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ON HADAMARD – DIRICHLET ALGEBRAS

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Abstract. S. Bhatt and R. Raina studied in [**1**] the behaviour of some fractional operators and Hadamard products on certain analytic functions on the unit disk. More generally, classes of analytic functions on the unit disk constitute a matter of actual intensive research. So, it is desirable to dispose of an adequate theoretic frame which allow relatively simple and expeditious results on this subject. Recently one of the authors considered topics on the structure of Hadamard algebras (cf. [**3**], [**4**]). In this article our aim is to consider Dirichlet spaces, which constitute well known Hilbert spaces, endowed with an abelian unitary Banach algebra structure induced by a Hadamard type product. The maximal ideal space, complex Hadamard homomorphisms, reproducing kernels, the generating function and spectra of their elements are determined.

1. Introduction

For $-1 < \alpha \leq 0$ let $\mathcal{D}_\alpha$ be the weighted pre – Hilbert space of analytic functions on the open unit disk for which

$$\int_D |df/dz|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} < \infty,$$

where $dA(z)$ is Lebesgue area measure on the unit disk, with norm given by

$$\|f\|^2_{\mathcal{D}_\alpha} = |f(0)|^2 + \int_D |df/dz|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} \tag{1}$$

and inner product

$$\langle f, g\rangle_{\mathcal{D}_\alpha} = f(0)\ \overline{g(0)} + \int_D df/dz \ \overline{dg/dz}\ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi}. \tag{2}$$

In the sequel we will denote by $\mathfrak{X}(D)$ to the space of analytic functions on the unit disk endowed with the compact open topology or that of uniform convergence on compact subsets of D. We will also consider the weighted Bergman space $A^2_\alpha(D) = \mathfrak{X}(D) \cap L^2(D,\ (1-|z|^2)^\alpha\ dA(z)/\pi)$ considered as a subspace of $L^2(D,\ (1-|z|^2)^\alpha\ dA(z)/\pi)$.

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2. The Hilbert structure

Theorem 1. *$(\mathcal{D}_\alpha,\ \langle,\rangle_{\mathcal{D}_\alpha})$ is a Hilbert space.*

Proof. It is clear that $\mathcal{D}_\alpha$ is a complex vector space and that $\langle,\rangle_{\mathcal{D}_\alpha}$ fulfils all properties of an inner product. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $\mathcal{D}_\alpha$, $n,\ m \in \mathbb{N}$. Then

$$\|f_n - f_m\|^2_{\mathcal{D}_\alpha} \geq |f_n(0) - f_m(0)|^2 + \|df_n/dz - df_m/dz\|^2_{A_0^2(D)}\,. \tag{3}$$

Moreover, if C is a compact subset of D, $\lambda \in C$, $0 < r < 1 - |\lambda|$ and $F \in \mathcal{D}_\alpha$ we have

$$\begin{aligned} |F(\lambda)| &= \left| \frac{1}{\pi\ r^2} \int_{|z-\lambda|\leq r} F(z)\ dA(z) \right| \\ &\leq \frac{1}{\pi\ r^2} \int_{|z-\lambda|\leq r} |F(z)|\ dA(z) \\ &\leq \left(\frac{1}{\pi\ r^2} \int_D |F(z)|^2\ dA(z) \right)^{1/2} \\ &= \|F\|_{A_0^2(D)} / r \end{aligned}$$

and so

$$|F(\lambda)| \leq \|F\|_{A_0^2(D)} / (1 - |\lambda|) \leq \|F\|_{A_0^2(D)} / dist(C, \mathbb{C}{-}D). \tag{4}$$

By (3) and (4) we obtain

$$\begin{aligned} \|f_n - f_m\|_{\mathcal{D}_\alpha} &\geq \|df_n/dz - df_m/dz\|_{A_0^2(D)} \\ &\geq dist(C, \mathbb{C}{-}D)\ \max_C\ |df_n/dz - df_m/dz| \end{aligned}$$

and hence $\{df_n/dz\}_{n\geq 1}$ is a uniform Cauchy sequence in $\mathfrak{X}(D)$. Since $\mathfrak{X}(D)$ is a Fréchet space there is $g \in \mathfrak{X}(D)$ such that $\{df_n/dz\}_{n\geq 1}$ converges to g in $\mathfrak{X}(D)$. Then

$$f_n(w) - f_n(0) \to \int_0^w g(z)\ dz, \tag{5}$$

convergence being also uniform on compact subsets of D in (5). By (3) we can define

$$f(w) = \lim_{n\to\infty}\ f_n(0) + \int_0^w g(z)\ dz, \qquad w \in D.$$

Thus $f_n \to f$ uniformly on compact subsets of D and $f \in \mathfrak{X}(D)$. Now, let $N \in \mathbb{N}$ be such that $\|f_n - f_m\|_{\mathcal{D}_\alpha} \leq 1$ for all positive integers $m,\ n$ greater than N. Then

$$\|f_n\|_{\mathcal{D}_\alpha} \leq \|f_n - f_N\|_{\mathcal{D}_\alpha} + \|f_N\|_{\mathcal{D}_\alpha} \leq 1 + \|f_N\|_{\mathcal{D}_\alpha}$$

if $n \geq N$, i.e.

$$\int_D |df_n/dz|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} \leq \left(1+\|f_N\|_{\mathcal{D}_\alpha}\right)^2 - |\ f_n(0)|$$

and by Fatou's lemma we obtain

$$\begin{aligned}\int_D |g(z)|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} \quad &\leq \quad \liminf \int_D |df_n/dz|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} \\ &\leq \quad \left(1+\|f_N\|_{\mathcal{D}_\alpha}\right)^2 - |\ f(0)|\,,\end{aligned}$$

so $f \in \mathcal{D}_\alpha$. Finally, let $\varepsilon > 0$ and $M \in \mathbb{N}$ such that $\|f_n - f_m\|_{\mathcal{D}_\alpha} \leq \varepsilon$ if $n,\ m$ are positive integers greater than M. If $0 < r < 1$ we get

$$|f_n(0) - f_m(0)|^2 + \int_{|z|\leq r} |df_n/dz - df_m/dz|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} \leq \varepsilon^2 \tag{6}$$

and if $n \to \infty$ in (6) then

$$|f(0) - f_m(0)|^2 + \int_{|z|\leq r} |df/dz - df_m/dz|^2 \ (1-|z|^2)^\alpha \ \frac{dA(z)}{\pi} \leq \varepsilon^2. \tag{7}$$

If $r \to 1^-$ then (7) yields $\|f_m - f\|_{\mathcal{D}_\alpha} \leq \varepsilon$ whenever $m \geq M$, i.e. $f_m \to f$ in $\mathcal{D}_\alpha$. □

3. The Banach structure

The object of this section is to introduce a Banach structure on $\mathcal{D}_\alpha$. So, if $f,\ g \in \mathcal{D}_\alpha$ we define their formal Hadamard product $f \odot g \in \mathcal{D}_\alpha$ as

$$(f \odot g)\,(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\ g^{(n)}(0)}{n!^2}\ z^n, \qquad z \in D.$$

Let $h = \sum_{n=0}^{\infty} c_n \ z^n$ be in $\mathcal{D}_\alpha$, $0 < r < 1$. Since h' converges absolutely on compact subsets of D we have

$$\int_{r\overline{D}} |h'\ (z)|^2\, (1-|z|^2)^\alpha \frac{dA(z)}{\pi} =$$

$$\begin{aligned} &= \sum_{n\geq 1, m\geq 1} nc_n \ m\overline{c_m} \int_{r\overline{D}} z^{n-1}\overline{z}^{m-1}(1-|z|^2)^\alpha \frac{dA(z)}{\pi} \\ &= \sum_{n\geq 1, m\geq 1} nc_n \ m\overline{c_m} \ \ 2\delta_{n,m} \int_0^r \rho^{n+m-1} \ (1-\rho^2)^\alpha \ d\rho \\ &= \sum_{n=1}^{\infty} n^2 \ |c_n|^2 \ r^{2(n+\alpha)} \ Be(n, \alpha+1). \end{aligned} \tag{8}$$

If $r \to 1^-$ by (1) and (8) we obtain

$$\|h\|^2_{\mathcal{D}_\alpha} = |c_0|^2 + \sum_{n=1}^{\infty} n^2 \ |c_n|^2 \ Be(n, \alpha+1). \tag{9}$$

In particular, $h = 0$ if and only if $h(z) = 0$ for all $z \in D$.

Remark 2. *If* $f \in \mathfrak{X}(D)$ *we write*

$$r_f = \overline{\lim} \ \sqrt[n]{\left|\frac{f^{(n)}(0)}{n!}\right|}.$$

Since f *converges absolutely and uniformly on compact subsets of* D *then* $|z| \ r_f < 1$ *if* $z \in D$. *Thus* $r_f \leq 1$. *On the other hand, if the above condition holds it is immediate that* $f \in \mathfrak{X}(D)$. *In other words,* $\mathfrak{X}(D)$ *is the set of holomorphic functions* f *at zero such that* $r_f \leq 1$.

Remark 3. *If* $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$ *are sequences of non negative real numbers then* $\overline{\lim} \ |a_n \ b_n|^{1/n} \leq \overline{\lim} \ |a_n|^{1/n} \ \overline{\lim} \ |b_n|^{1/n}$. *Since the Taylor series expansion is unique whenever it exists, by Remark 2 the Hadamard product is closed within* $\mathfrak{X}(D)$, *i.e.* $\odot : \mathfrak{X}(D) \times \mathfrak{X}(D) \to \mathfrak{X}(D)$. *Moreover,* $r_{f\odot g} \leq r_f \ r_g$ *if* $f, \ g \in \mathfrak{X}(D)$.

Proposition 4. *(cf.* [**2**]*) If* $f, \ g \in \mathfrak{X}(D)$ *then*

$$(f \odot g)(w) = \frac{1}{2\pi i} \int_{|z|=r} f(z) \ g\left(\frac{w}{z}\right) \ \frac{dz}{z}, \qquad |w| < r < 1.$$

Corollary 5. *If* $f, \ g \in \mathfrak{X}(D)$ *then*

$$\begin{aligned} (f \odot g)^{(n)}(w) &= \frac{1}{2\pi i} \int_{|z|=r} f(z) \ g^{(n)}\left(\frac{w}{z}\right) \ \frac{dz}{z^{n+1}}, \qquad |w| < r < 1. \\ &= \begin{cases} w^{-n} \left[f(z) \odot (z^n \ g^{(n)}(z))\right](w) & if \quad w \neq 0, \\ \left[f^{(n)}(0) \ g^{(n)}(0)\right]/n! & if \quad w = 0. \end{cases} \end{aligned}$$

Theorem 6. *If* $-1 < \alpha \leq 0$ *the space* $\mathcal{D}_\alpha$ *is an abelian Banach algebra without unit respect to Hadamard product.*

Proof. By Th. 1 we know that $(\mathcal{D}_\alpha, \|.\|_{\mathcal{D}_\alpha})$ is a Banach space. On the other hand, we observe that $Be(1, \ 1+\alpha) = 1/(1+\alpha)$, i.e. $Be(1, \ 1+\alpha) \geq 1$. If $n^2 \ Be(n, \ 1+\alpha) \geq 1$ for an integer $n \geq 1$ then

$$\begin{aligned} (n+1)^2 \ Be(n+1, \ 1+\alpha) &= \frac{(n+1)^2}{n \ (n+1+\alpha)} \ n^2 \ Be(n, \ 1+\alpha) \\ &\geq \frac{(n+1)^2}{n \ (n+1+\alpha)} \end{aligned}$$

and $(n+1)^2 \ Be(n+1, \ 1+\alpha) \geq 1$. Therefore $n^2 \ Be(n, \ 1+\alpha) \geq 1$ for all positive integers. In consequence, if $f = \sum_{n=0}^{\infty} a_n \ z^n$ and $g = \sum_{n=0}^{\infty} b_n \ z^n$ are two elements of $\mathcal{D}_\alpha$ and $N \in \mathbb{N}$ then

$$|a_0\ b_0|^2 + \sum_{n=1}^{N} n^2\ |a_n\ b_n|^2\ Be(n,\alpha+1) \leq$$

$$\begin{aligned} &\leq \left[|a_0|^2 + \sum_{n=1}^{N} n^2\ |a_n|^2\ Be(n,\alpha+1)\right]\left[|b_0|^2 + \sum_{n=1}^{N} n^2\ |b_n|^2\ Be(n,\alpha+1)\right] \\ &\leq \|f\|^2_{\mathcal{D}_\alpha}\ \|g\|^2_{\mathcal{D}_\alpha} \end{aligned}$$

and by (9) it follows that $f \odot g \in \mathcal{D}_\alpha$ and $\|f \odot g\|_{\mathcal{D}_\alpha} \leq \|f\|_{\mathcal{D}_\alpha}\ \|g\|_{\mathcal{D}_\alpha}$.

Finally, let us suppose that $g \in \mathcal{D}_\alpha$ is a unit. If $f \in \mathcal{D}_\alpha$ and $w \in D$ then

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{|t|=(1+w)/2} f(t)\ \left[g(\frac{w}{t})\ \frac{1}{t} - \frac{1}{t-w}\right]\ dt \\ &= \frac{1}{2\pi i} \int_{|z|=2|w|/(1+|w|)} f(\frac{w}{z})\ \left[g(z) - \frac{1}{1-z}\right]\ dz. \end{aligned} \tag{10}$$

If $0 < r < 1$, $f = z^n$, $n = 1,\ 2,\ \ldots$ from equation (10) we obtain that

$$\frac{1}{2\pi i} \int_{|z|=2|w|/(1+|w|)} \left[g(z) - \frac{1}{1-z}\right]\ \frac{dz}{z^n} = 0.$$

But $w \to 2\,|w|\,/(1+|w|)$ is a surjective function between D and $[0,1)$ and whence

$$\frac{1}{2\pi i} \int_{|z|=r} \left[g(z) - \frac{1}{1-z}\right]\ \frac{dz}{z^n} = 0$$

whenever $0 \leq r < 1$. In consequence

$$\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} \left[g(z) - \frac{1}{1-z}\right]_{z=0} = 0$$

for all $n \in \mathbb{N}$, i.e. $g(z) = 1/(1-z)$. But the series $\sum_{n=1}^{\infty} n^2\ Be(n,\alpha+1)$ diverges as we already know that its general term is greater than one, i.e. $g \notin \mathcal{D}_\alpha$. □

Corollary 7. *The set* $\mathcal{B}_\alpha = \{z^n\ [\delta_{n0} + n^2\ Be(n,\ \alpha+1)]^{-1/2}\}_{n\geq 0}$ *is an orthonormal basis of* $\mathcal{D}_\alpha$.

Proof. It is easy to see that $\mathcal{B}_\alpha$ is an orthonormal set and by (9) the Bessel – Parseval equality holds. □

Remark 8. *We observe that* α *can not be greater than zero in Th.6. For instance,* $\|z \odot z\|_{\mathcal{D}_\alpha} = \|z\|_{\mathcal{D}_\alpha} = (1+\alpha)^{-1/2}$, *i.e.* $\|z \odot z\|_{\mathcal{D}_\alpha} > \|z\|^2_{\mathcal{D}_\alpha}$ *if* $\alpha > 0$.

4. The spectral analysis

Remark 9. *Let $\mathcal{D}_\alpha \times \mathbb{C}$ be the Banach algebra obtained from $\mathcal{D}_\alpha$ by the usual method of adjunction of a unit. We will also denote this algebra by $\mathcal{D}_\alpha$.*

Theorem 10. *Given $(f, a) \in \mathcal{D}_\alpha$ we write*

$$\langle (f,a), \varkappa_p \rangle = \begin{cases} f^{(p)}(0)/p! + a & if \quad p \in \mathbb{N}_0, \\ \\ a & if \quad p = \infty. \end{cases}$$

Then $\mathfrak{X}_\alpha = \{\varkappa_p\}_{p=0}^{\infty}$ is the set of all complex valued homomorphisms on $\mathcal{D}_\alpha$, i.e. $\mathfrak{X}$ is the maximal ideal space of $\mathcal{D}_\alpha$.

Proof. Let $f \in \mathcal{D}_\alpha$ be represented by $f = \sum_{n=0}^{\infty} f^{(n)}(0)/n!\ z^n$ as an element of $\mathfrak{X}(D)$. By applying (9) it follows that

$$\lim_{N\to\infty} \left\| f - \sum_{n=0}^{N} f^{(n)}(0)/n!\ z^n \right\|_{\mathcal{D}_\alpha} = 0. \tag{11}$$

So, if $\varkappa$ is a non zero complex valued homomorphism on $\mathcal{D}_\alpha$ then it must be bounded and

$$\langle (f,a), \varkappa \rangle = a + \sum_{n=0}^{\infty} f^{(n)}(0)/n!\ \langle (z^n, 0), \varkappa \rangle . \tag{12}$$

Since $(z^k, 0) \odot (z^h, 0) = \delta_{kh}\ (z^h, 0)$ for each $k,\ h \in \mathbb{N}_0$ we have

$$\left\langle (z^k,0), \varkappa \right\rangle\ \left\langle (z^h,0), \varkappa \right\rangle = \left\langle (z^k,0) \odot (z^h,0), \varkappa \right\rangle = \delta_{kh}\ \left\langle (z^h,0), \varkappa \right\rangle .$$

Since $\varkappa \neq 0$ there must be a unique $p \in \mathbb{N}_0$ such that $\left\langle (z^k,0), \varkappa \right\rangle = \delta_{kp},\ k \in \mathbb{N}_0$. Therefore we write $\varkappa = \varkappa_p$ and the claim follows. □

Corollary 11. *If $f \in \mathcal{D}_\alpha$ and $p \in \mathbb{N}$ then*

$$f^{(p)}(0)/\left(\alpha+1\right)_p = f(0)\ \delta_{0p}/p + \int_D f'\ (z)\ \overline{z}^{p-1}\ (1 - |z|^2)^\alpha\ \frac{dA(z)}{\pi}.$$

Proof. With the above notation, by (12) we obtain

$$\begin{aligned} \langle (f,a), (z^p, 1) \rangle_{\mathcal{D}_\alpha} &= a + \sum_{n=0}^{\infty} f^{(n)}(0)/n!\ \langle (z^n, 0), (z^p, 0) \rangle_{\mathcal{D}_\alpha} \\ &= a + \frac{p\ f^{(p)}(0)}{(\alpha+1)_p},\ a \in \mathbb{C}. \end{aligned}$$

In particular, if $a = 0$ the result follows by (2). □

Corollary 12. *The classical Dirac's delta functional is given as $\langle f, 1 \rangle_{\mathcal{D}_\alpha} = f(0)$. Moreover, with the notation of Th. 10 we have*

$$\langle (f,a), \varkappa_p \rangle = \begin{cases} \left\langle (f,a), (\binom{\alpha+p}{p}\ \frac{z^p}{p}, 1) \right\rangle_{\mathcal{D}_\alpha} & if \quad p \in \mathbb{N}, \\ \\ \langle (f,a), (0,1) \rangle_{\mathcal{D}_\alpha} & if \quad p = \infty. \end{cases}$$

5. Generating functions and reproducing kernels

Theorem 13. *$\mathcal{D}_\alpha$ is a functional Hilbert space, it has the generating function*

$$\kappa_\alpha(z) = 1 + \sum_{n=1}^{\infty} \binom{n+\alpha}{n} \frac{z^n}{n}, \ z \in D, \tag{13}$$

and the reproducing kernel

$$K_\alpha(z,t) = \kappa_\alpha(\overline{z}\ t) = 1 + \int_0^t \left[(1-\overline{z}\ w)^{-\alpha-1} - 1\right] \frac{dw}{w}, \ (z,t) \in D \times D. \tag{14}$$

Proof. By D'Alembert test the series in (13) converges absolutely on D. If $(z,w) \in D \times D$ then

$$\frac{\partial}{\partial w} \sum_{n=1}^{\infty} \binom{n+\alpha}{n} \frac{(\overline{z}\ w)^n}{n} = \overline{z} \sum_{n=1}^{\infty} \binom{n+\alpha}{n} (\overline{z}\ w)^{n-1} = \frac{(1-\overline{z}\ w)^{-\alpha-1} - 1}{w}$$

and

$$\left| \frac{(1-\overline{z}\ w)^{-\alpha-1} - 1}{w} \right| \leq \sum_{n=1}^{\infty} \binom{n+\alpha}{n} |z|^n = (1-|z|)^{-\alpha-1} - 1.$$

Then the function K_α is well defined and

$$\int_D \left| \frac{(1-\overline{z}\ w)^{-\alpha-1} - 1}{w} \right|^2 (1-|w|^2)^\alpha \ \frac{dA(w)}{\pi} \leq \frac{\left[(1-|z|)^{-\alpha-1} - 1\right]^2}{1+\alpha},$$

i.e. the function $w \to \sum_{n=1}^{\infty} \binom{n+\alpha}{n} (\overline{z}\ w)^n / n$ belongs to $\mathcal{D}_\alpha$ for each $z \in D$, i.e. $t \to K_\alpha(z,t)$ belongs to $\mathcal{D}_\alpha$ for each $z \in D$. Now, $\mathcal{D}_\alpha$ is a functional Hilbert space because if $f \in \mathcal{D}_\alpha$ and $z \in D$ then

$$f(z) = \left\langle f(w), 1 + \sum_{n=1}^{\infty} \frac{(\overline{z}\ w)^n}{n^2\ Be(n, \alpha+1)} \right\rangle_{\mathcal{D}_\alpha} = \langle f(w), K_\alpha(z,w) \rangle_{\mathcal{D}_\alpha}$$

and so

$$|f(z)| \leq \|f\|_{\mathcal{D}_\alpha} \sqrt{1 + \frac{\left[(1-|z|)^{-\alpha-1} - 1\right]^2}{1+\alpha}}.$$

Finally, $\kappa_\alpha(z) = \sum_{n=0}^{\infty} z^n / \|z^n\|^2_{\mathcal{D}_\alpha}$ and (13) follows from Corollary 7. $\square$

6. On units and spectra

Theorem 14. *An element $(f,a) \in \mathcal{D}_\alpha$ is invertible if and only if*

(i) $0 \notin \{a\} \cup \left\{a + f^{(n)}(0)/n!\right\}_{n \geq 0}$ *and (ii)* $\overline{\lim} \left| f^{(n)}(0) / (f^{(n)}(0) + n!\ a) \right| \leq 1$.

Proof. If (f,a) is a unit and $(g,b) \in \mathcal{D}_\alpha$ is its inverse then

$$(f,a) \odot (g,b) = (f \odot g + ag + bf, ab) = (0,1).$$

In particular $b = a^{-1}$ and if $n \in \mathbb{N}_0$ then

$$\frac{f^{(n)}(0)\ g^{(n)}(0)}{n!^2} + a\ \frac{g^{(n)}(0)}{n!} + a^{-1}\ \frac{f^{(n)}(0)}{n!} = 0,$$

or

$$(\frac{f^{(n)}(0)}{n!}+a)\ \frac{g^{(n)}(0)}{n!}=-a^{-1}\ \frac{f^{(n)}(0)}{n!} \tag{15}$$

and (i) holds. Since $g\in\mathfrak{X}(D)$ then (ii) follows by (15) and Remark 2. Conversely, let us define

$$g(z)=-\frac{1}{a}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{f^{(n)}(0)+n!\ a}\ z^n,\ z\in D.$$

Then $g\in\mathfrak{X}(D)$ and $f\odot g+a\ g+(1/a)\ \ f=0$. We must prove that dg/dw belongs to $A^2_\alpha(D)$. Since $a\ dg/dw=-a^{-1}\ df/dw-d(f\odot g)/dw$ it will be enough to see that $[z\ df/dz\odot g(z)](w)/w\in A^2_\alpha(D)$. For, by Corollary 5 there exist numbers $0<\delta<1$ and $K>0$ such that $|d(f\odot g)/dw|\le K$ if $|w|<\delta$. Furthermore, by (ii) there is $N\in\mathbb{N}$ such that $\left|f^{(n)}(0)/(f^{(n)}(0)+n!\ a)\right|\le 1$ if $n\ge N$. Since functions $f,\ g$ as well as all of their derivatives converge uniformly on closed subsets of D if $\delta<r<1$ we have

$$\int_{\delta\le|w|\le r}\left|\frac{1}{2\pi i}\int_{|z|=(1+|w|)/2}df/dz\ g(w/z)\ dz\right|^2\frac{(1-|w|^2)^\alpha}{|w|^2}\ \frac{dA(w)}{\pi}=$$

$$\begin{aligned}
&= \int_{\delta\le|w|\le r}\left|\sum_{n=0}^{\infty}(1+n)\ \frac{f^{(1+n)}(0)}{(1+n)!}\ \frac{f^{(n)}(0)\ w^n}{f^{(n)}(0)+n!\ a}\right|^2\frac{(1-|w|^2)^\alpha}{|w|^2}\frac{dA(w)}{\pi}\\
&= \sum_{n=0}^{\infty}\left|(1+n)\ \frac{f^{(1+n)}(0)}{(1+n)!}\ \frac{f^{(n)}(0)}{f^{(n)}(0)+n!\ a}\right|^2\ Be(n,\alpha+1)\ (r-\delta)^{n+\alpha}\\
&\le 2\sum_{n=N}^{\infty}(1+n)^2\ \left|\frac{f^{(1+n)}(0)}{(1+n)!}\right|^2\ Be(n+1,\alpha+1)\ +\\
&\quad+\sum_{n=0}^{N-1}\left|(1+n)\ \frac{f^{(1+n)}(0)}{(1+n)!}\ \frac{f^{(n)}(0)}{f^{(n)}(0)+n!\ a}\right|^2\ Be(n,\alpha+1)\ (r-\delta)^{n+\alpha}\\
&\le 2\ \|f\|^2_{\mathcal{D}_\alpha}+\sum_{n=0}^{N-1}\left|(1+n)\frac{f^{(1+n)}(0)}{(1+n)!}\frac{f^{(n)}(0)}{f^{(n)}(0)+n!\ a}\right|^2 Be(n,\alpha+1)(r-\delta)^{n+\alpha}.
\end{aligned}$$

By the monotone convergence theorem if $r\to 1^-$ we deduce that

$$\begin{aligned}
&\int_D|d(f\odot g)/dw|^2\ (1-|w|^2)^\alpha\frac{dA(w)}{\pi}\le K\ \frac{1-(1-\delta^2)^{\alpha+1}}{\alpha+1}+2\ \|f\|^2_{\mathcal{D}_\alpha}+\\
&\quad+\sum_{n=0}^{N-1}\left|(1+n)\frac{f^{(1+n)}(0)}{(1+n)!}\frac{f^{(n)}(0)}{f^{(n)}(0)+n!\ a}\right|^2 Be(n,\alpha+1)(1-\delta)^{n+\alpha}<\infty
\end{aligned}$$

and $dg/dw\in A^2_\alpha(D)$. In consequence, (f,a) becomes invertible and $(f,a)^{-1}=(g,1/a)$. □

Corollary 15. *If* $(f,a) \in \mathcal{D}_\alpha$, *its spectrum is:*

$$\sigma(f,a) = cl\ \left\{a + f^{(n)}(0)/n!\right\}_{n\geq 0}.$$

Corollary 16. *Let* $f \in \mathcal{D}_\alpha$ *and let* $\mathfrak{h}_f(g) = f \odot g$, $g \in \mathcal{D}_\alpha$. *Then* $\mathfrak{h}_\alpha$ *is a compact operator on* $\mathcal{D}_\alpha$, $\sigma(\mathfrak{h}_f) = cl\ \left\{f^{(n)}(0)/n!\right\}_{n\geq 0}$ *and* $\sigma_p(\mathfrak{h}_f) = \left\{f^{(n)}(0)/n!\right\}_{n\geq 0}$. *If* $0 \notin \sigma_p(\mathfrak{h}_f)$ *then* $0 \in \sigma_c(\mathfrak{h}_f)$.

Proof. We have

$$\left\|\mathfrak{h}_f - \mathfrak{h}_{\sum_{n=0}^N f^{(n)}(0)/n!\ z^n}\right\|_{\mathcal{B}(\mathcal{D}_\alpha)} \leq \left\|f - \sum_{n=0}^N f^{(n)}(0)/n!\ z^n\right\|_{\mathcal{D}_\alpha},\ N \geq 0.$$

Since each $\mathfrak{h}_{\sum_{n=0}^N f^{(n)}(0)/n!\ z^n}$ has finite rank by (11) the operator $\mathfrak{h}_f$ becomes compact. It is now easy to determine the spectrum and the point spectrum of $\mathfrak{h}_f$. If $0 \notin \sigma_p(\mathfrak{h}_f)$ then

$$\mathcal{R}(\mathfrak{h}_f) = \left\{g \in \mathcal{D}_\alpha : \quad \sum_{n=0}^\infty \left|\frac{h^{(n)}(0)}{f^{(n)}(0)}\right| n^2\ Be(n,\alpha+1) < +\infty\right\}.$$

In particular, $f \notin \mathcal{R}(\mathfrak{h}_f)$. Moreover, by (11) we have

$$\lim_{N\to\infty}\left\|g - \mathfrak{h}_f\left(\sum_{n=0}^N \frac{h^{(n)}(0)}{f^{(n)}(0)}\ z^n\right)\right\|_{\mathcal{D}_\alpha} = 0,$$

so $\mathcal{R}(\mathfrak{h}_f)$ is dense in $\mathcal{D}_\alpha$ and $0 \in \sigma_c(\mathfrak{h}_f)$. □

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