i^4 \neq 0$ for an index $i \geq 1$ let $\alpha_1^4 = 1$ and $\alpha_i^4 = 0$, $\forall i \geq 2$ and $\alpha_{1j}^1 = 0$ by (1). We obtain two algebras isomorphic $\mathfrak{g}_{2m+1}^{42}$ and $\mathfrak{g}_{2m+1}^{43}$.
   (b) If $d_i = 0$ $\forall i$ we obtain two even dimensional algebras respectively isomorphic to $\mathfrak{g}_{2m}^{44}$ and $\mathfrak{g}_{2m}^{45}$.

□

Remark 25. *Observe that the algebras $\mathfrak{g}_{2m+1}^{42}$ and $\mathfrak{g}_{2m+1}^{43}$ are central extensions of the five dimensional filiform algebras $\mathfrak{l}_5^1$ and $\mathfrak{l}_5^2$.*

Corollary 26. *Any nonsplit $(n-5)$-filiform 2-abelian Lie algebra is isomorphic to one of the laws $\mathfrak{g}^i$, $i \in \{1, .., 45\}$.*

Remark 27. *As we have seen that a $(n-5)$-filiform Lie algebra is either 1- or 2-abelian, the global classification follows from determining the isomorphism classes of the 1-abelian ones.*

1.1. Characteristically nilpotent $(n-5)$-filiform Lie algebras

The first example of a nilpotent Lie algebra all whose derivations are nilpotent was given by Dixmier and Lister in 1957 [**7**], as an answer to a question formulated by Jacobson [**9**] two years earlier. This new class of Lie algebras was soon recognized to be very important, and called characteristically nilpotent, as they verify a certain sequence for derivations which is a kind of generalization of the central descending sequence for nilpotent Lie algebras ([**7**], [**10**]).

Definition 28. A Lie algebra $\mathfrak{g}$ is called characteristically nilpotent if the Lie algebra of derivations $Der(\mathfrak{g})$ is nilpotent.

Remark 29. *It is easily seen that the original definition given by Dixmier and Lister is equivalent to the given above* [**10**].

Remark 30. *It is trivial to verify that there do not exist characteristically nilpotent, $(n-p)$-filiform Lie algebras for indexes $p = 1, 2$. For $p = 3, 4$, it has been shown that these algebras have rank $r \geq 1$* [**2**]*, and that almost any of these laws is the nilradical of a solvable, rigid law.*

Lemma 31. *Let $\mathfrak{g}$ be an $(n-5)$-filiform,1-abelian Lie algebra. Then $rank(\mathfrak{g}) \geq 1$.*

Proof. If $\dim C^1\mathfrak{g} = 4$, the assertion follows immediately from the linear system (S) associated to the algebra, as this system admits nontrivial solutions. If $\dim C^1\mathfrak{g} = 5$, the only case for which the system could have zero solution is $\beta_2 = \beta_{3,1} = 1$, and the distinct values of $\left(\alpha_i^2, \alpha_i^3, \alpha_i^4\right)$. For any of these starting conditions it is routine to prove the existence of a nonzero semisimple derivation. □

Lemma 32. *Let* $\mathfrak{g} \in \mathfrak{h}_2$. *If* $\alpha_{ij}^1 \neq 0$ *for* $1 \leq i, j \leq n-6$ *such that*

$$\beta_{1,k} = \beta_{3,k} = \alpha_k^t = 0, \ k = i, j, \ t = 2, 3, 4$$

then rank $(\mathfrak{g}) \geq 1$.

Proof. Consider the endomorphism defined by

$$d(Y_i) = Y_i, \ d(Y_j) = -Y_j$$

and zero over the undefined images, where $(X_1, .., X_6, Y_1, .., Y_{n-6})$ is the dual basis of $(\omega_1, .., \omega_6, \theta_1, .., \theta_{n-6})$. Clearly d is a nonzero semisimple derivation of $\mathfrak{g}$. $\square$

Proposition 33. *A* $(n-5)$*-filiform Lie algebra* $\mathfrak{g}_n$ *is characteristically nilpotent if and only if it is isomorphic to one of the following laws:*

$$\mathfrak{g}_7^{11}, \ \mathfrak{g}_9^{15}, \ \mathfrak{g}_7^{17}, \ \mathfrak{g}_7^{24,\alpha} \ (\alpha \neq 0), \ \mathfrak{g}_7^{27}, \ \mathfrak{g}_9^{36}, \ \mathfrak{g}_7^{37}, \ \mathfrak{g}_7^{41}$$
$$\mathfrak{g}_8^{12}, \ \mathfrak{g}_8^{16}, \ \mathfrak{g}_8^{25,\alpha} \ (\alpha \neq 0), \ \mathfrak{g}_8^{26}, \ \mathfrak{g}_8^{34}, \ \mathfrak{g}_8^{39}$$

Corollary 34. *There are characteristically nilpotent Lie algebras* $\mathfrak{g}_n$ *with nilpotence index* 5 *for the dimensions* $n = 7, 8, 9, 14, 15, 16, 17, 18$ *and* $n \geq 21$.

Proof. As the sum of characteristically nilpotent algebras is characteristically nilpotent [**11**], the assertion follows from the previous proposition. $\square$

Remark 35. *In fact, for any* $n \geq 7$ *there exist characteristically nilpotent Lie algebras of nilindex* 5*. However, the algebras to be added are not p-filiform any more* [**3**].

1.2. Nilradicals of rigid algebras as factors of k-abelian Lie algebras

The second application of k-abelian Lie algebras is of interest for the theory of rigid Lie algebras. In this paragraph we prove the existence, by giving a family for dimensions $2m+2$ $(m \geq 4)$, of $(m-1)$-abelian Lie algebras $\mathfrak{g}$ of characteristic sequence $(2m-1, 2, 1)$ all whose factor algebras $\frac{\mathfrak{g}}{C^k\mathfrak{g}}$ $(k \geq m)$ are isomorphic to the nilradical of a solvable rigid law.

For $m \geq 4$ let $\mathfrak{g}_m$ be the Lie algebra whose Maurer-Cartan equations are

$$\begin{aligned}
d\omega_1 &= d\omega_2 = 0 \\
d\omega_j &= \omega_1 \wedge \omega_{j-1}, \ 3 \leq j \leq 2m-1 \\
d\omega_{2m} &= \omega_1 \wedge \omega_{2m-1} + \sum_{j=2}^{m} (-1)^j \, \omega_j \wedge \omega_{2m+1-j} \\
d\omega_{2m+1} &= \omega_2 \wedge \omega_3 \\
d\omega_{2m+2} &= \omega_1 \wedge \omega_{2m+1} + \omega_2 \wedge \omega_4
\end{aligned}$$

It is elementary to verify that this algebra has a characteristic sequence $(2m-1, 2, 1)$. Moreover, it is $(m-1)$-abelian, for the exterior product $\omega_m \wedge \omega_{m+1}$ proves that $\left[C^{m-2}\mathfrak{g}_m, C^{m-2}\mathfrak{g}_m\right] \neq 0$ and $\left[C^{m-1}\mathfrak{g}_m, C^{m-1}\mathfrak{g}_m\right] = 0$.

Notation 36. The dual basis of $\{\omega_1, .., \omega_{m2+2}\}$ will be denoted as: $(X_1, .., X_{2m+2})$.

Lemma 37. *For any* $4 \leq m \leq k \leq 2m-2$ *the factor algebra* $\frac{\mathfrak{g}_m}{C^k \mathfrak{g}_m}$ *has equations*

$$
\begin{aligned}
d\bar{\omega}_1 &= d\bar{\omega}_2 = 0 \\
d\bar{\omega}_j &= \bar{\omega}_1 \wedge \bar{\omega}_{j-1},\ 3 \leq j \leq k+1 \\
d\bar{\omega}_{2m+1} &= \bar{\omega}_2 \wedge \bar{\omega}_3 \\
d\bar{\omega}_{2m+2} &= \bar{\omega}_1 \wedge \bar{\omega}_{2m+1} + \bar{\omega}_2 \wedge \bar{\omega}_4
\end{aligned}
$$

where $\bar{\omega}_j = \omega_j \mod C^k \mathfrak{g}_m$. *Moreover, this algebra is* 1*-abelian of characteristic sequence* $(k, 2, 1)$.

The proof is trivial.

Proposition 38. *For any* $4 \leq m \leq k$ *the algebra* $\frac{\mathfrak{g}_m}{C^k \mathfrak{g}_m}$ *is isomorphic to the nilradical of a solvable, rigid Lie algebra* $\mathfrak{r}_{m,k}$.

Proof. Let $\mathfrak{r}_{m,k} = \frac{\mathfrak{g}_m}{C^k \mathfrak{g}_m} \oplus \mathfrak{t}_k$ be the semidirect product of $\frac{\mathfrak{g}_m}{C^k \mathfrak{g}_m}$ by the torus $\mathfrak{t}_k$ defined by its weights :

$$
\begin{gathered}
\lambda_1,\ \lambda_2 + (k-1)\lambda_1,\ \lambda_j = \lambda_2 + (k-3+j)\lambda_1\ \ (3 \leq j \leq k+1) \\
\lambda_{2m+1} = 2\lambda_2 + (2k-1)\lambda_1,\ \lambda_{2m} = 2\lambda_2 + 2k\lambda_1
\end{gathered}
$$

over the basis $\{\bar{X}_1, .., \bar{X}_{k+1}, \bar{X}_{2m+1}, \bar{X}_{2m+2}\}$ dual to $\{\bar{\omega}_1, .., \bar{\omega}_{k+1}, \bar{\omega}_{2m+1}, \bar{\omega}_{2m+2}\}$. Then the law is given by

$$
\begin{aligned}
&\left[V_1, \bar{X}_1\right] = \bar{X}_1,\ \left[V_1, \bar{X}_j\right] = (k-1)\bar{X}_j\,(3 \leq j \leq k+1), \\
&\left[V_1, \bar{X}_{2m+1}\right] = (2k-1)\bar{X}_{2m+1},\ \left[V_1, \bar{X}_{2m+2}\right] = 2k\bar{X}_{2m+2} \\
&\left[V_2, \bar{X}_j\right] = \bar{X}_j\,(2 \leq j \leq k+1),\ \left[V_2, \bar{X}_j\right] = 2\bar{X}_j\,(j = 2m+1, 2m+2) \\
&\left[\bar{X}_1, \bar{X}_j\right] = \bar{X}_{j+1}\,(2 \leq j \leq k),\ \left[\bar{X}_2, \bar{X}_3\right] = a\bar{X}_{2m+1},\ \left[\bar{X}_2, \bar{X}_4\right] = b\bar{X}_{2m+1}
\end{aligned}
$$

Thus the only nonzero brackets not involving the vector $\bar{X}_1$ are

$$
\left[\bar{X}_2, \bar{X}_3\right] = a\bar{X}_{2m+1},\ \left[\bar{X}_2, \bar{X}_4\right] = b\bar{X}_{2m+1}
$$

Now Jacobi implies $a = b$, and by a change of basis $a = 1$. Thus the law is rigid, and $\mathfrak{t}_k$ is a maximal torus of derivations of $\frac{\mathfrak{g}_m}{C^k \mathfrak{g}_m}$. □

Corollary 39. *For any* $4 \leq m \leq k$ *the factor algebra*

$$
\frac{\left(\frac{\mathfrak{g}_m}{C^k \mathfrak{g}_m}\right)}{\left\langle \bar{X}_{2m+2} \right\rangle}
$$

is 1*-abelian of characteristic sequence* $(k, 1, 1)$ *and isomorphic to the nilradical of a solvable rigid law* $\mathfrak{s}_{m,k}$*. Moreover*

$$\mathfrak{s}_{m,k} \simeq \frac{\mathfrak{r}_{m,k}}{\left\langle \bar{X}_{2m+2} \right\rangle}$$

2. Other 2-abelian nilpotent Lie algebras

In this final section we show by examples how the concept of k-abelianity arises naturally in the study of parabolic subalgebras of classical Lie algebras.

Let E_6 be the simple exceptional Lie algebra of dimension 78. Let Φ be a root system respect to a Cartan subalgebra $\mathfrak{h}$ and $\Delta = \{\alpha_1, .., \alpha_6\}$ a basis of fundamental roots. Recall that the standard Borel subalgebra of E_6 is given by

$$B(\Delta) = \mathfrak{h} + \sum_{\alpha \in \Phi^+} L_\alpha$$

where L_α is the weight space associated to the root α. Recall also that any parabolic subalgebra $\mathfrak{p}$ is determined, up to isomorphism, by a subsystem $\Delta_1 \subset \Delta$ such that $\mathfrak{p}$ is conjugated to the subalgebra

$$P(\Delta_1) = \mathfrak{h} + \sum_{\alpha \in \Phi_1 \cup \Phi_2^+} L_\alpha$$

where Φ_1 is the set of roots expressed in terms of $\Delta \backslash \Delta_1$ and $\Phi_2^+ = \Phi^+ \cap (\Phi \backslash \Phi_1)$. It is elementary to see that the nilradical is

$$\mathfrak{n}(\Delta_1) = \sum_{\alpha \in \Phi_2^+} L_\alpha$$

and called (E_6, Δ_1)-nilalgebra.

Let us consider the following subsets $\Delta_1 \subset \Delta$:

$$\mathcal{L} = \left\{ \begin{array}{c} \{\alpha_1, \alpha_4\}, \{\alpha_4, \alpha_6\}, \{\alpha_3, \alpha_5\}, \{\alpha_3, \alpha_4\}, \{\alpha_4, \alpha_5\}, \{\alpha_2, \alpha_3\}, \\ \{\alpha_2, \alpha_5\}, \{\alpha_2, \alpha_4\}, \{\alpha_1, \alpha_2, \alpha_3\}, \{\alpha_2, \alpha_5, \alpha_6\}, \\ \{\alpha_1, \alpha_2, \alpha_5\}, \{\alpha_2, \alpha_3, \alpha_6\}, \{\alpha_1, \alpha_4, \alpha_6\}, \\ \{\alpha_1, \alpha_2, \alpha_6\}, \{\alpha_1, \alpha_3, \alpha_5\}, \{\alpha_3, \alpha_5, \alpha_6\} \end{array} \right\}$$

As known, the maximal root of E_6 is

$$\delta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \sum_{i=1}^{6} k_{a_i} \alpha_i$$

We define the Δ_1-height of δ as

$$h_{\Delta_1}(\delta) = \sum_{\alpha_i \in \Delta_1} k_{a_i}$$

and the subsets

$$\Delta_1(k) = \left\{ \alpha \in \Phi_2^+ \mid h_{\Delta_1}(\alpha) = k \right\}$$

Proposition 40. *For any $\Delta_1 \in \mathcal{L}$ the (E_6, Δ_1)-nilalgebra $\mathfrak{n}(\Delta_1)$ is 2-abelian.*

Proof. As known, for the ideals $C^k\mathfrak{n}$ of the descending central sequence we have

$$C^k\mathfrak{n} = \sum_{\substack{\alpha\in\Delta_1(j)\\ j\geq k+1}} L_\alpha$$

Thus, if the derived subalgebra is not abelian, it suffices to show the existence of two roots $\alpha, \beta \in \Delta_1(2)$ such that $\alpha+\beta \in \Phi_2^+$ and that for any two roots $\gamma, \varepsilon \in \Delta_1(3)$ we have $\gamma+\varepsilon \notin \Phi_2^+$. Moreover, let $\delta_1 = \sum_{i=1}^{6} \alpha_i \in \Phi$

1. $\Delta_1 = \{\alpha_1, \alpha_4\}$: take $\alpha = \delta_1 - \alpha_5 - \alpha_6,\ \beta = \delta - \alpha_1 - \alpha_2 - \alpha_4;\ \alpha+\beta = \delta$
2. $\Delta_1 = \{\alpha_3, \alpha_5\}$: $\alpha = \delta_1,\ \beta = \delta_1 - \alpha_1 - \alpha_6 + \alpha_4;\ \alpha+\beta = \delta$
3. $\Delta_1 = \{\alpha_4, \alpha_5\}$: $\alpha = \delta_1 - \alpha_1 - \alpha_2 - \alpha_6,\ \beta = \delta_1;\ \alpha+\beta = \delta - \alpha_2 - \alpha_4$
4. $\Delta_1 = \{\alpha_2, \alpha_3\}$: $\alpha = \delta_1,\ \beta = \delta_1 - \alpha_1 - \alpha_2 - \alpha_6;\ \alpha+\beta = \delta - \alpha_2 - \alpha_4$
5. $\Delta_1 = \{\alpha_2, \alpha_4\}$: $\alpha = \delta_1,\ \beta = \delta_1 - \alpha_1 - \alpha_6 + \alpha_4;\ \alpha+\beta = \delta$
6. $\Delta_1 = \{\alpha_1, \alpha_2, \alpha_3\}$: $\alpha = \delta - \delta_1,\ \beta = \delta_1 - \alpha_2;\ \alpha+\beta = \delta - \alpha_2$
7. $\Delta_1 = \{\alpha_1, \alpha_2, \alpha_5\}$: $\alpha = \delta - \delta_1,\ \beta = \delta_1 - \alpha_2;\ \alpha+\beta = \delta - \alpha_2$
8. $\Delta_1 = \{\alpha_1, \alpha_3, \alpha_6\}$: $\alpha = \delta_1 - \alpha_6,\ \beta = \delta_1 - \alpha_1 - \alpha_2;\ \alpha+\beta = \delta - \alpha_2 - \alpha_4$
9. $\Delta_1 = \{\alpha_1, \alpha_4, \alpha_6\}$: $\alpha = \delta_1 - \alpha_6,\ \beta = \delta_1 - \alpha_1 - \alpha_2;\ \alpha+\beta = \delta - \alpha_2 - \alpha_4$
10. $\Delta_1 = \{\alpha_1, \alpha_2, \alpha_6\}$: $\alpha = \delta_1 - \alpha_1,\ \beta = \delta_1 + \alpha_4;\ \alpha+\beta = \delta$
11. $\Delta_1 = \{\alpha_1, \alpha_3, \alpha_5\}$: $\alpha = \delta_1 - \alpha_2 - \alpha_6,\ \beta = \delta_1 + \alpha_4;\ \alpha+\beta = \delta - \alpha_2$

Let X_α, X_β be generators of the weight spaces L_α and L_β : then the preceding relations show that

$$[X_\alpha, X_\beta] = X_{\alpha+\beta} \neq 0$$

proving that the derived subalgebra is not abelian.
Finally, it is trivial to see that for any subset Δ_1 listed above we have

$$\left[C^2\mathfrak{n}(\Delta_1), C^2\mathfrak{n}(\Delta_1)\right] = 0$$

Observe that the eleven considered cases in fact cover all the subsets enumerated in $\mathcal{L}$, since we are only interested on isomorphism classes. Therefore we only need to prove the result for the pairwise non-isomorphic algebras. ☐

Remark 41. *This example suggests to study the k-abelianity properties of graded Lie algebras. This procedure, establishing concrete properties on the filtration associated to the central descending sequence, can be used to classify naturally graded Lie algebras for specific characteristic sequences* [**4**].

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